Braid representations and Connections with Knot Theory

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1 Introduction

I feel bad for anybody who happens to stumble across this document. This is something that I'm using more than anything to get practice writing in LaTeX . There is very little and poor explanation on almost every topic.

2 Braid Basics

2.1 Presentation

The Braid group is presented as follows:

$$B_n = \langle \sigma_1, \sigma_2 \ldots, \sigma_{n-1} | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| > 1 \rangle \quad (1)$$

Geometrically, the σ_i represent an over-crossing string in $\mathbb{R}^3$. Braids were originally used by Emil Artin in 1925 in relationship with knot theory. In this talk, I will be using several alternate definitions for the Braid group the proof of their equivalence can be found in [1].

2.2 Artin Representation

The braid group acts on the free group on n generators F_n as follows:

$$\sigma_i\left(x_j\right)=\begin{cases} x_i x_{i+1} x_i^{-1} & i=j \\ x_i & i=j+1 \\ x_j & otherwise \end{cases} \tag{2}$$

It can be shown that this representation is faithful. More generally, the Braid group can be viewed as the set of automorphisms of F_n that

1. $\beta x_i = A_i x_{\tau(i)} A_i^{-1}$ for some A_i in F_n and τ in Σ_n.
2. $\beta(x_1 x_2 \cdots x_n) = x_1 x_2 \cdots x_n$

2.3 Mapping Class Group definition

The most useful realization of the braid group is as a certain mapping class group of the n-times punctured disk (denote by D_n throughout this document). To be precise, consider the group of diffeomorphisms of the disk D which fix the boundary pointwise and preserve a finite set of n points (setwise), P_n called "punctures". This space will denoted by Diff($\langle D_n, P_n\rangle$, fix ∂D_n). Of course this space if too large to be of any immediate use, so we mod out by the subgroup of diffeomorphisms which are isotopic to the identity (since if g_1 and f_1 are isotopic to f_2, g_2 respecitevely then $f_1 \circ g_1$ is isotopic to $f_2 \circ g_2$ this space actually is a normal subgroup). In other words, this is the orbit space Diff($\langle D_n, P_n\rangle$, fix $\partial D_n)/\sim$ where $\sim$ is the relation $f \sim g \Leftrightarrow f$ is isotopic to g via an isotopy fixing the boundary and permuting punctures. We define the above as Mod(D_n, fix ∂D_n). Notice that

1. If we had originally decided to consider homeomorphisms or even analytic maps instead of diffeomorphisms, our group would not be any different. This is because analytic maps for a dense open subset of the space of homeomorphisms. See the second chapter of Hirsch's book on differential topology [5].
2. The permutation of P_n determined by the diffeomorphism is an invariant of the isotopy class of the map.

It is surprisingly intuitive to see the equivalence of this group and the braid group. The key is the following lemma:

Lemma 1 (Alexander's Trick) *Any homeomorphism of the disk which fixes the boundary is isotopic to identity via an isotopy which at every stage fixes the boundary.*

Proof See [6]. Note that here we do not require than any subsets are also fixed as we did in the case above). Essentially, since the homeomorphism fixes the boundary, we can slowly shrink the turbulent area to a point.

□

Theorem 2 *Mod(D_n, fix ∂D_n)* $\simeq B_n$.

Proof See [REF NEEDED]. Since any element of Mod(D_n, fix ∂D_n) is isotopic to the identity through homeomorphisms which fix the boundary (again, I stress that this does not say that the element is trivial as a mapping class), we can form the mapping cylinder of this map. Tracing where the punctures go in this motion gives us the braid.

I define the inverse of this process in an intuitive (hand-waving) way. Imagine the disk to be made of rubber and the braids to made of strong wire. Simply push the wire braids through the rubber and follow the motion of the surrounding rubber. This defines out inverse map.

□

3 The Burau Representation

There are two ways of viewing the Burau representation of the braid group. The first one is with use of free differential calculus and the Artin representation, and the second uses the mapping class group idea and deals with automorphisms of the homology of a certain covering space of the n times punctured disk.

3.1 Free calculus

Free calculus was introduced by Ralph H. Fox in a series of papers published in the Annals in 1952 [2]. The seemingly innocent definition has led to many powerful results spanning many aspects of Algebraic topology and knot theory.

Consider the free group on n-generators $F_n :=< x_1, \ldots, x_n| >$ and form the group ring $\mathbb{Z}F_n$ of formal linear combinations of elements in F_n with multiplication defined as in F_n and extended by distributivity. We define a *derivation* to be a map $D : \mathbb{Z}F_n \to \mathbb{Z}F_n$ such that:

1) D is a linear map of $\mathbb{Z}$ modules

2) $D(uv) = D(u)\epsilon(v) + uD(v)$ for all u, v in $\mathbb{Z}F_n$.

Where ϵ is defined by $\epsilon(x) = 1$ for all x in F_n and extended by linearity.

It should be pointed out that this is **not** the definition we would expect from ordinary calculus. As of right now, I do not know how the theory would be different or less useful if the derivation were to be defined in the usual way.

Clearly, it suffices to show how the derivation acts on the basis $x_1 \ldots x_n$. The derivations $\Delta_i : x_i \mapsto \delta_{i,j}$, as with the case with the calculus on manifolds, play the role of basis of all derivations.

Lemma 3 (Free differential calculus basic Lemma) *1) The maps Δ_i are derivations*

2) The maps $\Delta_1, \Delta_2, \ldots, \Delta_n$ form a basis for all (Fox) derivations on $\mathbb{Z}F_n$ (in particular, the space of derivations form a free $\mathbb{Z}F_n$ module).

Proof See [1] or [3]. Note that 1) is obvious.

□

To further the analogy with calculus we denote our maps Δ_i by $\frac{\partial}{\partial x_i} w\,(x_1, \ldots, x_n)$. These maps satisfy a product rule (by definition), a chain rule and a fundamental theorem of free calculus (see below). However, while many of the same theorems apply, the actual calculations have a different feel to them because of the "strange" product rule present.

Example (Some basic sample calculations).

Take $w(x_1, x_2, x_3) = x_1^2 x_2^{-1} + x_3^3$ in F_3. Then

$$\begin{aligned} \frac{\partial}{\partial x_1} w\,(x_1, x_2) &= \frac{\partial}{\partial x_1}(x_1^2 x_2^{-1} + x_3^3) \\ &= \frac{\partial}{\partial x_1}(x_1^2 x_2^{-1}) + \frac{\partial}{\partial x_1} x_3^3 \\ &= 1 + x_1 \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial}{\partial x_2} w\,(x_1, x_2) &= \frac{\partial}{\partial x_2}(x_1^2 x_2^- 1 + x_3^3 \\ &= \frac{\partial}{\partial x_2}(x_1^2 x_2^{-1}) + \frac{\partial}{\partial x_2} x_3^3 \\ &= -x_1^2 x_2^- 1 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial x_3} w\,(x_1, x_2) &= \frac{\partial}{\partial x_3}(x_1^2 x_2^{-1} + x_3^3) \\ &= \frac{\partial}{\partial x_3}(x_1^2 x_2^{-1}) + \frac{\partial}{\partial x_3} x_3^3 \\ &= 1 + x_3 + x_3^2. \end{aligned}$$

□

The immediate power of the free calculus is that it gives a way to form a representation of an automorphism of the free group F_n. This representation is relies on a so called chain rule.

Lemma 4 (The chain rule for Jacobian matrices) *For bases $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n)$, and $(z_1, \ldots, z_n)$ of F_n, we have the following:*

$$\frac{\partial x_i}{\partial z_k} \quad = \quad \sum_{j=1}^{n} \frac{\partial x_i}{\partial y_j} \frac{\partial y_j}{\partial z_k}. \tag{3}$$

Proof See [1] for a straight forward proof or [3] for a slicker approach.

□

In Dynamical systems, a useful invariant (homomorphism from monoid of differential continuous functions) to $\mathbb{R}$) comes from evaluating the derivative of a function f at a fixed point p. This is a direct consequence for the usual calculus chain rule. Similarly, in the case of free calculus, we get a homomorphism compatible with groups of functions that have a certain kind of fixed point.

Notation 5

1. We denote A_ψ:=$\{\phi : \psi \circ \phi = \psi\ \}$ where ψ is a given function $\psi : F_n \to G$ and G is any group.
2. For $\phi \in \mathrm{Aut}(F_n)$ define $\|\phi\|$ as the matrix whose i, j th component is $\partial\phi(x_j)\ /\ \partial x_i$ and for $\phi \in A_\psi \subset \mathrm{Aut}(F_n)$ define $\|\phi\|^\psi$ as $\|\phi\|$ with all of its entries mapped via ψ into G.

Since automorphisms and choosing basis for a free group are the same thing, by the chain rule we get:

Theorem 6 (Magnus) *The map $\phi \mapsto \|\phi\|$ of $A_\phi \to Mat(n, \mathbb{Z}G)$ (where $Mat(n, \mathbb{Z}G)$ is the group of invertible $n \times n$ matrices over $\mathbb{Z}G$) is a representation of A_ϕ.*

Proof Use the chain rule and upon evaluation use invariance.

□

3.2 The topological representation

In this section, I will show the Burau representation can be arrived at by a topological approach using homology theory. Throughout this chapter, I will be viewing the braid group as in the mapping class group definition. Recall that in this definition, a braid is represented is an isotopy class of a diffeomorphism (since diffeomorphisms form a dense open subset of the space of homeomorphisms on any manifold it does not really matter if we consider homeomorphisms or diffeomorphisms, and I may change between the two freely) the space of all of the disk preserving the boundary pointwise and n "punctures" set-wise.

Before we even look at anything having anything to do with braids, let's consider the following which is of a purely algebraic topological nature. We know that the n-times punctured disk has fundamental group which is free on n generators. We define the *winding number* as the map $\phi : \pi_1(D_n) \to \mathbb{Z} = \langle t \rangle$ given by $x_i \mapsto t$ where t is taken as the generator of $\mathbb{Z}$ and x_i is one of these n generators. Geometrically, the generators x_i are loops which start at a given basepoint on the boundary and encircle a puncture once in a counter-clockwise manner before returning to the basepoint and ϕ is the usual winding number one would get by doing a path integral around such a curve.

From covering space theory, there exists a (unique up to homeomorphism) covering space $\tilde{D}_n$ of D_n with $\pi_1 \tilde{D}_n = \ker\phi$ whose space of deck transformations are precisely im $\phi = \mathbb{Z} = \langle t \rangle$. By definition, these deck transformations act on the space $\tilde{D}_n$ by permuting each fiber lying over D_n. This action descends to an action on the level of homology.

There are two things here that will allow us to form the topological version of the Burau representation:

1. $H_1(\tilde{D}_n)$ is a free $\mathbb{Z}[t, t^{-1}]$ module on n-generators.
2. The homeomorphisms $\beta \in B_n : D_n \to D_n$ lift to homeomorphisms $\tilde{\beta} : \tilde{D}_n \to \tilde{D}_n$.

Note that this turns out to be the *reduced* Burau representation since we are dealing with (n-1) $\times$ (n-1) matrices.

Now, the second statement above is a consequence of the general lifting lemma from topology. Note this theorem applies because the action of the braid on F_n preservers ϕ. The first statement can be proved in a geometric setting using the long exact sequence from homology (see [4]), however, I will use the free calculus developed in the previous section.

4 The Interaction between Topological and Differential Methods

The theorem below gives us the best possible dictionary between the language of free calculus and the topological covering space theory developed in the previous section.

Theorem 7 *There is a natural bijective correspondence between words w in $\pi_1 D_n$ and the words $\sum_{j=1}^{n} \frac{\partial w}{\partial x_i} x_i$ in $H_1(\tilde{D}_n)$.*

Proof This proof is geometric in nature. When lifting $\tilde{x_i x_j}$, unfortunately we do not get $\tilde{x_i}\tilde{x_j}$. The problem is that

$\square$