Solutions for 8.3

4.) $A=\sum_{j=1}^{\infty}\frac{100}{3^j-4j}=\sum_{j=1}^{\infty}a_j$. Note the terms in this series are approaching $\frac{100}{3^j}$ by the dominant term concept. This suggests we do a limit comparison with $\sum_{j=1}^{\infty}c_j=\sum_{j=1}^{\infty}\left(\frac{1}{3}\right)^j$.

$$\frac{a_j}{c_j}=\frac{100}{3^j-4j}\cdot\frac{3^j}{1}\rightarrow\frac{100\cdot 3^j}{3^j}=100<\infty$$ so by the limit comparison test A converges.

6.) $A=\sum_{j=5}^{\infty}\left(\frac{20+j}{3j+10}\right)^j=\sum_{j=5}^{\infty}a_j$. Here it is not clear whether $a_j\rightarrow 0$, allowing a possibility for convergence.

Letting $L=\lim_{j\to\infty}\left(\frac{20+j}{3j+10}\right)^j$

$$\ln L=\lim_{j\to\infty}\ln\left(\frac{20+j}{3j+10}\right)^j=\lim_{j\to\infty}j\cdot\ln\left(\frac{20+j}{3j+10}\right)=\lim_{j\to\infty}j\cdot\ln\left(\frac{1}{3}\right)=-\infty \qquad\Rightarrow\qquad L=e^{-\infty}=0$$

So this series may converge. A appears to be similar the series $C=\sum_{j=5}^{\infty}\left(\frac{1}{3}\right)^j=\sum_{j=5}^{\infty}c_j$, a convergent geometric series, so perhaps a limit comparison will work to show convergence.

$$\frac{a_j}{c_j}=\left(\frac{20+j}{3j+10}\right)^j\cdot\frac{3^j}{1}=\left(\frac{(j+20)\cdot 3}{3j+10}\right)^j=\left(\frac{3j+60}{3j+10}\right)^j$$

Letting $L=\lim_{j\to\infty}\frac{a_j}{c_j}$ we have: $$\ln L=\lim_{j\to\infty}\ln\left(\frac{3j+60}{3j+10}\right)^j=\lim_{j\to\infty}\frac{\ln\left(\frac{3j+60}{3j+10}\right)}{1/j}$$

Applying L'Hopital's rule: $$\ln L=\lim_{j\to\infty}\frac{\left(\frac{3j+10}{3j+60}\right)\cdot\left(\frac{3(3j+10)-3(3j+60)}{(3j+10)^2}\right)}{-1/j^2}$$

$$\ln L=\lim_{j\to\infty}\frac{\left(\frac{3j+10}{3j+60}\right)\cdot\left(\frac{-150}{(3j+10)^2}\right)}{-1/j^2}=\lim_{j\to\infty}\left(\frac{(450j+1500)j^2}{(3j+60)(3j+10)^2}\right)=\frac{450}{27}=\frac{50}{3}.$$

So $L=e^{(50/3)}<\infty$ and thus A is a convergent series.

8.) $A=\sum_{j=10}^{\infty}\frac{2+\cos j}{3^j}=\sum_{j=10}^{\infty}a_j$ Note the terms in this series are bounded above by $\frac{3}{3^j}$ and bounded below by $\frac{1}{3^j}$, since $-1 \le \cos j \le 1$. This suggests we do a direct comparison with $C=\sum_{j=10}^{\infty}c_j=\sum_{j=10}^{\infty}3\cdot\left(\frac{1}{3}\right)^j$, known to converge since it is a multiple of the tail of a convergent geometric series. That is noting $0\le a_j \le c_j, \forall j$ implies A converges.

12.) $A=\sum_{j=1}^{\infty}\sqrt{j}=\sum_{j=1}^{\infty}a_j$ Here $a_j=\sqrt{j}\to\infty$ so by the divergence test A diverges.

16.) $A=\sum_{j=0}^{\infty}\frac{(-1)^j}{1+e^j}=\sum_{j=0}^{\infty}a_j$. Alternating series between + and – terms, so test for abs. conv. With $A_p=\sum_{j=0}^{\infty}\left|a_j\right|=\sum_{j=0}^{\infty}\frac{1}{1+e^j}=\sum b_j$. To test this use the Limit Comparison Test, comparing to the known convergent, geometric series $C=\sum_{j=0}^{\infty}c_j=\sum_{j=0}^{\infty}\left(\frac{1}{e}\right)^j$, with absolute value of ratio less than one.

$\frac{b_j}{c_j}=\frac{1}{1+e^j}\cdot\frac{e^j}{1}=\frac{e^j}{1+e^j}\to 1=L$ and since $L<\infty \Rightarrow$ both C and A_p converge, so A is absolutely convergent.

20.) Compare values of $S_{100}=\sum_{j=2}^{100}\frac{1}{\ln j}$ to $S_{200}=\sum_{j=2}^{200}\frac{1}{\ln j}$ and explain result.

Note S_{100} = Σ(1/ln(j),j,2,100)= sum(seq(1/ln(j),j,2,100)) = 29.991

and S_{200} = Σ(1/ln(j),j,2,200)= sum(seq(1/ln(j),j,2,200)) = 50.043.

The sequence of partial sums appears to be increasing rather rapidly. This suggests that $S_n=\sum_{j=2}^{n}\frac{1}{\ln j}$ is diverging and hence $S=\sum_{j=2}^{\infty}\frac{1}{\ln j}=\sum a_j$ is divergent. To prove this compare S to $D=\sum_{j=2}^{\infty}\frac{1}{j}=\sum d_j$, a divergent series since it is a tail of the known divergent harmonic series.

$$j>\ln j \Rightarrow \frac{1}{j}<\frac{1}{\ln j} \text{ or } d_j<a_j$$, so by the comparison test S diverges.