Special Section:

Polynomial Functions and Models

1 Identify Polynomial Functions and Their Degree

A **polynomial function** is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad \textbf{(1)}$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers and n is a nonnegative integer. The domain is the set of all real numbers.

EXAMPLE

Identifying Polynomial Functions

Determine which of the following are polynomial functions. For those that are, state the degree; for those that are not, tell why not.

(a) $f(x) = 2 - 3x^4$

(b) $g(x) = \sqrt{x}$

(c) $h(x) = \dfrac{x^2 - 2}{x^3 - 1}$

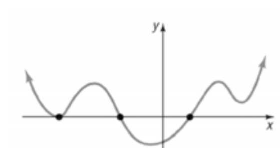
(d) $F(x) = 0$

(e) $G(x) = 8$

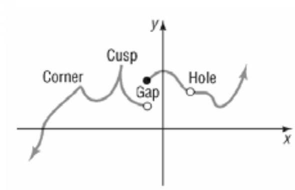
(f) $H(x) = -2x^3(x-1)^2$

Summary of the Properties of the Graphs of Polynomial Functions

Degree	Form	Name	Graph
No degree	$f(x) = 0$	Zero function	The x-axis
0	$f(x) = a_0, \quad a_0 \neq 0$	Constant function	Horizontal line with y-intercept a_0
1	$f(x) = a_1x + a_0, \quad a_1 \neq 0$	Linear function	Nonvertical, nonhorizontal line with slope a_1 and y-intercept a_0
2	$f(x) = a_2x^2 + a_1x + a_0, \quad a_2 \neq 0$	Quadratic function	Parabola: Graph opens up if $a_2 > 0$; graph opens down if $a_2 < 0$



(a) Graph of a polynomial function: smooth, continuous



(b) Cannot be the graph of a polynomial function

2 Graph Polynomial Functions Using Transformations

A **power function of degree** n is a function of the form

$$f(x) = ax^n$$

where a is a real number, $a \neq 0$, and $n > 0$ is an integer.

Examples of power functions are

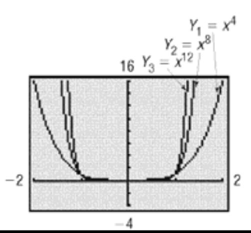
$f(x) = 3x$ (degree 1) $\quad f(x) = -5x^2$ (degree 2) $\quad f(x) = 8x^3$ (degree 3) $\quad f(x) = -5x^4$ (degree 4)

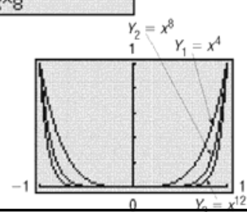
Exploration

Using your graphing utility and the viewing window $-2 \leq x \leq 2$, $-4 \leq y \leq 16$, graph the function $Y_1 = f(x) = x^4$. On the same screen, graph $Y_2 = g(x) = x^8$. Now, also on the same screen, graph $Y_3 = h(x) = x^{12}$. What do you notice about the graphs as the magnitude of the exponent increases? Repeat this procedure for the viewing window $-1 \leq x \leq 1$, $0 \leq y \leq 1$. What do you notice?

X	Y2	Y3
-1	1	1
1	1	1
.5	.00391	2.4E-4
.1	1E-8	1E-12
.01	1E-16	1E-24
.001	1E-24	1E-36
0	0	0

Y2■X^8



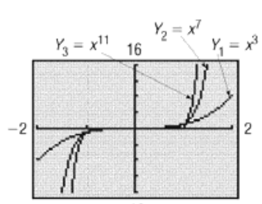


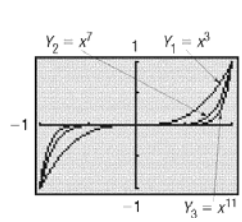
Properties of Power Functions, $f(x) = x^n$, n Is an Even Integer

1. The graph is symmetric with respect to the y-axis, so f is even.
2. The domain is the set of all real numbers. The range is the set of nonnegative real numbers.
3. The graph always contains the points $(0, 0)$, $(1, 1)$, and $(-1, 1)$.
4. As the exponent n increases in magnitude, the graph becomes more vertical when $x < -1$ or $x > 1$; but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

Exploration

Using your graphing utility and the viewing window $-2 \leq x \leq 2$, $-16 \leq y \leq 16$, graph the function $Y_1 = f(x) = x^3$. On the same screen, graph $Y_2 = g(x) = x^7$ and $Y_3 = h(x) = x^{11}$. What do you notice about the graphs as the magnitude of the exponent increases? Repeat this procedure for the viewing window $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. What do you notice?





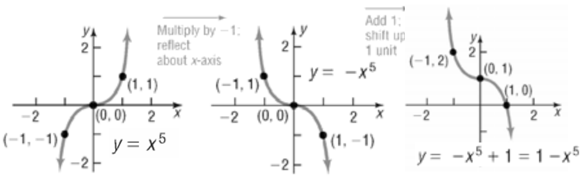
Properties of Power Functions, $f(x) = x^n$, n Is an Odd Integer

1. The graph is symmetric with respect to the origin, so f is odd.
2. The domain and the range are the set of all real numbers.
3. The graph always contains the points $(0, 0)$, $(1, 1)$, and $(-1, -1)$.
4. As the exponent n increases in magnitude, the graph becomes more vertical when $x < -1$ or $x > 1$, but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

EXAMPLE

Graphing Polynomial Functions Using Transformations

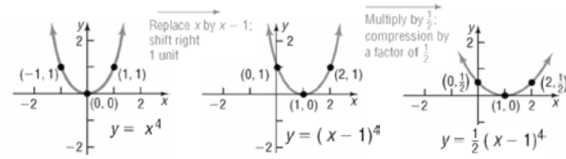
Graph: $f(x) = 1 - x^5$



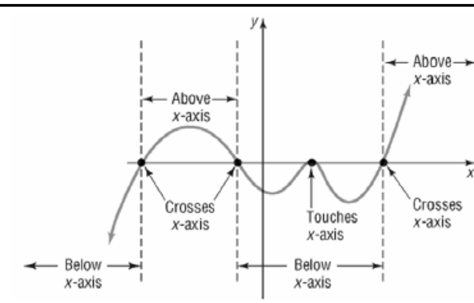
EXAMPLE

Graphing Polynomial Functions Using Transformations

Graph: $f(x) = \frac{1}{2}(x - 1)^4$



3 Identify the Zeros of a Polynomial Function and Their Multiplicity



If f is a polynomial function and r is a real number for which $f(r) = 0$, then r is called a (real) **zero of f**, or **root of f**. If r is a (real) zero of f, then

(a) r is an x-intercept of the graph of f.

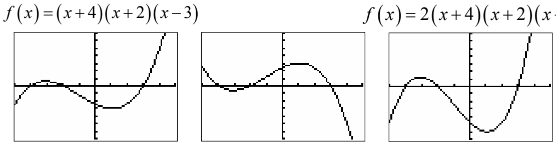
(b) $(x - r)$ is a factor of f.

EXAMPLE **Finding a Polynomial from Its Zeros**

Find a polynomial of degree 3 whose zeros are -4, -2, and 3.

$$f(x) = a(x+4)(x+2)(x-3)$$

$f(x) = (x+4)(x+2)(x-3)$ $\qquad$ $f(x) = 2(x+4)(x+2)(x-3)$



$f(x) = -(x+4)(x+2)(x-3)$

If $(x - r)^m$ is a factor of a polynomial f and $(x - r)^{m+1}$ is not a factor of f, then r is called a **zero of multiplicity m of f**.

EXAMPLE **Identifying Zeros and Their Multiplicities**

For the polynomial, list all zeros and their multiplicities.

$$f(x) = -2(x-2)(x+1)^3(x-3)^4$$

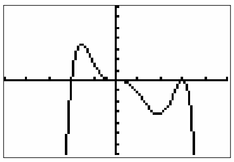
EXAMPLE

For the polynomial $-x^3(x-3)^2(x+2)$

(a) Find the x- and y-intercepts of the graph of f.

(b) Using a graphing utility, graph the polynomial.

(c) For each x-intercept, determine whether it is of odd or even multiplicity.



X	Y1
-3	-972
-2	0
-1	16
0	0
1	-12
2	-32
3	0

Y1 = -X^3(X-3)²(X…

X	Y1
0	0
1	-12
2	-32
3	0
4	-384
5	-3500
6	-15552

Y1 = -X^3(X-3)²(X…

If r Is a Zero of Even Multiplicity

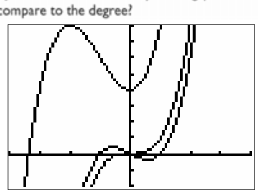
Sign of $f(x)$ does not change from one side of r to the other side of r. — Graph **touches** x-axis at r.

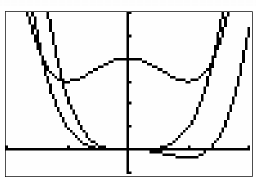
If r Is a Zero of Odd Multiplicity

Sign of $f(x)$ changes from one side of r to the other side of r. — Graph **crosses** x-axis at r.

Exploration

Graph $Y_1 = x^3$, $Y_2 = x^3 - x$, and $Y_3 = x^3 + 3x^2 + 4$. How many turning points do you see? How does the number of turning points relate to the degree? Graph $Y_1 = x^4$, $Y_2 = x^4 - \frac{4}{3}x^3$, and $Y_3 = x^4 - 2x^2$. How many turning points do you see? How does the number of turning points compare to the degree?





Theorem

If f is a polynomial function of degree n, then f has at most $n - 1$ turning points.

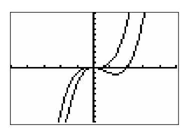
Exploration

For each pair of functions Y_1 and Y_2 given in parts (a), (b), and (c), graph Y_1 and Y_2 on the same viewing window. Create a TABLE or TRACE for large positive and large negative values of x. What do you notice about the graphs of Y_1 and Y_2 as x becomes very large and positive or very large and negative?

(a) $Y_1 = x^2(x-2)$; $Y_2 = x^3$

(b) $Y_1 = x^4 - 3x^3 + 7x - 3$; $Y_2 = x^4$

(c) $Y_1 = -2x^3 + 4x^2 - 8x + 10$; $Y_2 = -2x^3$



X	Y1	Y2
-6	-288	-216
-5	-175	-125
-4	-96	-64
-3	-45	-27
-2	-16	-8
-1	-3	-1
0	0	0

Y1 = X²(X-2)

X	Y1	Y2
1	-1	1
2	0	8
3	9	27
4	32	64
5	75	125
6	144	216
7	245	343

X=7

Calculator screens shown for part (a) only.

Theorem

End Behavior

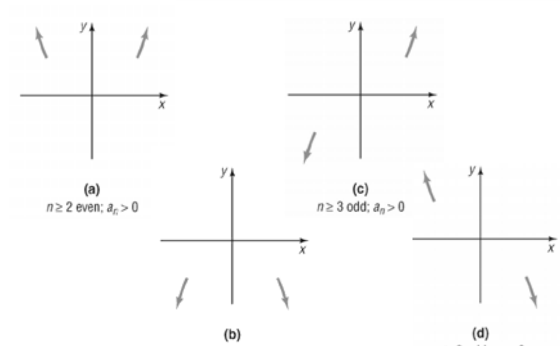
For large values of x, either positive or negative, that is, for large $|x|$, the graph of the polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

resembles the graph of the power function

$$y = a_n x^n$$

End Behavior $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$



Summary

Graph of a Polynomial Function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0$

- Degree of the polynomial f: n
- Maximum number of turning points: $n - 1$
- At a zero of even multiplicity: The graph of f touches the x-axis.
- At a zero of odd multiplicity: The graph of f crosses the x-axis.
- Between zeros, the graph of f is either above or below the x-axis.
- End behavior: For large $|x|$, the graph of f behaves like the graph of $y = a_n x^n$.

4 Analyze the Graph of a Polynomial Function

EXAMPLE

Analyzing the Graph of a Polynomial Function

For the polynomial: $f(x) = x^4 - 2x^3 - 8x^2$

(a) Find the degree of the polynomial. Determine the end behavior; that is, find the power function that the graph of f resembles for large values of $|x|$.

(b) Find the x- and y-intercepts of the graph of f.

(c) Determine whether the graph crosses or touches the x-axis at each x-intercept.

(d) Use a graphing utility to graph f.

(e) Determine the number of turning points on the graph of f. Approximate the turning points, if any exist, rounded to two decimal places.

(f) Use the information obtained in parts (a) to (e) to draw a complete graph of f by hand.

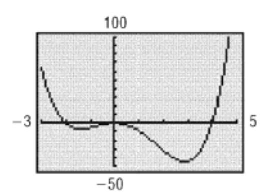
(g) Find the domain of f. Use the graph to find the range of f.

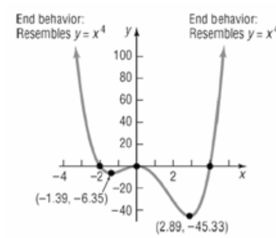
(h) Use the graph to determine where f is increasing and where f is decreasing.

EXAMPLE

Analyzing the Graph of a Polynomial Function

For the polynomial: $f(x) = x^4 - 2x^3 - 8x^2$





EXAMPLE

Using a Graphing Utility to Analyze the Graph of a Polynomial Function

For the polynomial $f(x) = x^3 + 2.48x^2 - 4.3155x + 2.484406$:

(a) Find the degree of the polynomial. Determine the end behavior: that is, find the power function that the graph of f resembles for large values of $|x|$.

(b) Graph f using a graphing utility.

(c) Find the x- and y-intercepts of the graph.

(d) Use a TABLE to find points on the graph around each x-intercept.

(e) Determine the local maxima and local minima, if any exist, rounded to two decimal places. That is, locate any turning points.

(f) Use the information obtained in parts (a)–(e) to draw a complete graph of f by hand. Be sure to label the intercepts, turning points, and the points obtained in part (d).

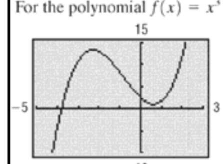
(g) Find the domain of f. Use the graph to find the range of f.

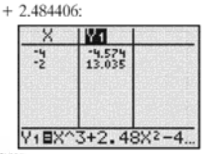
(h) Use the graph to determine where f is increasing and where f is decreasing.

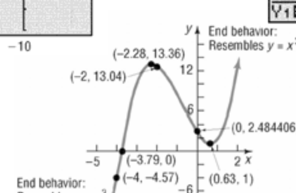
EXAMPLE

Using a Graphing Utility to Analyze the Graph of a Polynomial Function

For the polynomial $f(x) = x^3 + 2.48x^2 - 4.3155x + 2.484406$:







Summary

Steps for Graphing a Polynomial by Hand

To analyze the graph of a polynomial function $y = f(x)$, follow these steps:

STEP 1: End behavior: find the power function that the graph of f resembles for large values of x.

STEP 2: (a) Find the x-intercepts, if any, by solving the equation $f(x) = 0$.
(b) Find the y-intercept by letting $x = 0$ and finding the value of $f(0)$.

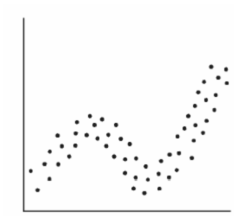
STEP 3: Determine whether the graph of f crosses or touches the x-axis at each x-intercept.

STEP 4: Use a graphing utility to graph f. Determine the number of turning points on the graph of f. Approximate any turning points rounded to two decimal places.

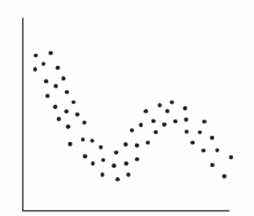
STEP 5: Use the information obtained in Steps 1 to 4 to draw a complete graph of f by hand.

5 Find the Cubic Function of Best Fit to Data

Data That Follow A Cubic Relation



$y = ax^3 + bx^2 + cx + d, a > 0$



$y = ax^3 + bx^2 + cx + d, a < 0$

EXAMPLE A Cubic Function of Best Fit

The data in Table 8 represent the average miles per gallon of vans, pickups, and sport utility vehicles (SUVs) in the United States for 1991–2001, where 1 represents 1991, 2 represents 1992, and so on.

(a) Draw a scatter diagram of the data using the year x as the independent variable and average miles per gallon M as the dependent variable. Comment on the type of relation that may exist between the two variables x and M.

(b) Using a graphing utility, find the cubic function of best fit $M = M(x)$ to these data.

(c) Graph the cubic function of best fit on your scatter diagram.

(d) Use the function found in part (b) to predict the average miles per gallon in 2002 $(x = 12)$.

Table on next slide.

Year, x	Average Miles per Gallon, M
1991, 1	17.0
1992, 2	17.3
1993, 3	17.4
1994, 4	17.3
1995, 5	17.3
1996, 6	17.2
1997, 7	17.2
1998, 8	17.2
1999, 9	17.0
2000, 10	17.4
2001, 11	17.6