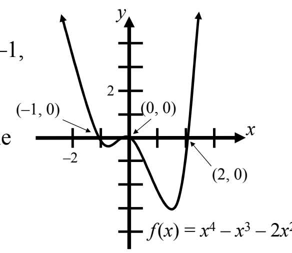
2.3: Graphing Polynomial Functions

Starter 2.3: Find all the real zeros of $f(x) = x^4 - x^3 - 2x^2$.

Factor completely:
$f(x) = x^4 - x^3 - 2x^2 = x^2(x + 1)(x - 2)$.

The real zeros are $x = -1$, $x = 0$, and $x = 2$.

These correspond to the x-intercepts.



Graphs of Polynomial Functions

We have learned how to graph functions with the following degrees:

0 *Example:* $f(x) = 2$ *horizontal line*

1 *Example:* $f(x) = 2x - 3$ *line*

2 *Example:* $f(x) = 2x^2 + 2x - 3$ *parabola*

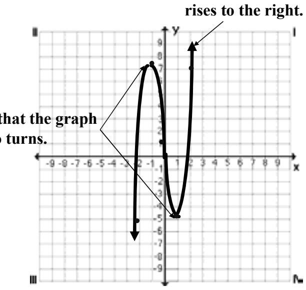
How do you graph polynomial functions with degrees higher than 2?

Example: **Graph $f(x) = 3x^3 - 9x + 1$**

We'll make a table of values, then graph...

Notice that the graph rises to the right.

Notice that the graph has two turns.



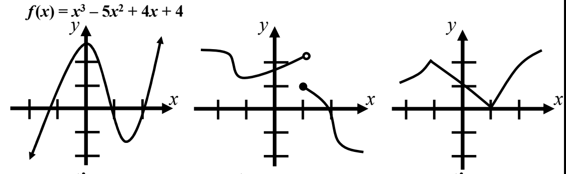
x	$f(x)$
-3	**-53**
-2	**-5**
-1	**7**
0	**1**
1	**-5**
2	**7**
3	**55**

Graphs of Polynomial Functions:

- ***are continuous (there are no breaks)***
- ***have smooth turns (shaped like an unbroken "sideways S"***
- ***with degree n, have at most n – 1 turns (in the previous example, the polynomial had a degree of 3, so there were 2 turns)***
- ***rise to the right if the leading coefficient is positive (in the previous example, the leading coefficient was 3, so the graph rose to the right).***
- ***fall to the right if the leading coefficient is negative.***

Graphs of polynomial functions are **continuous**. That is, they have no breaks, holes, or gaps.

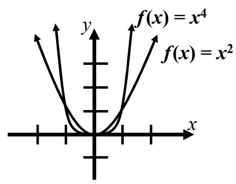
$f(x) = x^3 - 5x^2 + 4x + 4$

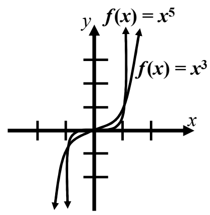


continuous smooth polynomial	not continuous not polynomial	continuous not smooth not polynomial

Polynomial functions are also **smooth** with rounded turns. Graphs with points or cusps are not graphs of polynomial functions.

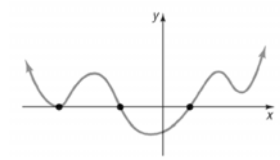
Polynomial functions of the form $f(x) = x^n$, $n \geq 1$ are called **power functions**.



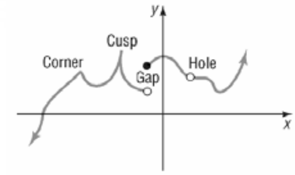


If n is ***even***, their graphs resemble the graph of $f(x) = x^2$.

If n is ***odd***, their graphs resemble the graph of $f(x) = x^3$.

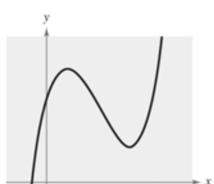


(a) Graph of a polynomial function: smooth, continuous

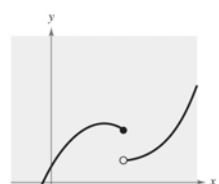


(b) Cannot be the graph of a polynomial function

Continuous and Discontinuous Polynomial Graphs

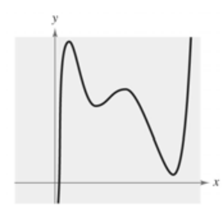


(a) Polynomial functions have continuous graphs.

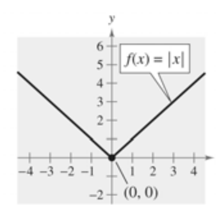


(b) Functions with graphs that are not continuous are not polynomial functions.

Polynomial Functions with Smooth and Sharp Turns



Polynomial functions have graphs with smooth rounded turns.



Graphs of polynomial functions cannot have sharp turns.

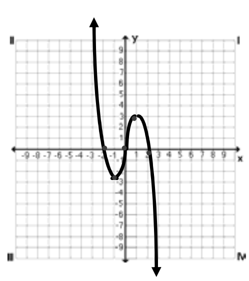
Example: Graph $f(x) = -x^3 + 4x$

Prior to graphing, predict the number of turns and the left and right behaviors of the graph…

The graph will have at most 2 turns (since it is degree 3)

The graph will fall to the right (since the leading coefficient is –1)

x	f(x)
-4	**48**
-3	**15**
-2	**0**
-1	**-3**
0	**0**
1	**3**
2	**0**
3	**-15**
4	**-48**

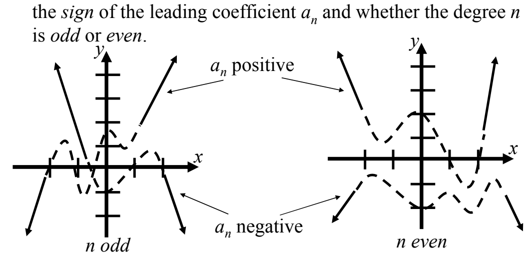


Leading Coefficient Test

As x grows positively or negatively without bound, the value $f(x)$ of the polynomial function

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \quad (a_n \neq 0)$$

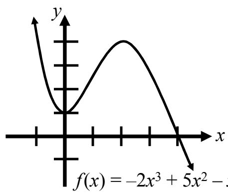
grows positively or negatively without bound depending upon the *sign* of the leading coefficient a_n and whether the degree n is *odd* or *even*.



Example: Describe the right-hand and left-hand behavior for the graph of $f(x) = -2x^3 + 5x^2 - x + 1$.

Degree	3	Odd
Leading Coefficient	-2	Negative

As $x \to +\infty, f(x) \to -\infty$ and as $x \to -\infty, f(x) \to +\infty$



A real number a is a **zero** of a function $y = f(x)$ if and only if $f(a) = 0$.

Real Zeros of Polynomial Functions
If $y = f(x)$ is a polynomial function and a is a real number then the following statements are equivalent.

1. a is a zero of f.
2. a is a solution of the polynomial equation $f(x) = 0$.
3. $x - a$ is a factor of the polynomial $f(x)$.
4. $(a, 0)$ is an x-intercept of the graph of $y = f(x)$.

A **turning (inflection) point** of a graph of a function is a point at which the graph changes from increasing to decreasing or *vice versa*.

A polynomial function of degree n has at most $n - 1$ *turning points* and at most n *zeros*.

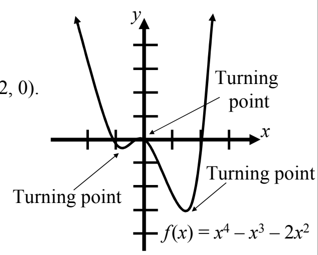
Example: Find all the real zeros and turning points of the graph of $f(x) = x^4 - x^3 - 2x^2$.

Factor completely: $f(x) = x^4 - x^3 - 2x^2 = x^2(x + 1)(x - 2)$.

The real zeros are $x = -1$, $x = 0$, and $x = 2$.

These correspond to the x-intercepts $(-1, 0)$, $(0, 0)$ and $(2, 0)$.

The graph shows that there are three turning points. Since the degree is four, this is the maximum number possible.

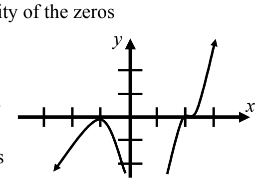


Repeated Zeros
If k is the largest integer for which $(x - a)^k$ is a factor of $f(x)$ and $k > 1$, then a is a **repeated zero** of **multiplicity** k.

1. If k is *odd* the graph of $f(x)$ crosses the x-axis at $(a, 0)$.
2. If k is *even* the graph of $f(x)$ touches, but does not cross through, the x-axis at $(a, 0)$.

Example: Determine the multiplicity of the zeros of $f(x) = (x - 2)^3(x + 1)^4$.

Zero	Multiplicity	Behavior
2	3 odd	crosses x-axis at (2, 0)
–1	4 even	touches x-axis at (–1, 0)

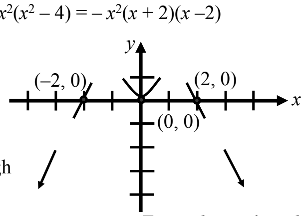


Example: Sketch the graph of $f(x) = 4x^2 - x^4$.

1. Write the polynomial function in standard form: $f(x) = -x^4 + 4x^2$
   The leading coefficient is *negative* and the degree is *even*.
   as $x \to \pm\infty, f(x) \to -\infty$
2. Find the zeros of the polynomial by factoring.
   $f(x) = -x^4 + 4x^2 = -x^2(x^2 - 4) = -x^2(x + 2)(x - 2)$

Zeros:
$x = -2, 2$ multiplicity 1
$x = 0$ multiplicity 2

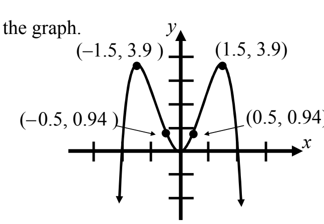
x-intercepts:
$(-2, 0)$, $(2, 0)$ crosses through
$(0, 0)$ touches only



Example continued

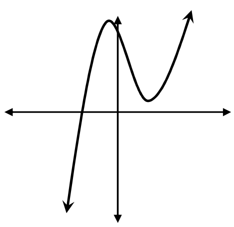
Example continued: Sketch the graph of $f(x) = 4x^2 - x^4$.

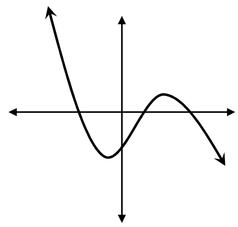
3. Since $f(-x) = 4(-x)^2 - (-x)^4 = 4x^2 - x^4 = f(x)$, the graph is symmetrical about the y-axis.
4. Plot additional points and their reflections in the y-axis: (1.5, 3.9) and (–1.5, 3.9), (0.5, 0.94) and (–0.5, 0.94)
5. Draw the graph.



Graph of f(x) = $ax^3 + bx^2 + cx + d$

$a > 0$ **$a < 0$**





Sketch the graph of f(x) = $(x + 1)(x - 1)(x - 2)$

1. ***Find and plot the zeros of the function.***
2. ***Perform a sign analysis of f(x) by testing one value of x from each of the intervals determined by the zeros.***
3. ***Sketch the graph: step 1 gives you the x-intercepts of the graphs, and step 2 tells you where the graph is above or below the x-axis.***

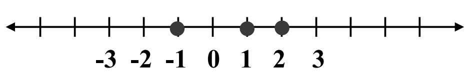
Sketch the graph of f(x) = $(x + 1)(x - 1)(x - 2)$

1. ***Find and plot the zeros of the function.***

$(x + 1)(x - 1)(x - 2) = 0$

$x = -1, \quad x = 1, \quad x = 2$

The zeros of *f* are -1, 1, and 2.



Sketch the graph of f(x) = $(x + 1)(x - 1)(x - 2)$

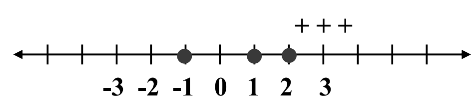
2. ***Perform a sign analysis of f(x) by testing one value of x from each of the intervals determined by the zeros.***

Interval: $x > 1$ ***Test:*** $x = 3$

$f(3) = (3 + 1)(3 - 1)(3 - 2)$

+ + + = +

signs of factors *sign of product*



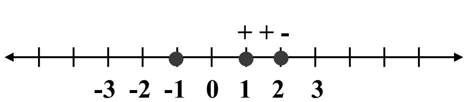
Sketch the graph of f(x) = $(x + 1)(x - 1)(x - 2)$

Interval: $1 < x < 2$ ***Test:*** $x = 1.5$

$f(1.5) = (1.5 + 1)(1.5 - 1)(1.5 - 2)$

+ + - = -

signs of factors *sign of product*



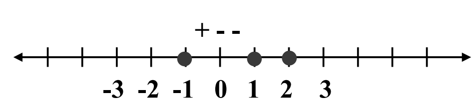
Sketch the graph of f(x) = $(x + 1)(x - 1)(x - 2)$

Interval: $-1 < x < 1$ ***Test:*** $x = 0$

$f(0) = (0 + 1)(0 - 1)(0 - 2)$

+ - - = +

signs of factors *sign of product*

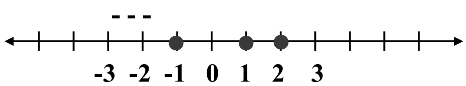


Sketch the graph of $f(x) = (x + 1)(x - 1)(x - 2)$

Interval: $x < -1$ *Test:* $x = -2$

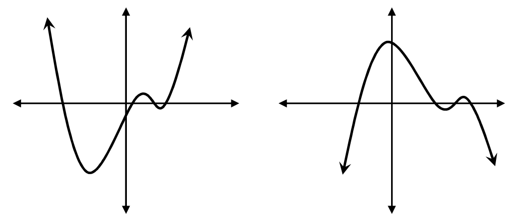
$f(-2) = (-2 + 1)(-2 - 1)(-2 - 2)$

$\underbrace{- \quad - \quad -}_{\text{signs of factors}} \; \underbrace{= \; -}_{\text{sign of product}}$



Graph of $f(x) = ax^4 + bx^3 + cx^2 + dx + e$

$a > 0$ $a < 0$



Effect of a Squared or Cubed Factor

- ***If a polynomial P(x) has a squared factor such as*** $(x - c)^2$***, then*** $x = c$ ***is a double root of*** $P(x) = 0$***. In this case, the graph of*** $y = P(x)$ ***is tangent to the x-axis at*** $x = c$***.***
- ***If a polynomial P(x) has a cubed factor such as*** $(x - c)^3$***, then*** $x = c$ ***is a triple root of*** $P(x) = 0$***. In this case, the graph of*** $y = P(x)$ ***flattens out around (c, 0) and crosses the x-axis at this point.***