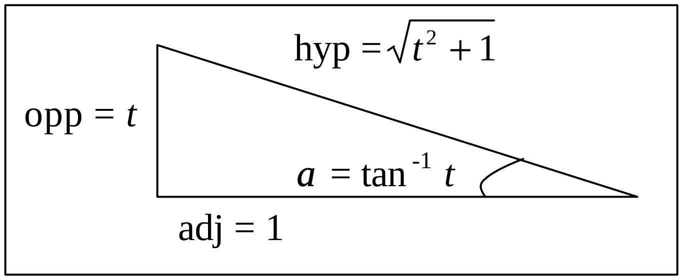
Weierstrass Substitution

This process applies to integrals in which the integrand is a rational function in sin(x) or cos(x) as in the general form $I = \int R(\sin x, \cos x)\,dx$ The substitution

$$t = \tan\frac{x}{2}, \quad x \in (-p, p) \qquad \Leftrightarrow \qquad x = 2\tan^{-1}(t), \quad t \in \Re,$$

changes the integral of a rational function in (sin x) or (cos x). Recalling the double angle identities:

$$\sin(2a) = 2\cdot\sin(a)\cdot\cos(a),$$
$$\cos(2a) = \cos^2(a) - \sin^2(a)$$



and using the right triangle shown as a mnemonic device, notice how:

$$\sin x = \sin(2\tan^{-1}(t)) = 2\cdot\sin(\tan^{-1}(t))\cdot\cos(\tan^{-1}(t)) = 2\cdot\frac{t}{\sqrt{t^2+1}}\cdot\frac{1}{\sqrt{t^2+1}} = \frac{2t}{t^2+1};$$

$$\cos x = \cos(2\tan^{-1}(t)) = \cos^2(\tan^{-1}(t)) - \sin^2(\tan^{-1}(t)) = \left(\frac{1}{\sqrt{t^2+1}}\right)^2 - \left(\frac{t}{\sqrt{t^2+1}}\right)^2 = \frac{1-t^2}{t^2+1};$$

$dx = \frac{2}{t^2+1}dt$. The integral will now have the form $I = \int R\left(\frac{2t}{t^2+1}, \frac{1-t^2}{t^2+1}\right)\frac{2}{t^2+1}dt$.

Another sometimes-useful substitution involves integrals of the form $I = \int f\left(\sqrt[n]{g(x)}\right)dx$. Here we let $\boxed{u = \sqrt[n]{g(x)} \quad \Leftrightarrow \quad u^n = g(x)}$. By the chain rule in differential form, we also have: $n\cdot u^{n-1}\cdot du = g'(x)\cdot dx \quad \Leftrightarrow \quad dx = \frac{n\cdot u^{n-1}}{g'(x)}\cdot du$. Thus this substitution is effective only if the differential expression $f\left(\sqrt[n]{g(x)}\right)\cdot dx$ may be expressed in the form, $h(u)\cdot du$, in which h is a recognizably integrable function.

Non-Elementary or Special Functions

These integrals, and many others, do not simplify in terms of elementary functions:

$\mathrm{erf}\,(x) = \frac{2}{\sqrt{p}}\int_0^x e^{-t^2}dt$, $\quad$ $\mathrm{FresnelC}(x) = \int_0^x \cos\left(\frac{p}{2}t^2\right)dt$, $\quad$ $\mathrm{FresnelS}(x) = \int_0^x \sin\left(\frac{p}{2}t^2\right)dt$

$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}dt$, $\quad$ $\mathrm{Ei}(x) = \int_0^x \frac{\sin t}{t}dt$.

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