Exam 1 Solutions

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1. (a) False (Consider the function $f(x) = x/x$ at 0.)
   (b) False (Again, Consider the function $f(x) = x/x$ at 0.)
   (c) False (Consider the function $f(x) = |x|$ evaluated at 0.)

2. (This was basically a quiz problem). First note that the function

$$f(x) = x^2 \tan\left(\frac{\pi x}{4}\right)$$

is continuous on $[0, 1]$. Moreover $f(0) = 0$ and $f(1) = 1$. Thus, by IVT the result holds.

3. (a) (Quotient Rule)

$$\begin{aligned} f'(x) &= \frac{0 \cdot (x^2 + 1) - 1 \cdot (2x)}{(1 + x^2)^2} \\ &= \frac{-2x}{(1 + x^2)^2} \end{aligned}$$

   (b) (Linearity or Additive Rule)

$$\begin{aligned} f'(x) &= \frac{(e^x)' + (1/e^x)'}{2} \\ &= \frac{(e^x) - (1/e^x)}{2}. \end{aligned}$$

   (c) (Product Rule then Linearity)

$$\begin{aligned} f'(x) &= (x^2 + 1)' \cdot (1 + e^x) + (x^2 + 1) \cdot (1 + e^x)' \\ &= 2x \cdot (1 + e^x) + (x^2 + 1) \cdot e^x \\ &= 2x + (x + 1)^2 e^x. \end{aligned}$$

(d) (Quotient Rule)

$$\begin{aligned} f'(x) &= \frac{(x^2+1)'(x+1) - (x^2+1)(x+1)'}{(x+1)^2} \\ &= \frac{2x(x+1) - (x^2+1)}{(x+1)^2} \\ &= \frac{x^2+2x-1}{(x+1)^2}. \end{aligned}$$

(e) (Quotient Rule and Power Rule)

$$\begin{aligned} f'(x) &= \frac{(\sqrt{x})'(x+1) - (\sqrt{x})(x+1)'}{(x+1)^2} \\ &= \frac{\frac{1}{2\sqrt{x}}(x+1) - (\sqrt{x})}{(x+1)^2}. \end{aligned}$$

4. (a)

$$\lim_{x\to a}\left(x^2+2\right) = a^2+2.$$

(b)

$$\begin{aligned} \lim_{x\to 0}\frac{\tan(x)}{x} &= \lim_{x\to 0}\frac{\sin(x)}{x\cos(x)} \\ &= \left(\lim_{x\to 0}\frac{\sin(x)}{x}\right)\left(\lim_{x\to 0}\frac{1}{\cos(x)}\right) \quad \text{(since both exist)} \\ &= 1\cdot 1 \\ &= 1. \end{aligned}$$

(c)

$$\begin{aligned} \lim_{x\to -1}\frac{x^2+2x+1}{x+1} &= \lim_{x\to -1}\frac{(x+1)^2}{x+1} \\ &= \lim_{x\to -1}(x+1) \\ &= 0. \end{aligned}$$

5.

$$
\begin{aligned}
(x^3)' :&= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h\to 0} 3x^2 + 3xh + h^2 \\
&= 3x^2.
\end{aligned}
$$

6. **Bonus!** (This was done in class on two separate occasions!!)

$$
\begin{aligned}
(\sin(x))' :&= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)(\cos(h) - 1)}{h} + \lim_{h\to 0} \frac{\cos(x)\sin(h)}{h} \\
&= \sin(x) \cdot \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos(x) \cdot \lim_{h\to 0} \frac{\sin(h)}{h} \\
&= \sin(x) \cdot 0 + \cos(x) \cdot 1 \\
&= \cos(x).
\end{aligned}
$$