Efficient Design of Low-Order H_∞ Optimal Controllers Using Evolutionary Algorithms and a Bisection Approach

A. Popov and **H. Werner**

Hamburg University of Technology
Institute of Control Systems

October 2006

Motivation

Computing the H_∞ norm

Evolutionary Algorithms

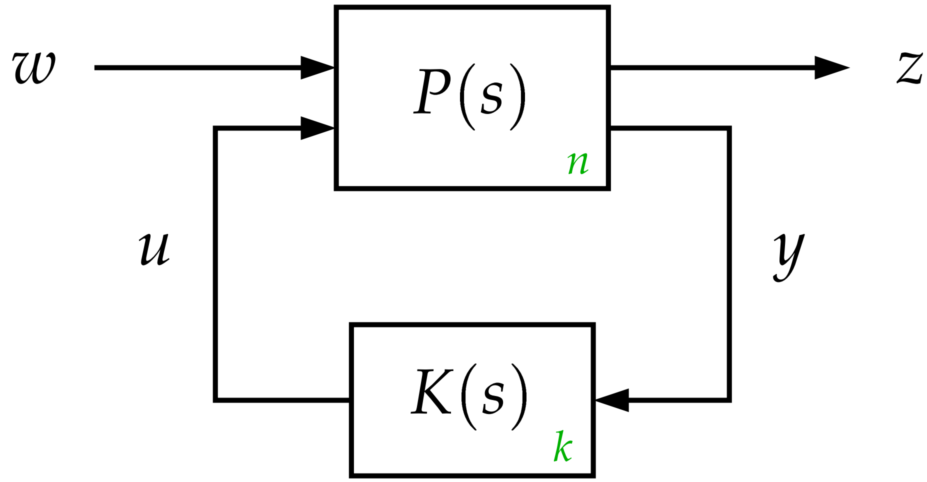
Population bisection

Numerical results

Conclusions

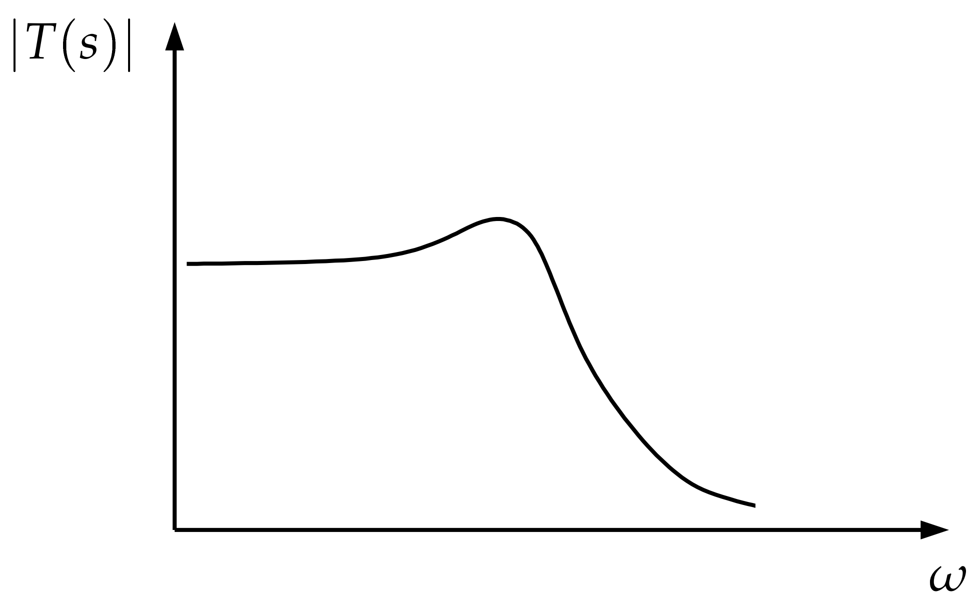
Motivation

- Often high order plant models are available
- The standard design methodologies lead to high order controllers
- Computational and numerical constraints limit the order of the applicable controller
- Evolutionary Algorithms (EA) are a powerful tool for attacking non-convex problems
- The computation cost associated with EA is high



Designing H_∞ controller of order $k < n$ is a non-convex problem
see e.g. [Skelton et al.(1997)]

Bisection method according to [Boyd et al.(1988)]

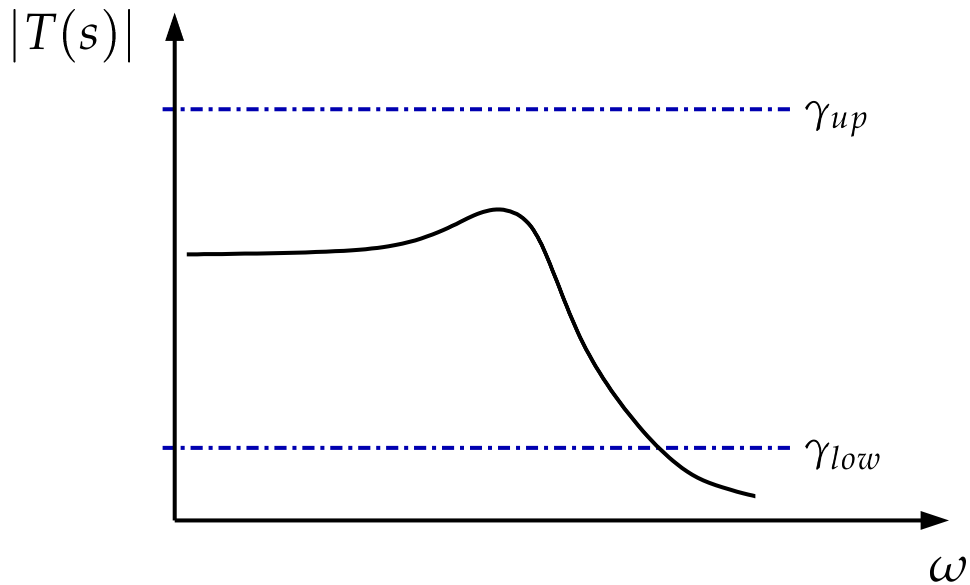


Eqs (1)

Bisection method

Bisection method according to [Boyd et al.(1988)]

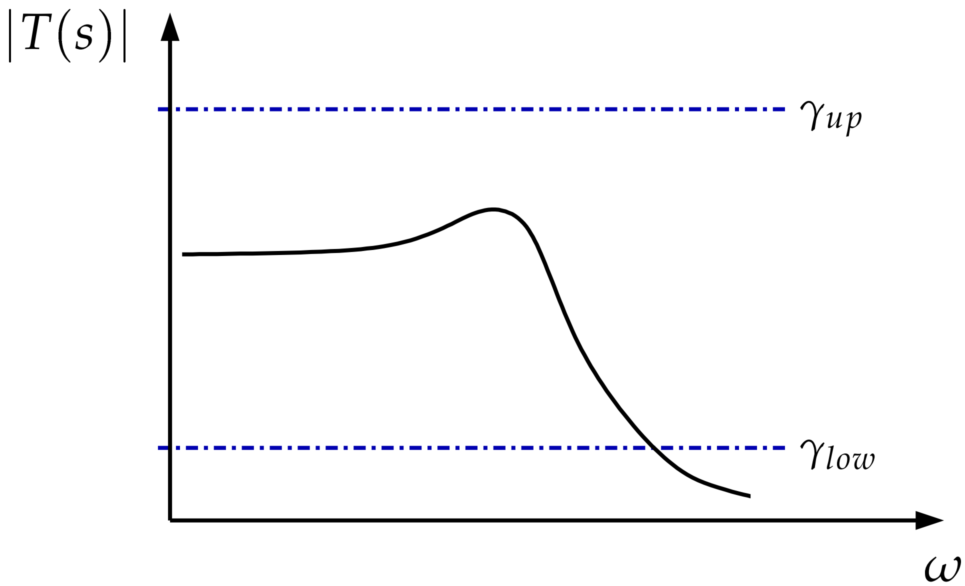
1. Calculate lower and upper bounds for the H_∞ norm using the Hankel singular values.



Bisection method

Bisection method according to [Boyd et al.(1988)]

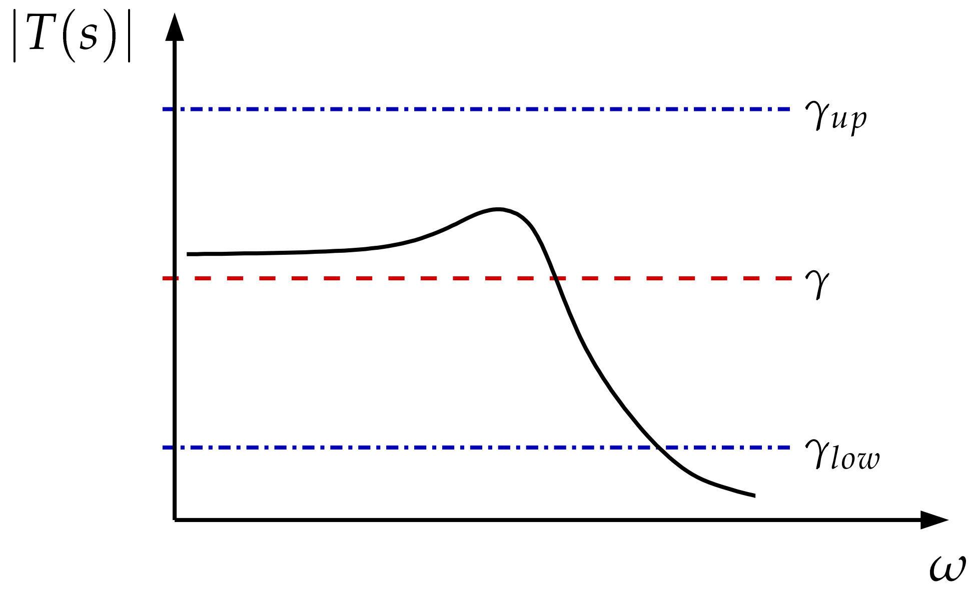
1. Calculate lower and upper bounds for the H_∞ norm using the Hankel singular values.
2. If boundaries closer than desired precision STOP



Bisection method

Bisection method according to [Boyd et al.(1988)]

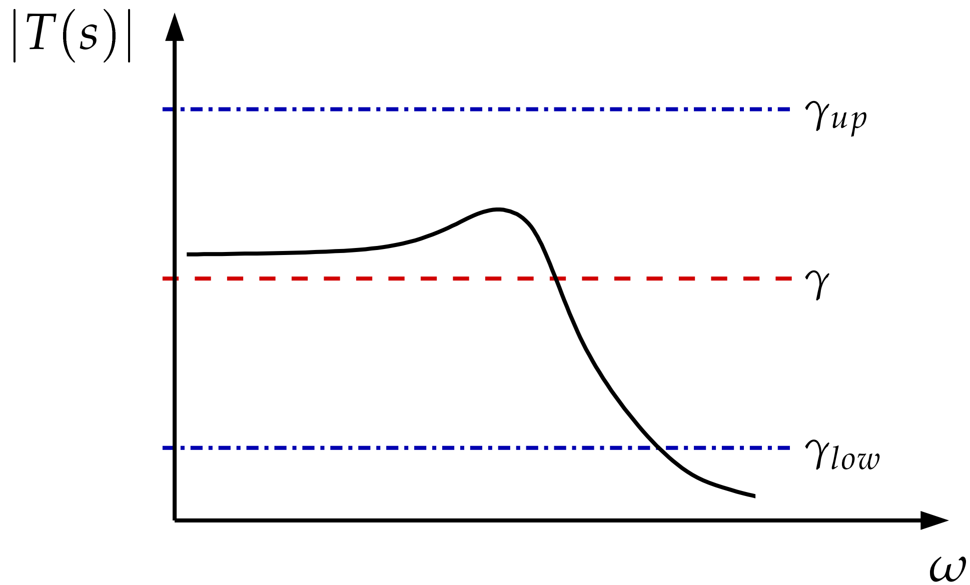
1. Calculate lower and upper bounds for the H_∞ norm using the Hankel singular values.
2. If boundaries closer than desired precision STOP
3. Take γ in the middle between lower and upper boundary.



Bisection method

Bisection method according to [Boyd et al.(1988)]

1. Calculate lower and upper bounds for the H_∞ norm using the Hankel singular values.
2. If boundaries closer than desired precision STOP
3. Take γ in the middle between lower and upper boundary.
4. Form the Hamiltonian matrix M_γ and compute its eigenvalues.



Bisection method

Bisection method according to [Boyd et al.(1988)]

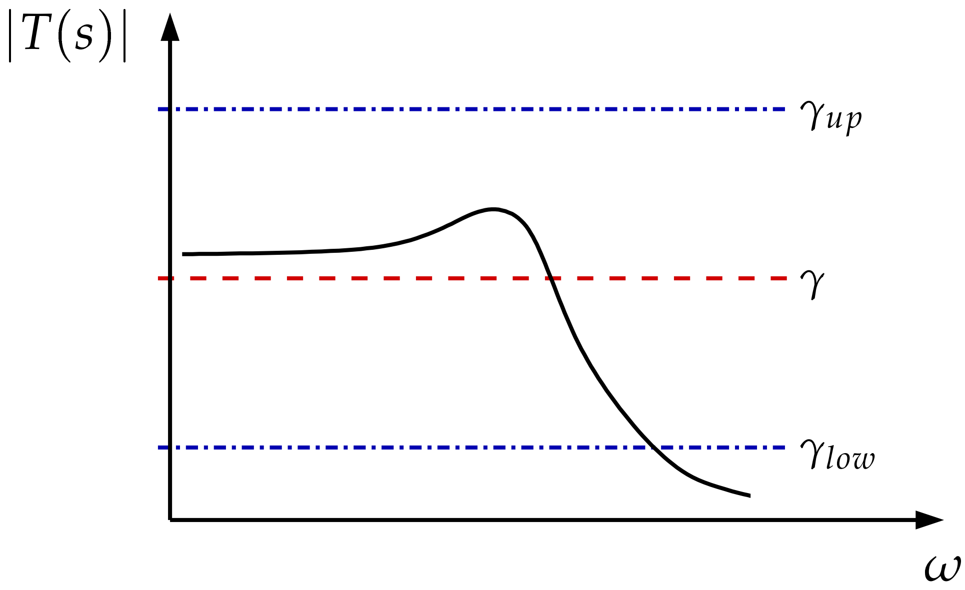
1. Calculate lower and upper bounds for the H_∞ norm using the Hankel singular values.
2. If boundaries closer than desired precision STOP
3. Take γ in the middle between lower and upper boundary.
4. Form the Hamiltonian matrix M_γ and compute its eigenvalues.
5. If M_γ has no imaginary eigenvalues.

$$\gamma_{up} = \gamma$$

else

$$\gamma_{low} = \gamma$$

GO TO step 2;



Bisection method

Bisection method according to [Boyd et al.(1988)]

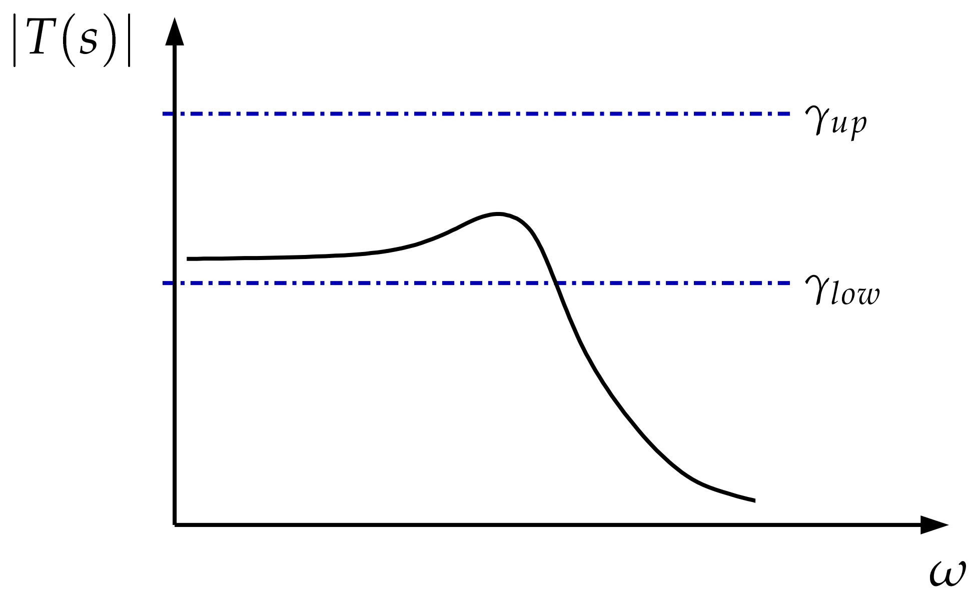
1. Calculate lower and upper bounds for the H_∞ norm using the Hankel singular values.
2. If boundaries closer than desired precision STOP
3. Take γ in the middle between lower and upper boundary.
4. Form the Hamiltonian matrix M_γ and compute its eigenvalues.
5. If M_γ has no imaginary eigenvalues.

$$\gamma_{up} = \gamma$$

else

$$\gamma_{low} = \gamma$$

GO TO step 2;



Bisection method

Bisection method according to [Boyd et al.(1988)]

1. Calculate lower and upper bounds for the H_∞ norm using the Hankel singular values.

2. If boundaries closer than desired precision STOP

3. Take γ in the middle between lower and upper boundary.

4. Form the Hamiltonian matrix M_γ and compute its eigenvalues.

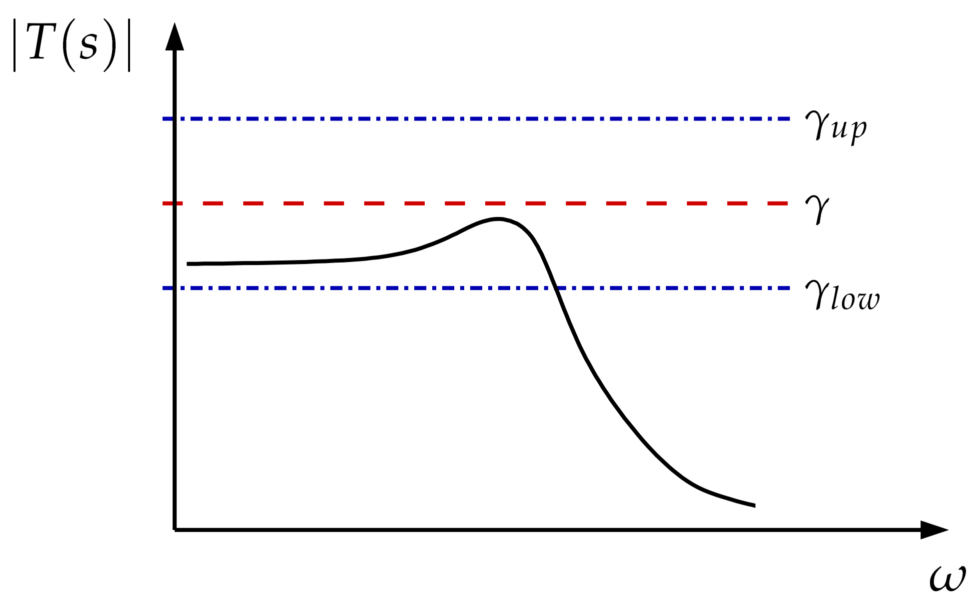
5. If M_γ has no imaginary eigenvalues.

$$\gamma_{up} = \gamma$$

else

$$\gamma_{low} = \gamma$$

GO TO step 2;



Bisection method

Bisection method according to [Boyd et al.(1988)]

1. Calculate lower and upper bounds for the H_∞ norm using the Hankel singular values.

2. If boundaries closer than desired precision STOP

3. Take γ in the middle between lower and upper boundary.

4. Form the Hamiltonian matrix M_γ and compute its eigenvalues.

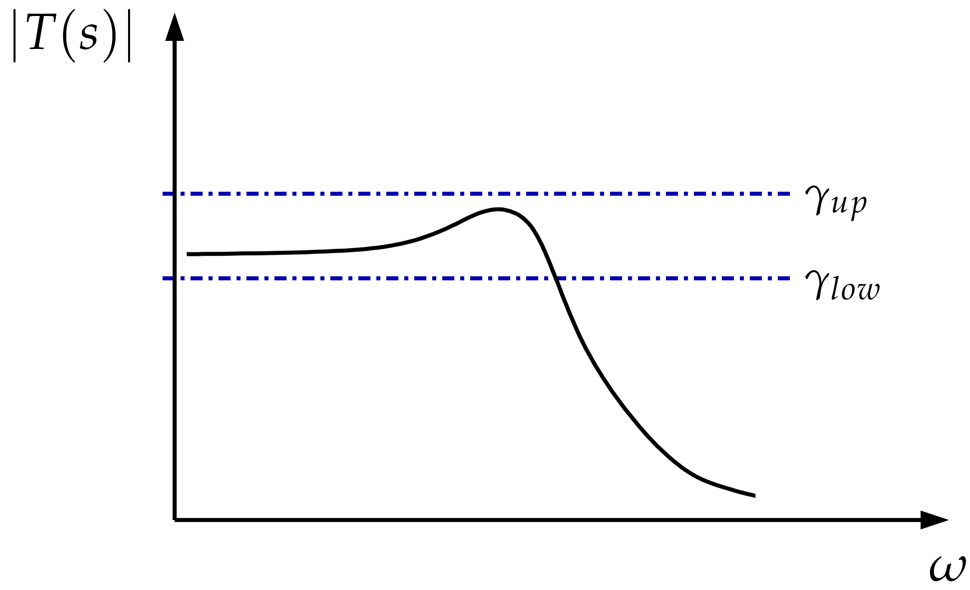
5. If M_γ has no imaginary eigenvalues.

$$\gamma_{up} = \gamma$$

else

$$\gamma_{low} = \gamma$$

GO TO step 2;



Bisection method

Bisection method according to [Boyd et al.(1988)]

1. Calculate lower and upper bounds for the H_∞ norm using the Hankel singular values.

2. If boundaries closer than desired precision STOP

3. Take γ in the middle between lower and upper boundary.

4. Form the Hamiltonian matrix M_γ and compute its eigenvalues.

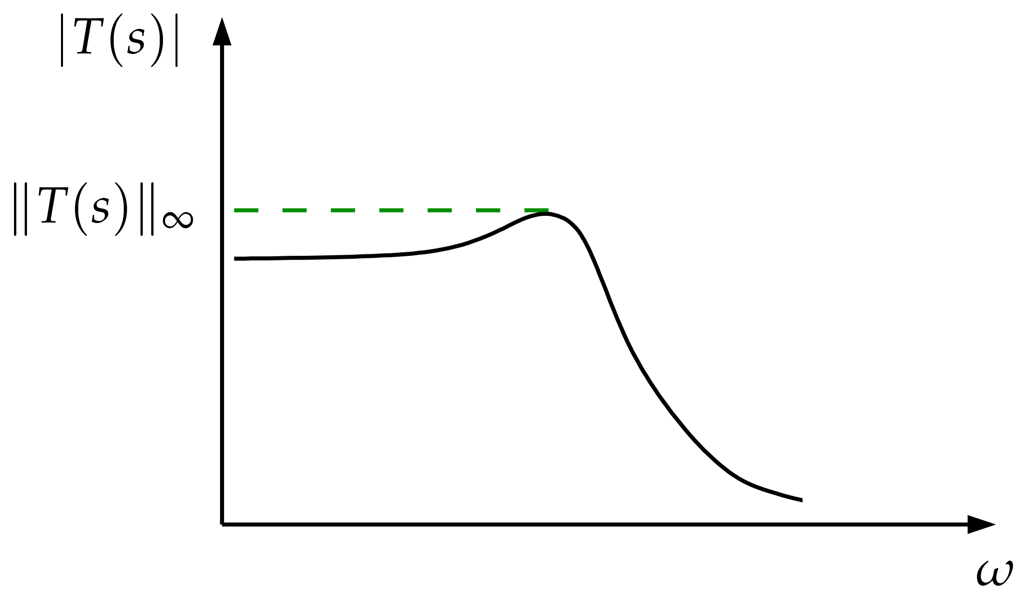
5. If M_γ has no imaginary eigenvalues.

$$\gamma_{up} = \gamma$$

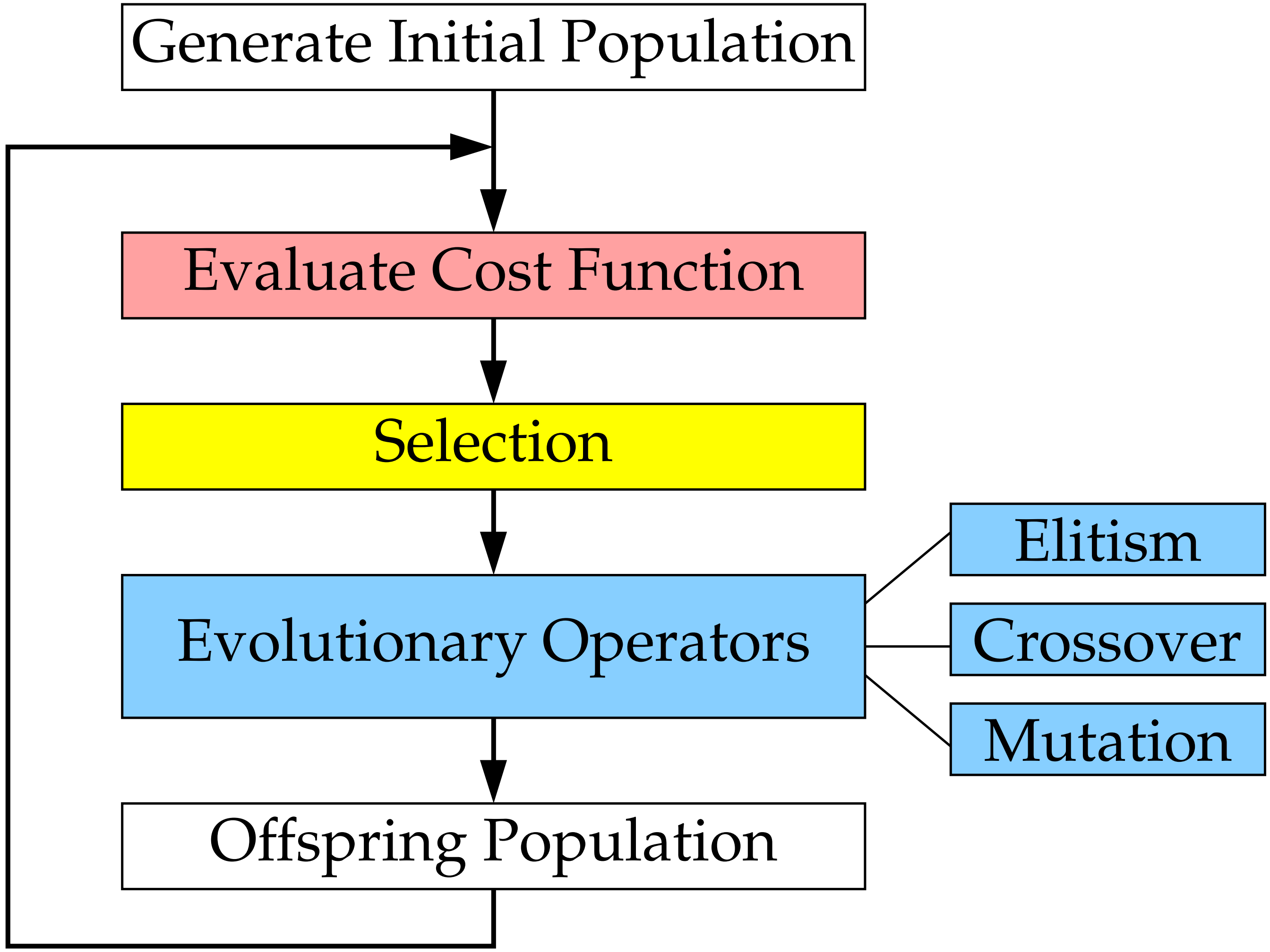
else

$$\gamma_{low} = \gamma$$

GO TO step 2;



Eqs (1)


Generate Initial Population
Evaluate Cost Function
Selection
Evolutionary Operators
Elitism
Crossover
Mutation
Offspring Population

	c_1	c_2	c_3
cost funct. $f(c)$	0.1	1	10

Function proportional

Ranking selection

rank	1	2	3

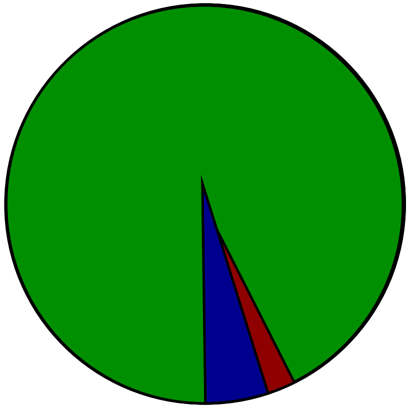
fitness $F(c)$

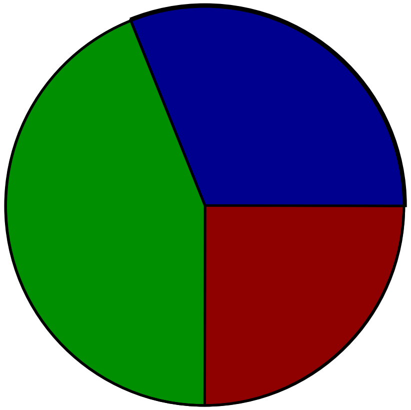
$$\frac{k}{f(c)}$$

0.9 0.09 0.009

$$\frac{k}{\sqrt{rank(c)}}$$

0.44 0.31 0.25





Motivation

Computing the H_∞ norm

Evolutionary Algorithms

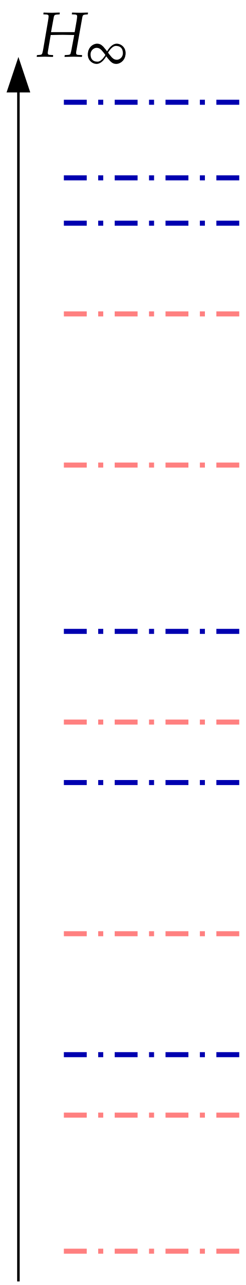
Population bisection

Numerical results

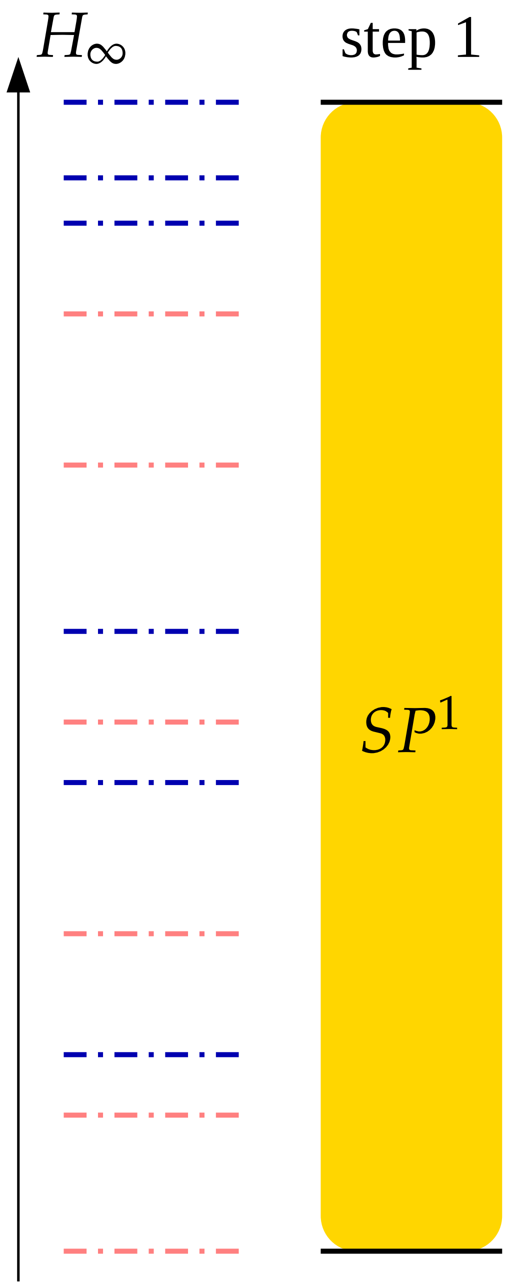
Conclusions

Population Bisection

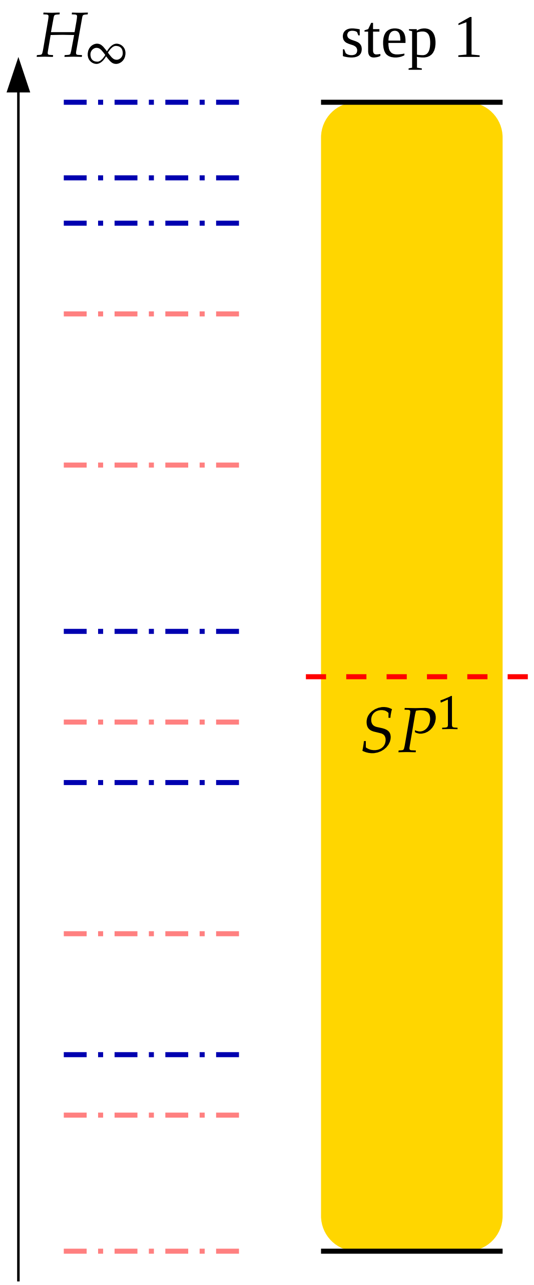
Population Bisection



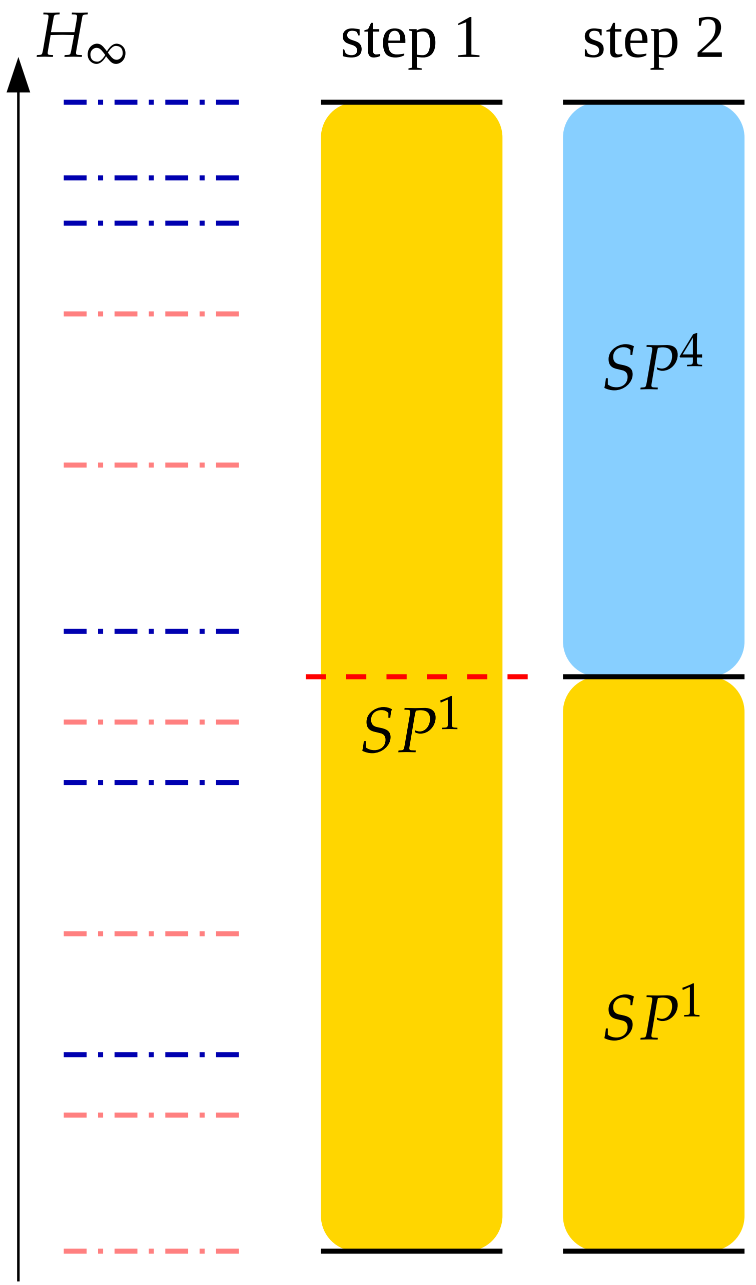
Population Bisection



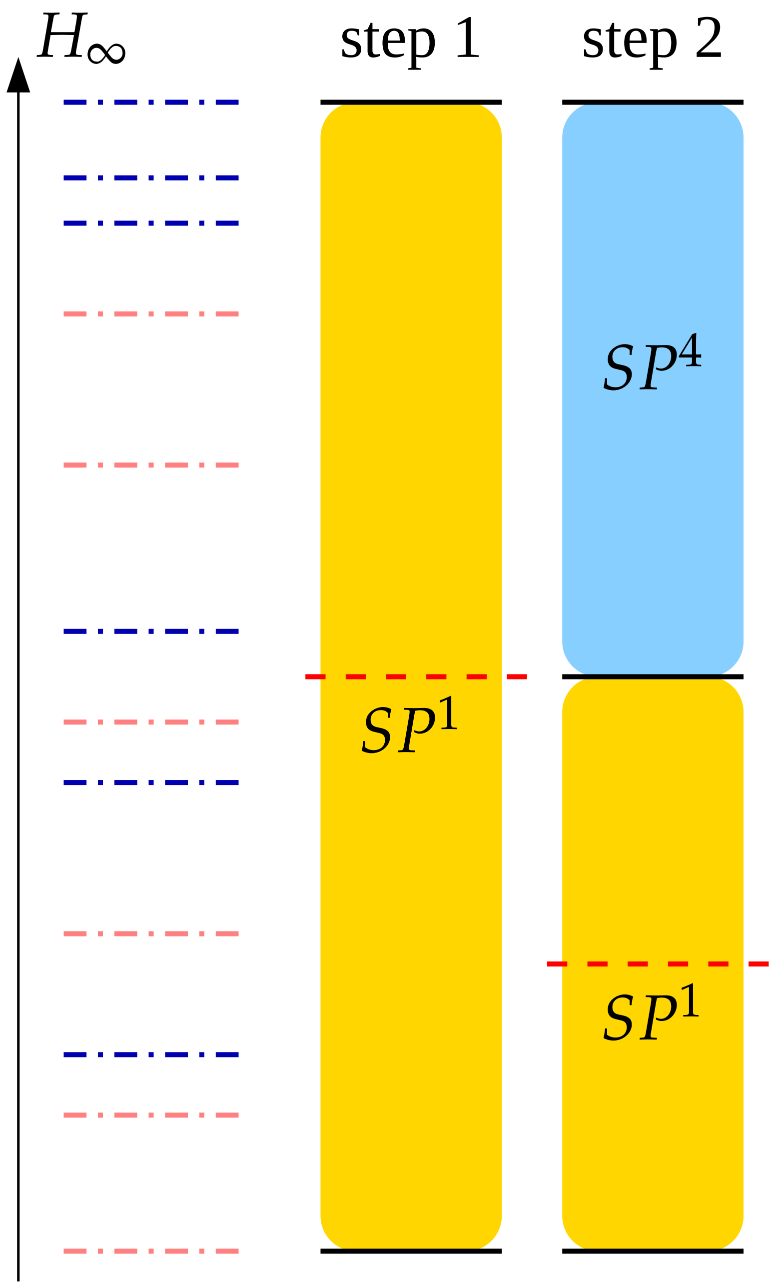
Population Bisection



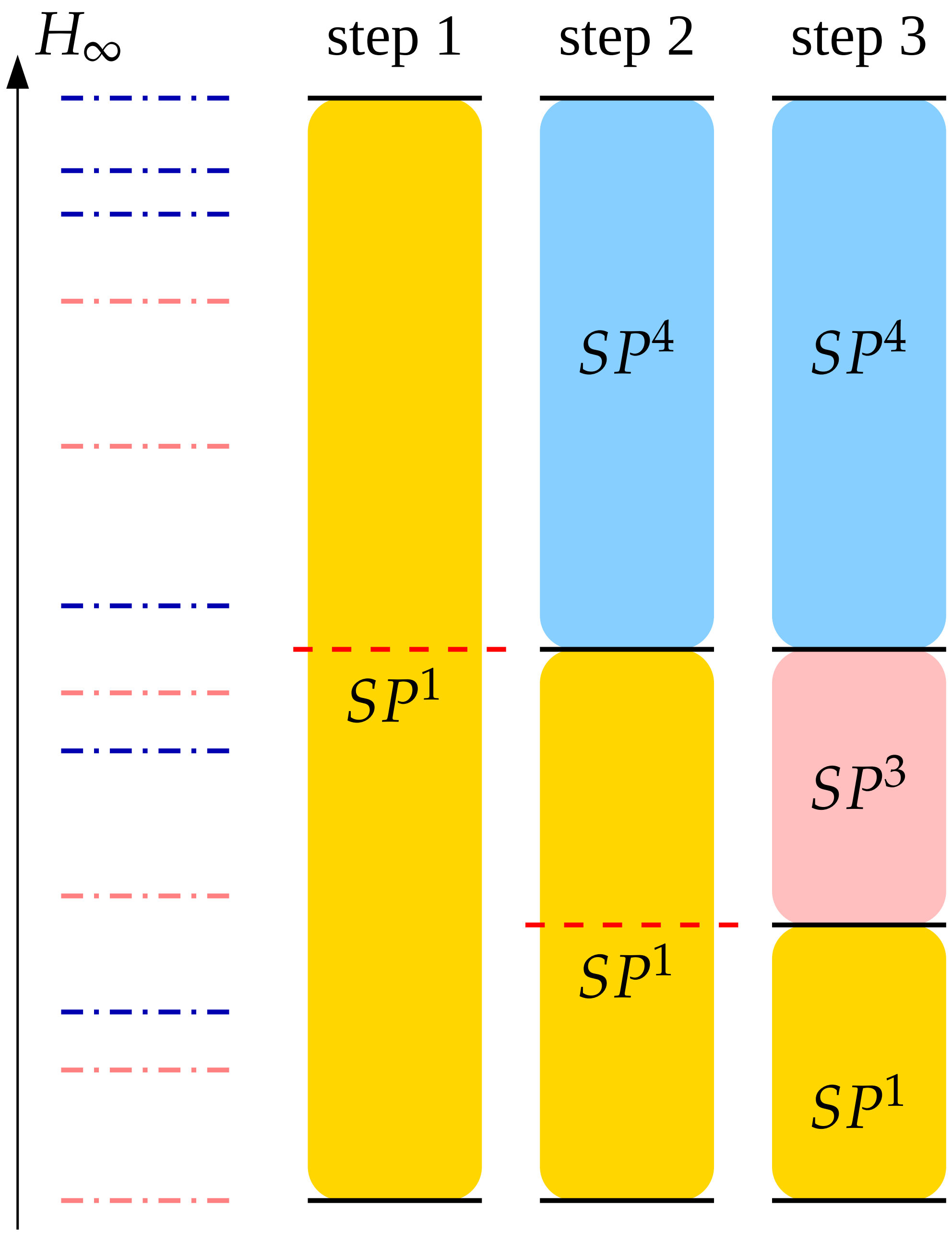
Population Bisection



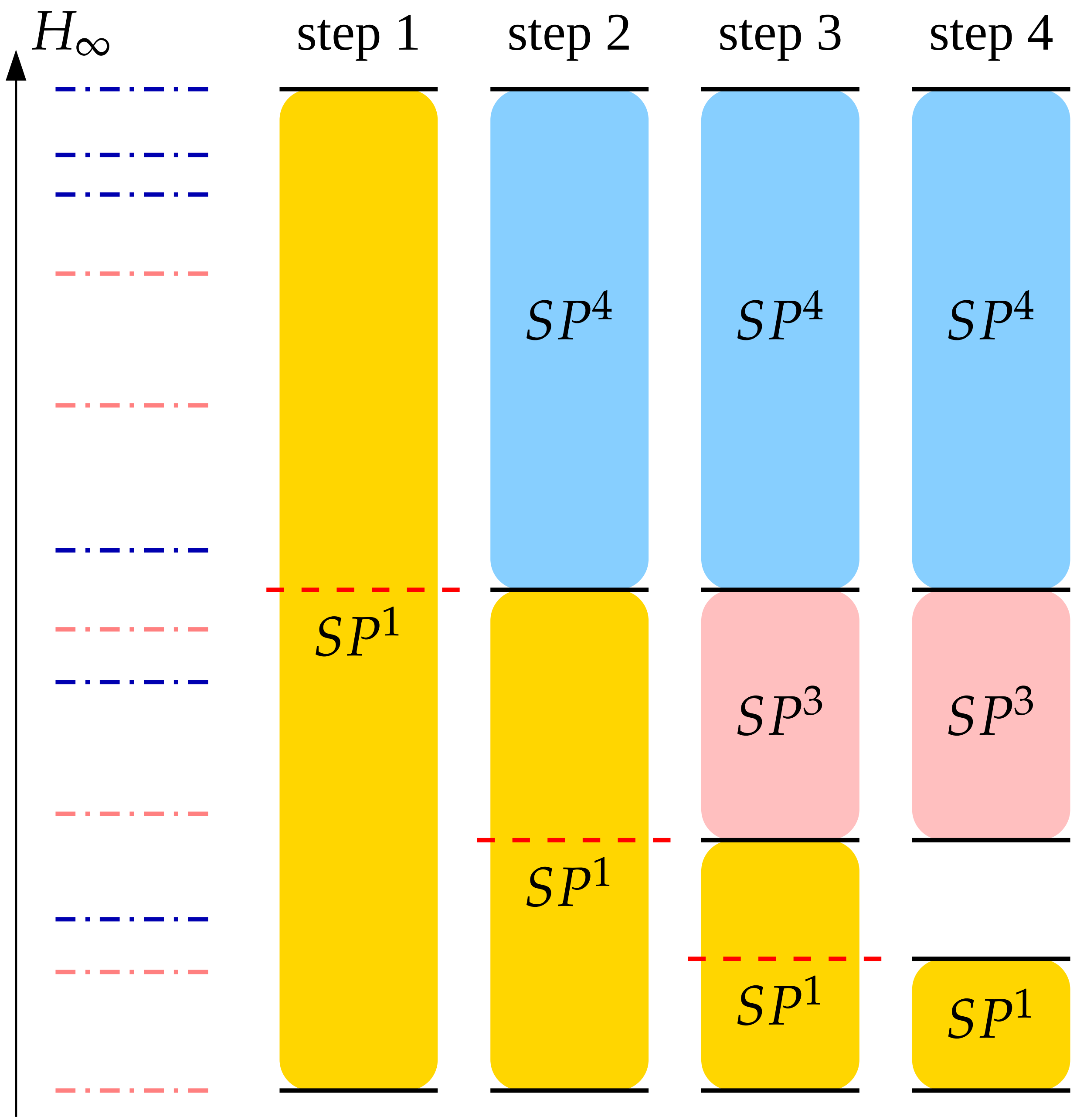
Population Bisection



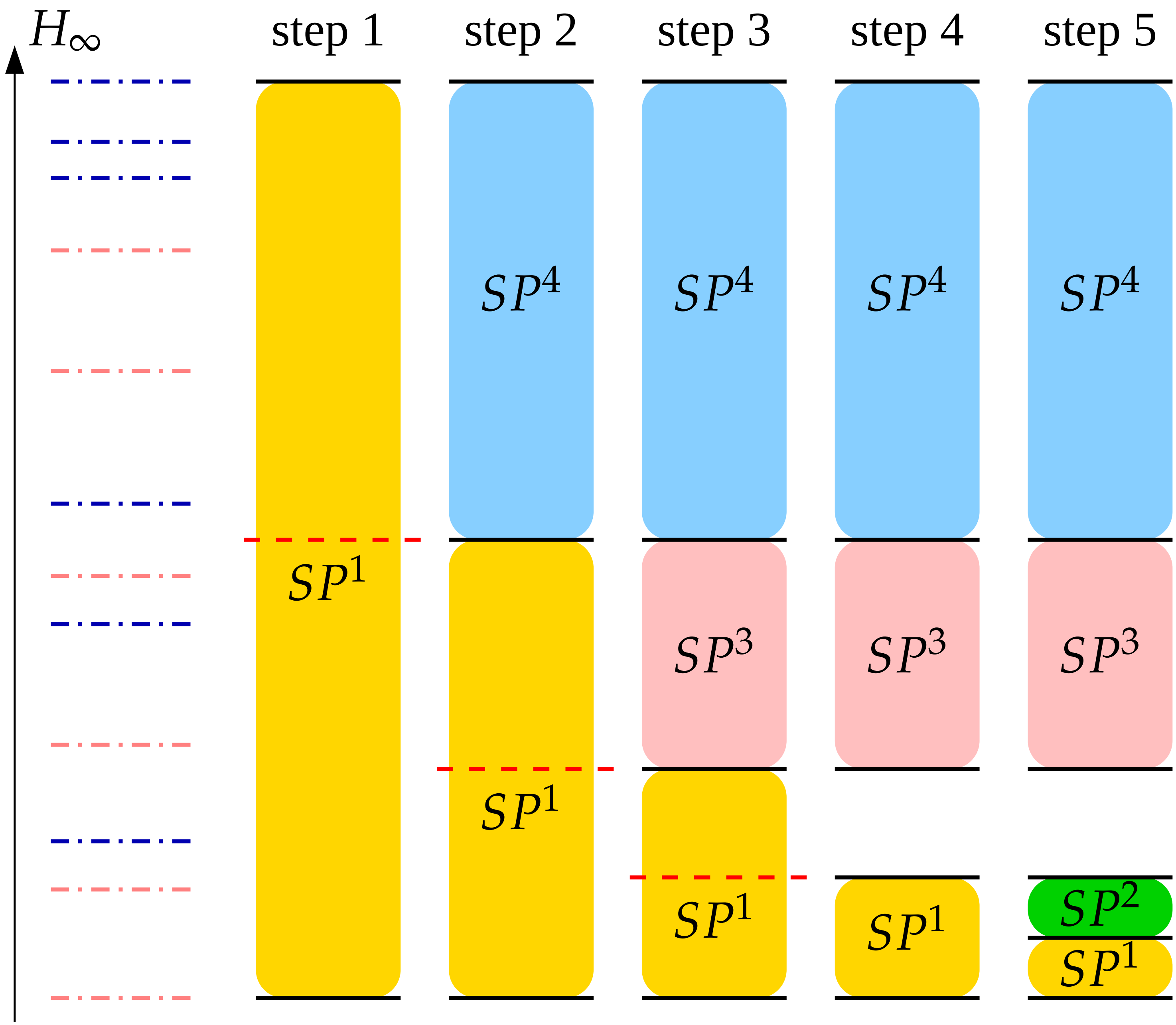
Population Bisection



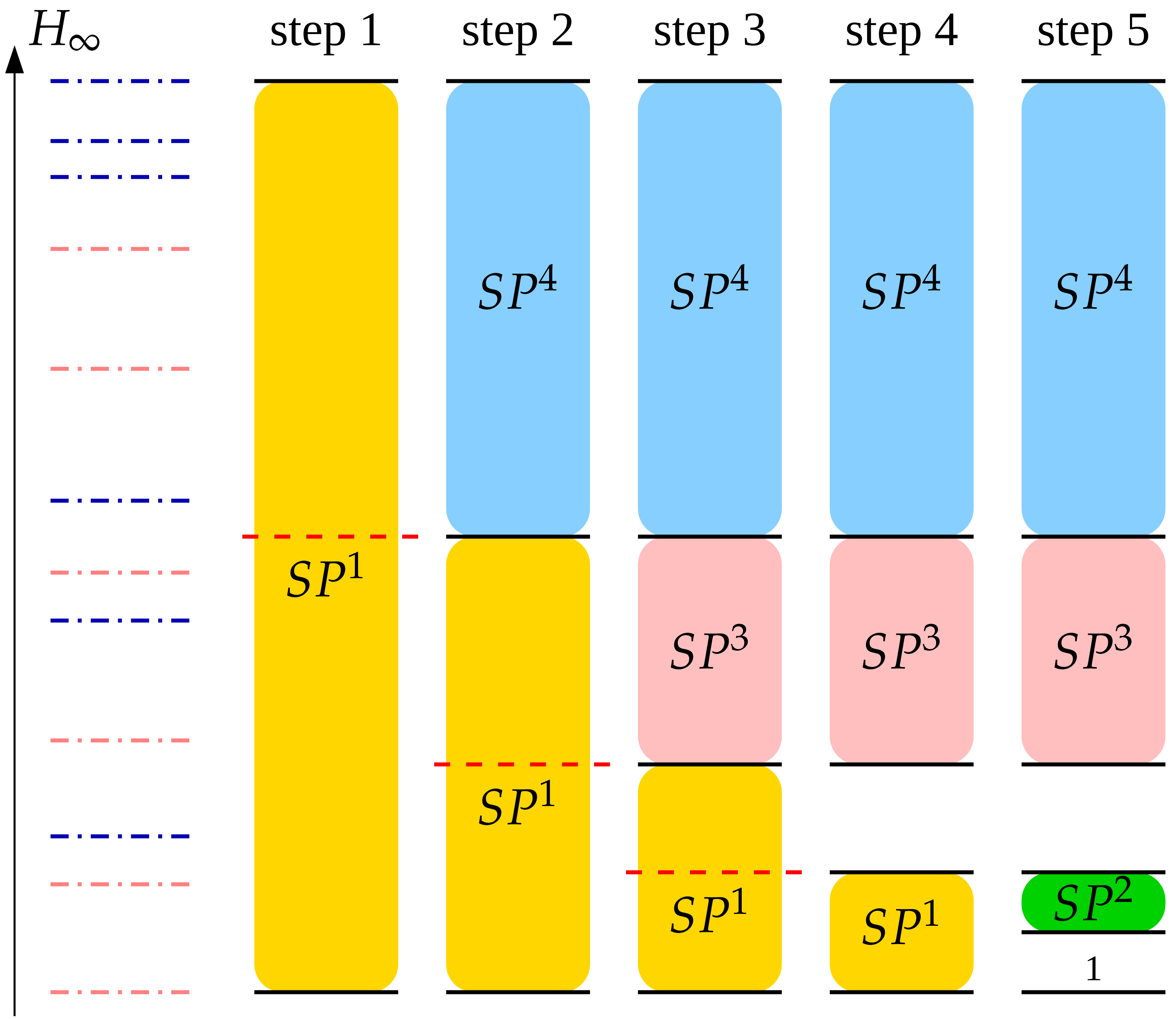
Population Bisection



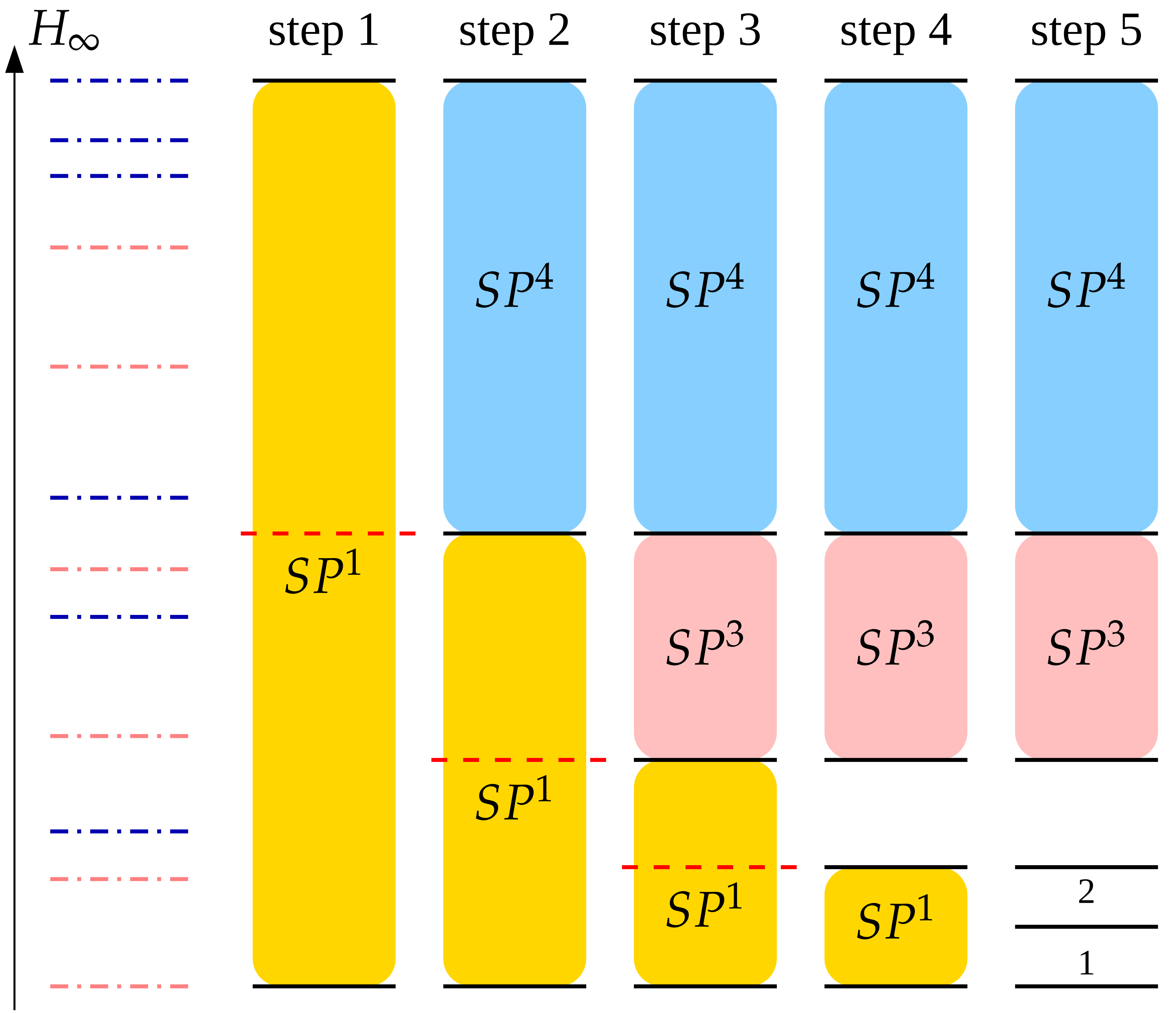
Population Bisection



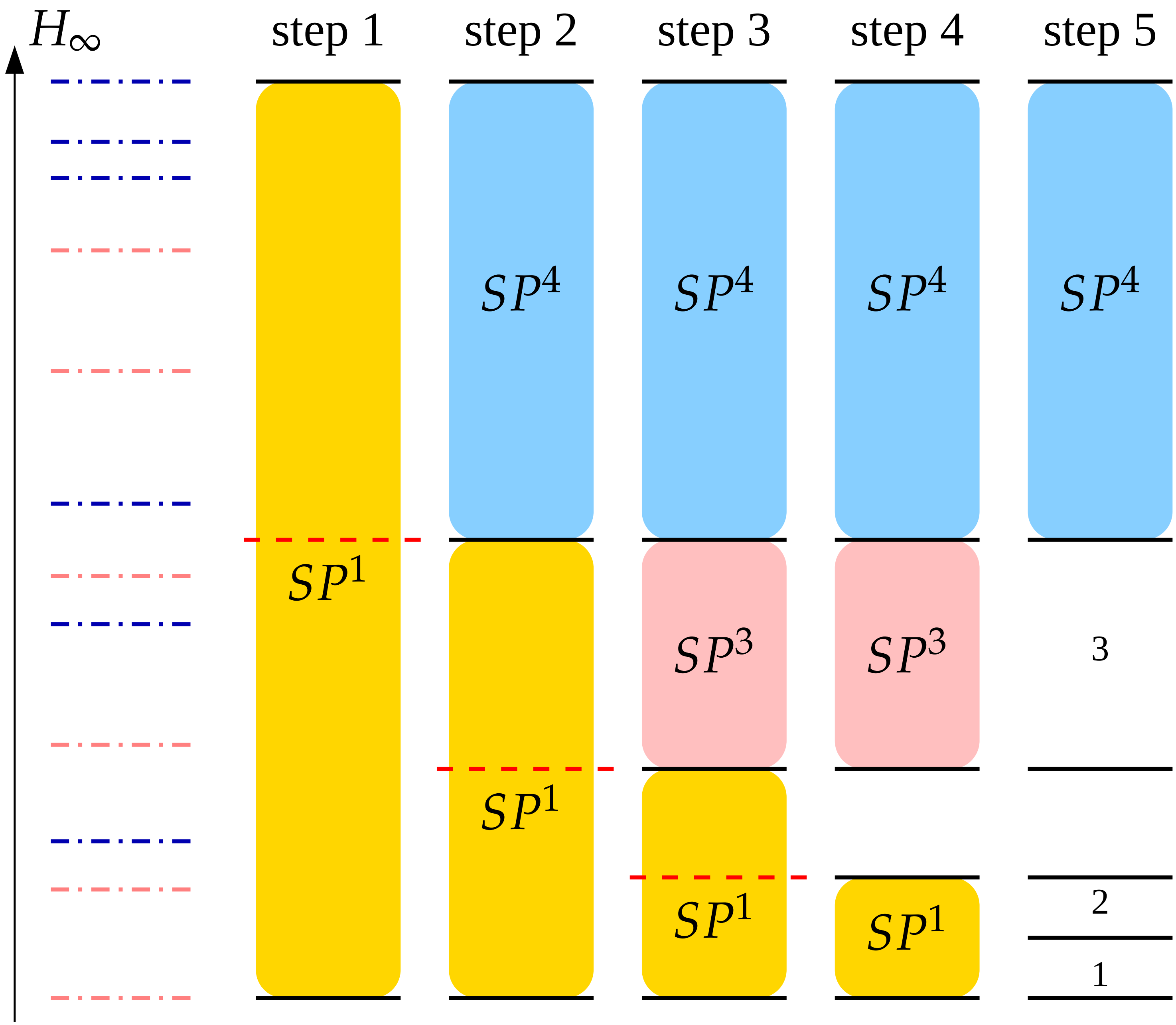
Population Bisection



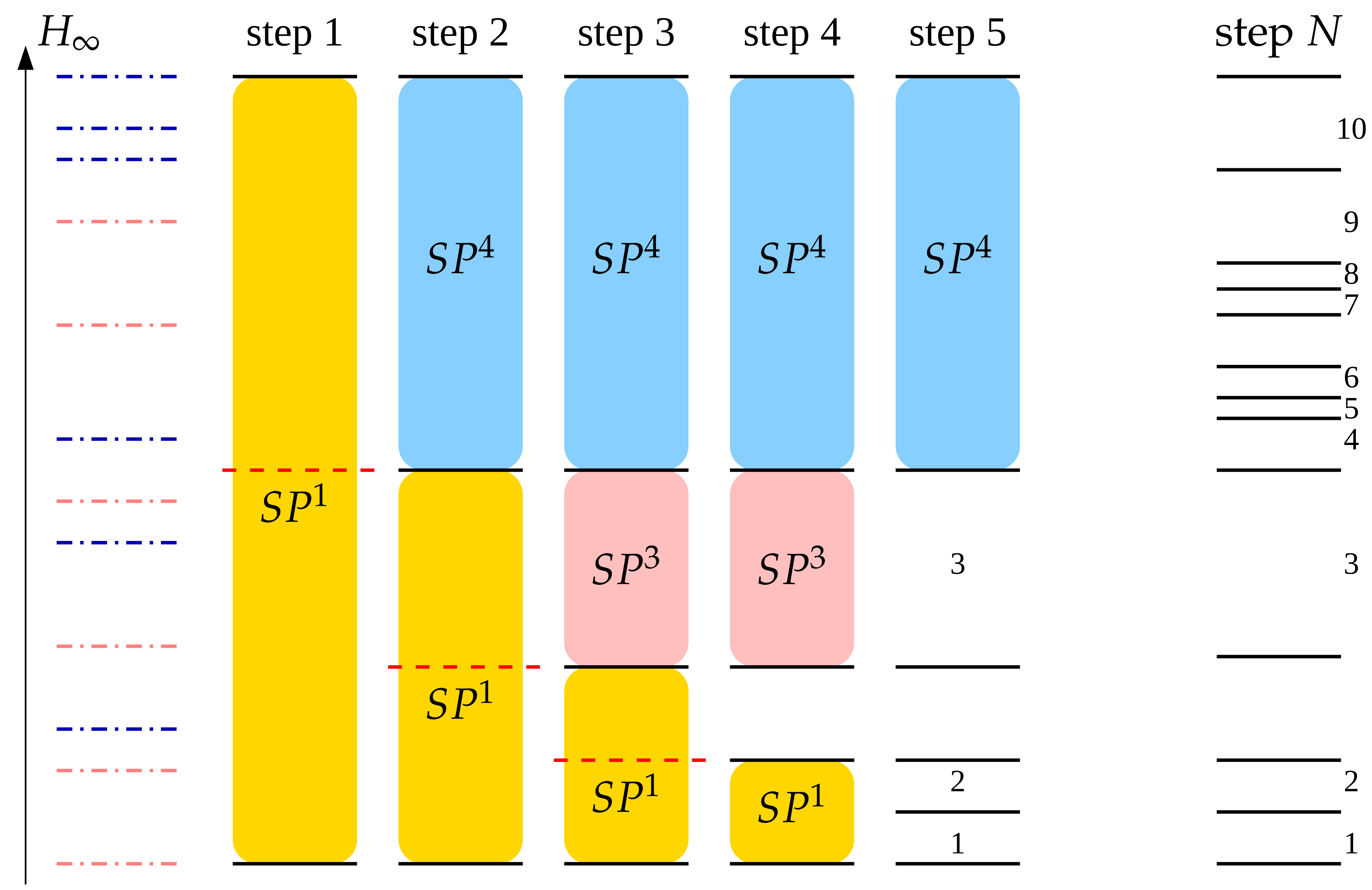
Population Bisection



Population Bisection



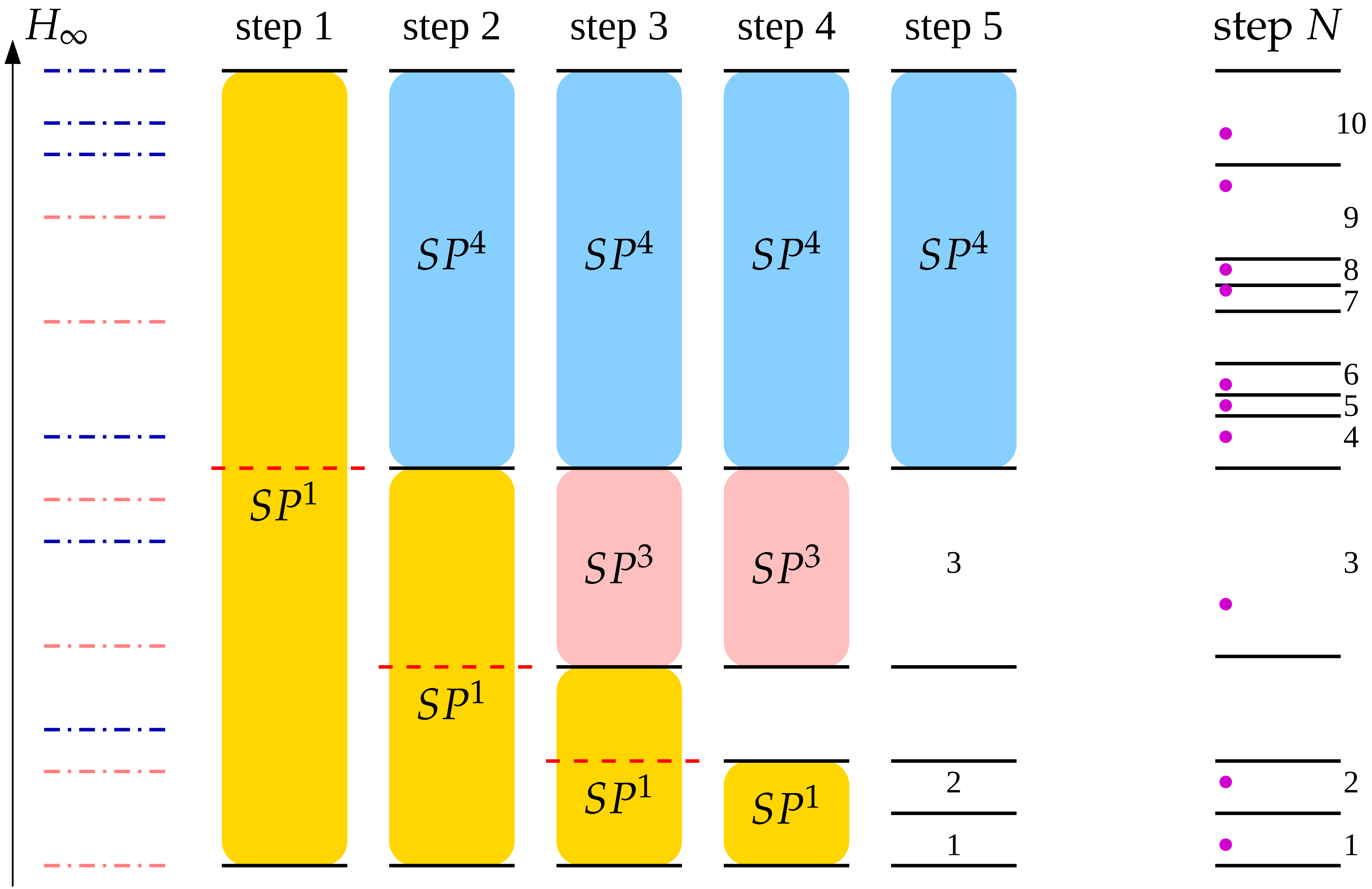
Population Bisection



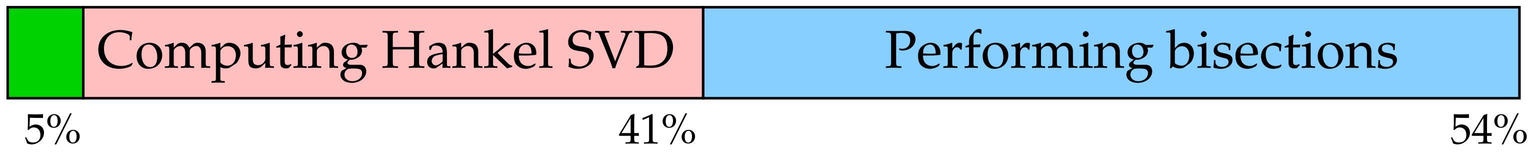
Technische Universität Hamburg-Harburg

Institut für Regelungstechnik

Population Bisection

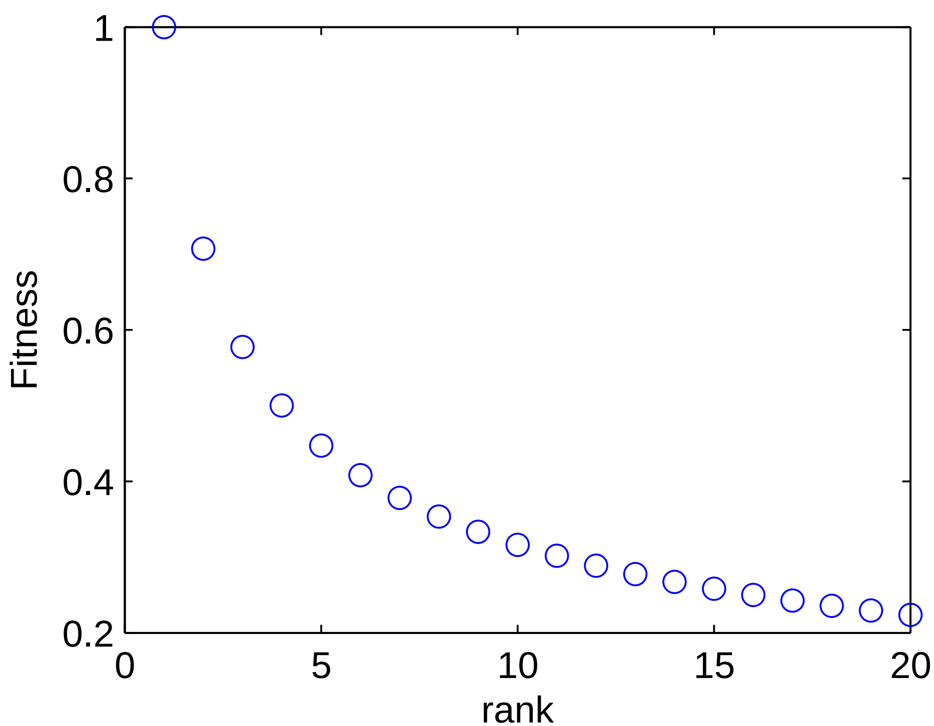


Improvements to population bisection



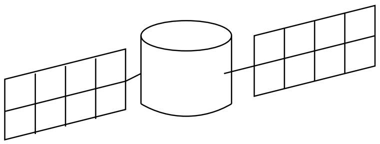
- Hankel SVD is computationally expensive

- Introduce "skip factor" $F(c^i) = \frac{1}{\sqrt{rank(c^i)}}$

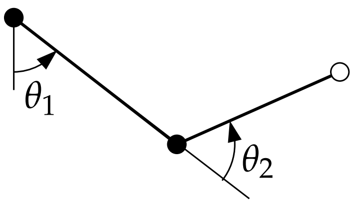


Design Problems

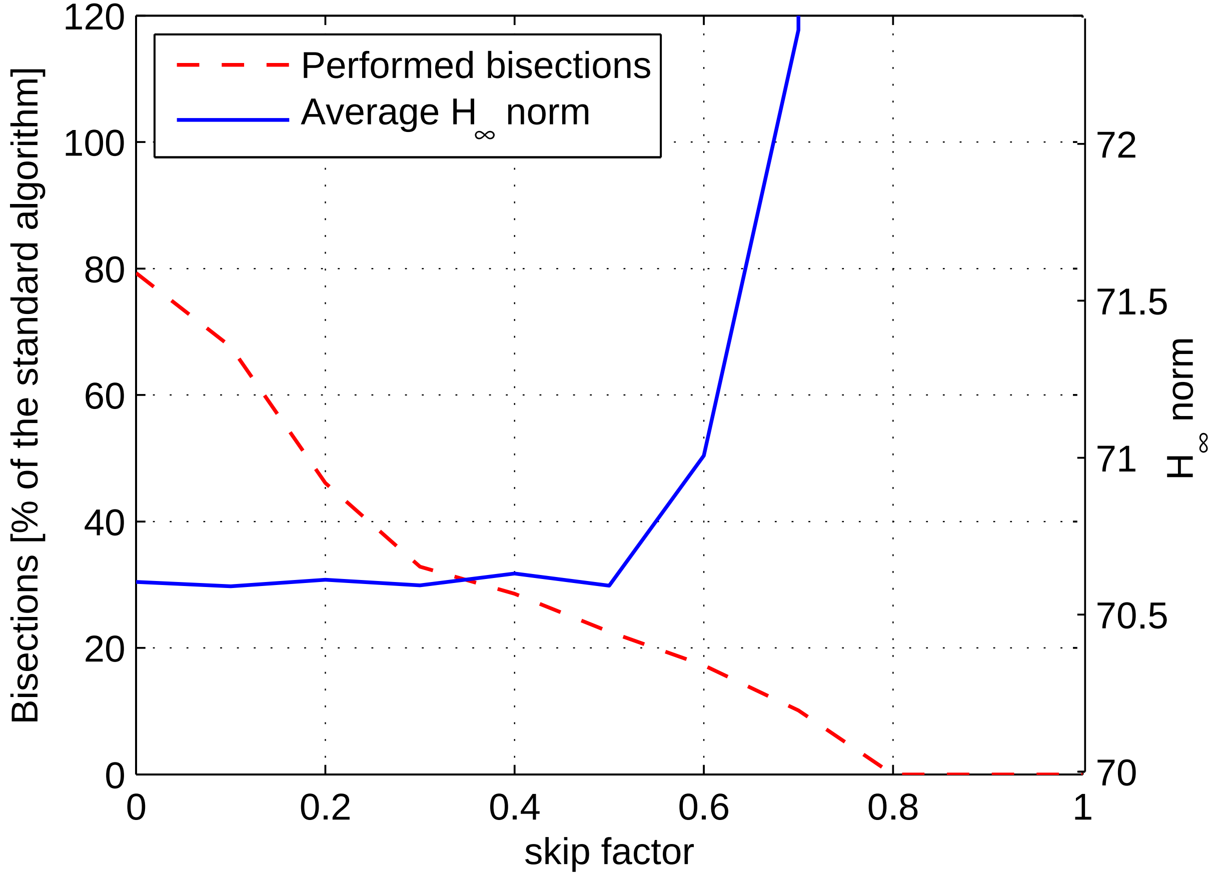
- Spinning Satellite (2×2, 8^{th} order) [Balas et al.(2001)]



- Two-link flexible manipulator (2×2, 14^{th} order) [Apkarian et al.(1996)]

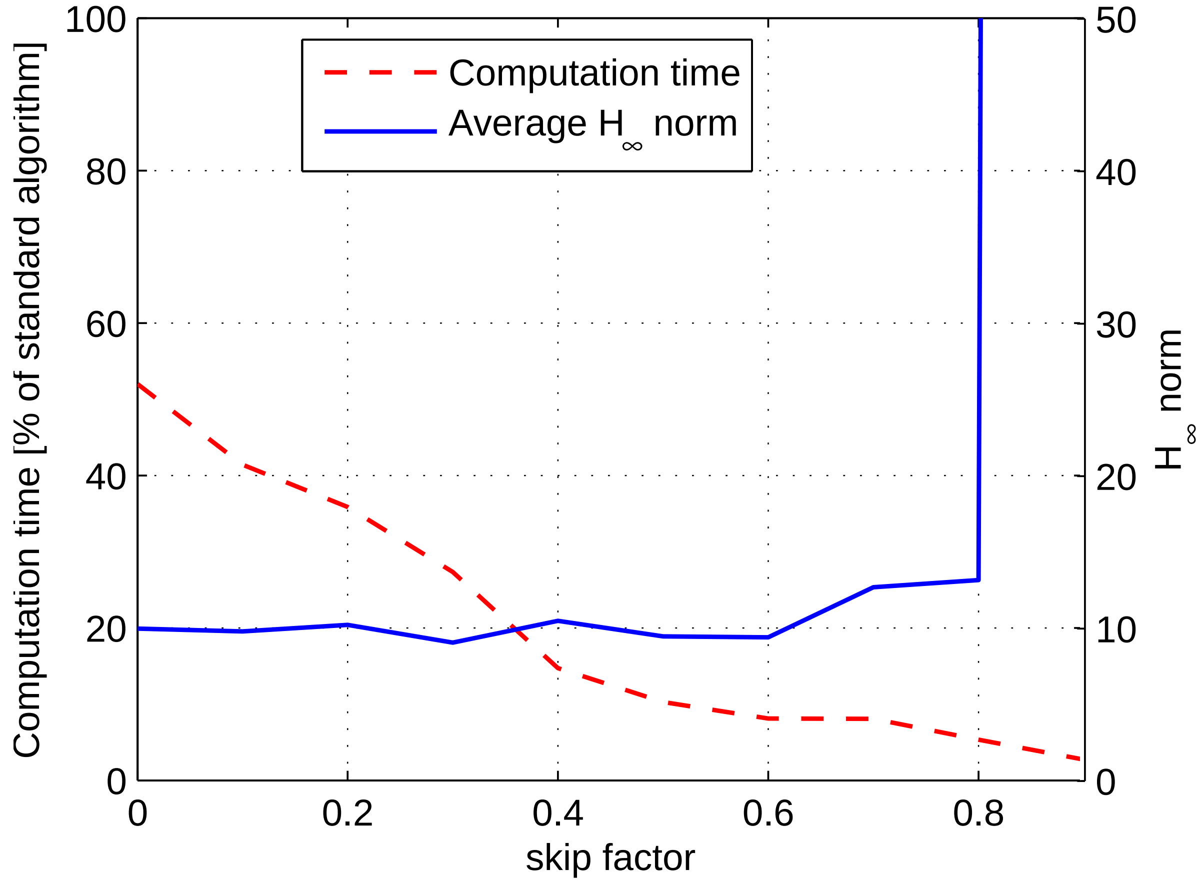


8^{th} order controller: H_∞ norm = 70.5351. With: 2^{nd} order:



`hinfmix`	8^{th} order	$t = 2.03$ sec.
population bisection	2^{nd} order	$t = 1.05$ sec.

14^{th} order controller: H_∞ norm = 3.82.   With 2^{nd} order:



Computational time with Hankel SVD boundaries 833 sec.

Motivation

Computing the H_∞ norm

Evolutionary Algorithms

Population bisection

Numerical results

Conclusions

Conclusions

- Developed generic population bisection technique

- Experimentally shown that the population bisection applied for H_∞ controller design:

  - reduces the number of necessary bisections with more than **20% without loss** of information!
  - can reduce the computation time down to **1/10**

TUHH
Technische Universität Hamburg-Harburg

Institut für Regelungstechnik

Thank you for your attention!

Q&A

Q&A: Bisection method
Q&A: Skip factor
Q&A: Optimization options
Q&A: Results for Spinning Satellite
Q&A: Results with and without the improvements
Q&A: Results for the 2-Mass-Spring Problem
Q&A: Results for the 2-Mass-Spring Problem - Graphics
Q&A: Results for 2-Link Flexible Manipulator

Bisection method according to [Boyd et al.(1988)]

1. Calculate lower and upper bound for the H_∞ norm

$$\gamma_{low} = \max\{\sigma_{\max}(D), \max\{\sigma_{H_i}\}\} \tag{1}$$

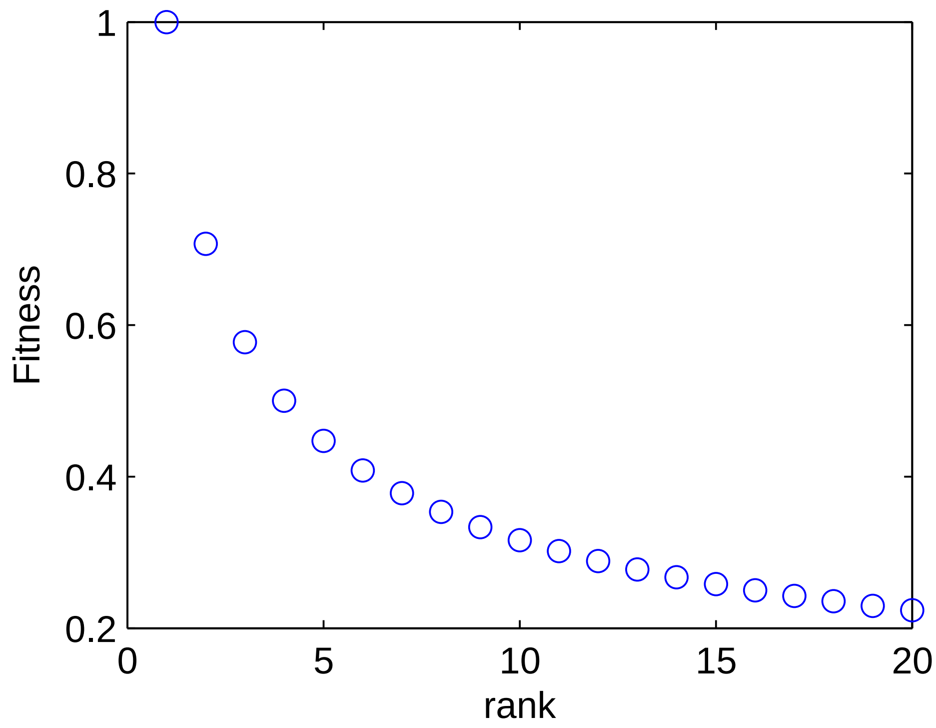
$$\gamma_{up} = \sigma_{\max}(D) + 2\sum_{i=1}^{n} \sigma_{H_i}$$

2. If $(\gamma_{up} - \gamma_{low}) \leq 2\epsilon\gamma_{low}$ STOP.
3. $\gamma = (\gamma_{low} + \gamma_{up})/2;$
4. Form M_γ and compute its eigenvalues

$$M_\gamma = \begin{bmatrix} A - BRD^TC & -\gamma BRB^T \\ \gamma C^TSC & -A^T + C^TDRB^T \end{bmatrix} \tag{2}$$

5. If M_γ has no imaginary eigenvalues $\gamma_{up} = \gamma,$
   else $\gamma_{low} = \gamma$
   GO TO step 2;

Q&A: Skip factor



$$\frac{1}{\sqrt{SP^k_{rank}}} - \frac{1}{\sqrt{SP^k_{rank} + SP^k_{Nctrl}}} < skip\_factor \qquad (3)$$

Q&A: Optimization options

- Evolutionary Algorithm: Genetic Algorithm Toolbox (GADS) of MATLAB [Mathworks(2005)]
- Controller representation: Companion canonical form

$$A_K = \begin{bmatrix} 0 & \dots & 0 & a_1 \\ 1 & \dots & 0 & a_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & a_k \end{bmatrix}; \ B_K = \begin{bmatrix} 1 & b_{12} & \dots & b_{1m} \\ 0 & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{k2} & \dots & a_{km} \end{bmatrix}$$

  C_K and D_K are full matrices.
- Computations performed on Pentium IV, 3GHz processor, 512 MB RAM.
- Initial range for each parameter between $[-1;1]$.
- Crossover rate = 0.8.
- Mutation shrink = 1.
- Mutation scale = 1.
- Crossover type - intermediate.
- Number of elite children = 2.

Average results of 10 optimization runs.
Population size = 20;   Number of generations = 100

Table 1: Results for the satellite problem

Skip factor	Bisections [% of stnd.]	H_∞ norm mean	H_∞ norm STD
0	79.27	70.6086	0.0460
0.1	67.62	70.5953	0.0574
0.2	46.06	70.6153	0.0526
0.3	32.86	70.5974	0.0541
0.4	28.57	70.6352	0.0651
0.5	22.55	70.5972	0.0360
0.6	17.25	71.0078	1.1755
0.7	10.09	72.3539	3.5015
0.8	0.00	216.6496	122.3274
0.9	0.00	202.9761	155.3374
1.0	0.00	634.4	1115.6

Q&A: Results with and without the improvements

Table 2: Influence of the modifications to the population bisection

Skip factor	Bisections [% of stnd.]	H_∞ norm mean	H_∞ norm STD
0 $\star$	82.30	70.5567	0.0172
0 a	80.35	70.6238	0.0777
0 b	79.61	70.5933	0.0286
0	79.27	70.6086	0.0460
0.3 $\star$	68.62	70.5488	0.0112
0.3 a	64.46	70.5521	0.0108
0.3 b	33.43	70.6126	0.0294
0.3	32.86	70.5974	0.0541

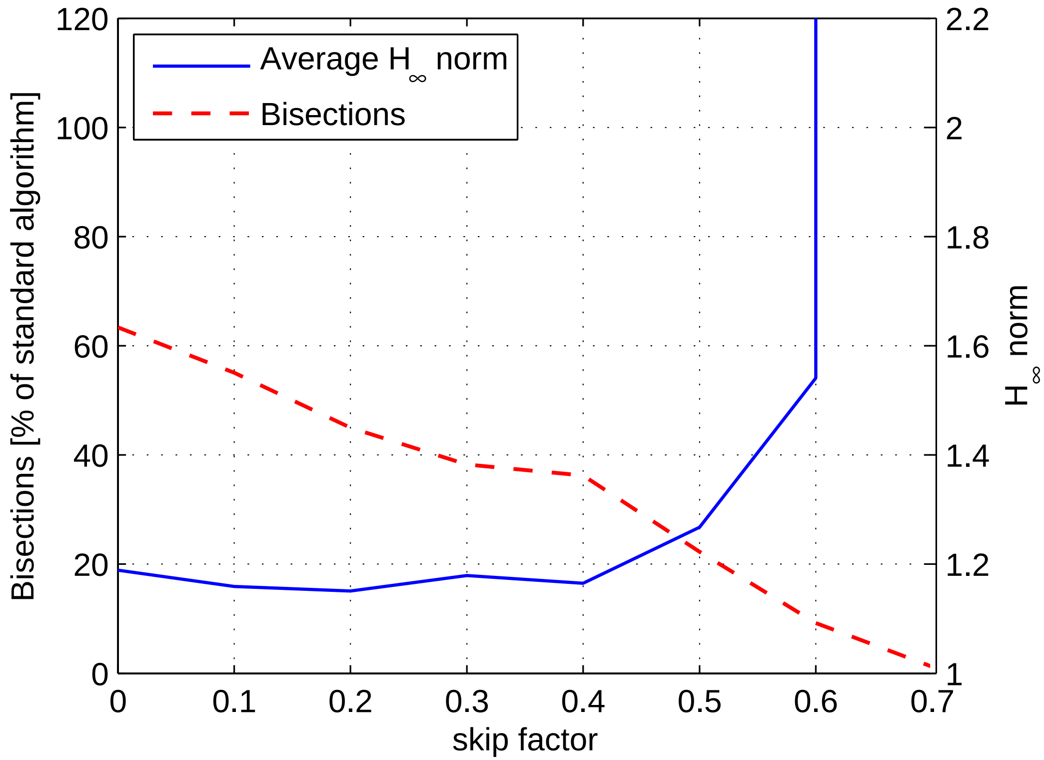
Q&A: Results for the 2-Mass-Spring Problem

Average results of 10 optimization runs.
Population size = 20; Number of generations = 1000

Table 3: Results for the two-mass-spring problem

Skip factor	Bisections [% of stnd.]	H_∞ norm mean	H_∞ norm STD
0	63.36	1.1891	0.1079
0.1	55.07	1.1593	0.0663
0.2	45.00	1.1506	0.0822
0.3	38.25	1.1792	0.1446
0.4	36.24	1.1651	0.0694
0.5	22.27	1.2674	0.0933
0.6	9.20	1.5408	0.3225
0.7	1.21	$1.79\,10^7$	$4.41\,10^7$

4^{th} order controller: H_∞ norm = 0.6014. With: 2^{nd} order:



Computation time with Hankel SVD boundaries 44.86 sec.
Computation time with "cheaper" boundaries 16.68 sec.

Q&A: Results for 2-Link Flexible Manipulator

Average results of 5 optimization runs.
Population size = 50; Number of generations = 2000

Table 4: Results for the two-link flexible manipulator problem

Skip factor	Time [sec]	H_∞ norm mean	H_∞ norm STD
0	432.79	9.9492	2.2616
0.1	344.93	9.7795	1.8187
0.2	298.68	10.1905	1.1601
0.3	227.65	9.0235	0.3303
0.4	122.67	10.4607	3.0890
0.5	85.47	9.4452	0.5784
0.6	67.36	9.3681	0.5942
0.7	67.18	12.6639	3.1250
0.8	44.19	13.1278	2.0433
0.9	22.73	1795.65	1780.50
1.0	24.49	9479.72	19862.77

References

[Apkarian et al.(1996)] P. Apkarian, G. Becker, P. Gahinet, and H. Kajiwara. LMI techniques in control engineering from theory to practice. In *Workshop notes CDC*, Kobe, Japan, 1996. IEEE.

[Balas et al.(2001)] G. Balas, J. Doyle, K. Glover, A. Packard, and R. Smith. *μ-Analysis and Synthesis Toolbox*. The MathWorks, fourth edition edition, 2001.

[Boyd et al.(1988)] S. Boyd, V. Balakrishnan, and P. Kobamba. On computing the h_∞ norm of a tranfer matrix. In *proceedings of American Control Conference 1988*. IEEE, IEEE, 1988.

[Mathworks(2005)] I. Mathworks. *Genetic Algorithm and Direct Search Toolbox For Use with Matlab*. The MathWorks, Inc., 2 edition, 2005.

[Skelton et al.(1997)] R. Skelton, T. Iwasaki, and K. Grigoriadis. *A unified approach to linear control design*. Taylor & Francis, London, UK, 1997.