Minimum Conditional Probability of Error Detection for MC-CDMA

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***Abstract*— Multi-Carrier Code Division Multiple Access (MC-CDMA) is an attractive choice for high speed wireless communication as it mitigates the problem of Intersymbol Interference (ISI) and also exploits frequency diversity. The data stream in MC-CDMA is spread using a user specific spreading sequence in the frequency domain and every chip travels on a different sub-carrier. At the receiver, the different sub-carriers have to be combined properly to form the decision variable. Minimum Mean Square Error (MMSE) detector is a popular choice for this, but as it tries to minimize the Mean Square Error (MSE) and not the Bit Error Rate (BER), the BER it attains is not optimum. In this paper, we propose a Minimum Probability of Error (MPOE) based linear multiuser detector, which tries to minimize the probability of error. A Minimum Conditional Probability of Error (MCPOE) detector is introduced for reducing the computational complexity of the MPOE detector. Theoretical and simulation results are presented for the MCPOE and MPOE detector and it can be seen that the MPOE detector outperforms the MMSE detector.**

I. Introduction

Multi-Carrier Code Division Multiple Access (MC-CDMA) is a fairly recent multiple access technique that has attracted much research during the past decade. A combination of Orthogonal Frequency Division Multiplexing (OFDM) and CDMA [1] [2], MC-CDMA combines the advantages of both the schemes. In an MC-CDMA system, the data stream of each user is spread using his spreading sequence and then each chip is transmitted over a different subcarrier. Therefore, like OFDM, MC-CDMA signals have attractive spectral characteristics. Their spectral energy is confined almost to the allocated bandwidth. Owing to this and other spectral characteristics, MC-CDMA allows for easier system coexistence and equalization. Moreover, like CDMA, by transmitting the same data bit over different sub-carriers, MC-CDMA exploits frequency diversity better than OFDM. Also due to the increased chip duration, the effect of Intersymbol Interference (ISI) on the transmitted signals is reduced. In an outdoor propagation environment, the residual ISI can be eliminated using sufficient guard interval.

This paper focuses on the downlink scenario, where the base station communicates with the mobile sets. Each user receives his own signal corrupted by noise and Multiple Access Interference (MAI) from the other users. Multiuser detection deals with the problem of the reliable demodulation of user's bits in such a scenario. Various detectors, such as MRC, ERC and MMSE [3] have already been proposed for the downlink case. Traditionally, adaptive filtering in communications has been developed based on the Mean Square Error (MSE) criterion [5]. Recently, [6] - [9], several linear detectors have been suggested for CDMA and OFDM based on the principle of Minimum Probability of Error (MPOE). As these detectors try to minimize the BER and not the MSE, they achieve a better BER than the MMSE detectors.

In this paper we propose a detector for MC-CDMA based on the MPOE criterion. The theoretical and simulation results for the same have been presented. It can be seen that the MPOE detector outperforms the MMSE detector in terms of BER, albeit at a higher computational complexity when in weight adaptation mode. Once in the decision directed mode, both MMSE and MPOE are linear detectors. The computational complexity of the MPOE detector can be reduced by using the Minimum Conditional Probability of Error (MCPOE) approach. The computational complexity of the MCPOE detector, in the weight adaptation mode is linear in the number of users. On the other side, the MCPOE detector requires a training sequence for weight adaptation.

The organization of the paper is as follows. Section II describes the signal model assumed. Section III derives the probability of error for MC-CDMA and arrives at the MPOE detector. Section IV discusses the MCPOE detector. Simulation results are presented in Section V. The conclusions are presented in Section VI.

II. Signal Model

In a MC-CDMA system, the incoming data bits are multiplied by the user specific spreading sequence and then each chip is transmitted on a different sub-carrier. The sub-carrier spacing is $\frac{F}{T_b}$, where F is an integer greater than or equal to one and T_b is the symbol duration. If $F = 1$, then the system conserves bandwidth and if $F > 1$, then the system uses more bandwidth, but at the same time achieves better frequency diversity. The spectra of the various sub-carriers overlap, but the information carried by the different sub-carriers can still be recovered due to the orthogonality of the sub-carriers. An efficient way of implementing the modulation digitally is to perform an Inverse Discrete Fourier Transform (IDFT) on the incoming symbols and then D/A convert the sequence obtained. The resulting analog waveform can be upconverted and then transmitted.

If the bit vector corresponding to the K users is written as

$$A = [a_0 \; a_1 \ldots \; a_{K-1} \;]^T \tag{1}$$

where $a_i \in \{+1, -1\}$ if user i is active and 0 otherwise and the orthonormal code matrix,

$$C = \begin{bmatrix} c_0[0] & c_1[0] & \ldots & c_{K-1}[0] \\ c_0[1] & c_1[1] & \ldots & c_{K-1}[1] \\ \ldots & \ldots & \ldots & \ldots \\ c_0[K-1] & c_1[K-1] & \ldots & c_{K-1}[K-1] \end{bmatrix} \tag{2}$$

where the i^{th} column corresponds to the spreading sequence of the i^{th} user, then the discrete time transmitted sequence can be written as

$$S = [s_0 \; s_1 \; \ldots \; s_{K-1} \;]^T = F^{-1}CA \tag{3}$$

where F^{-1} is the K point IDFT matrix. The actual analog waveform during the k^{th} symbol duration that is transmitted after upconversion is

$$s_k(t) = \sum_{m=0}^{K-1} \sum_{i=0}^{K-1} c_m[i] a_m[k] cos(2\pi f_c t + 2\pi i \frac{F}{T_b} t) \tag{4}$$

The effect of the channel on the transmitted signal can be written as

$$y[n] = (h * s)[n] + \eta, \quad 0 \leq n \leq K-1 \tag{5}$$

where $h[l]$, $0 \leq l \leq L-1$ is the discrete time channel impulse response and η is the additive white Gaussian noise (AWGN) . Because of the linear convolution of the transmitted signal with the channel impulse response, the received signal is affected by ISI and Interchannel Interference (ICI). Equation (5) can be written as [4]

$$\begin{aligned} y_i[n] &= \sum_{l=0}^{L-1} h[l] s_i[n-l]_K \sigma[n-l] \\ &+ \sum_{l=1}^{L-1} h[l] s_{i-1}[n-l]_K (1 - \sigma[n-l]) + \eta \\ &= y_i^{(i)}[n] + y_i^{(i-1)}[n] + \eta \end{aligned} \tag{6}$$

where $y_i[n]$ is the received signal in the i^{th} symbol duration and $y_i^{(i)}[n]$ and $y_i^{(i-1)}[n]$ are the contributions to it from the symbols $s_i[n]$ and $s_{i-1}[n]$, transmitted in the i^{th} and $(i-1)^{th}$ duration respectively. σ is the Heaviside function and $[n]_K$ is n modulo K. It can be seen that due to the multipath, the first $L-1$ symbols of received signal are distorted by ISI from the previous symbol. The received signals after the DFT can be written as

$$Y_i = H_{ICI} S_i + H_{ISI} S_{i-1} + N \tag{7}$$

where

$$\begin{aligned} H_{ICI}(k,m) &= \sum_{l=0}^{L-1} H_l^i(m-k) e^{-j\frac{2\pi}{K}lm} \\ H_l^i(m-k) &= \frac{1}{K} \sum_{n=0}^{K-1} h[l]\sigma[n-l] e^{j\frac{2\pi}{K}n(m-k)} \end{aligned} \tag{8}$$

and

$$\begin{aligned} H_{ISI}(k,m) &= \sum_{l=1}^{L-1} H_l^{i-1}(m-k) e^{-j\frac{2\pi}{K}lm} \\ H_l^{i-1}(m-k) &= \frac{1}{K} \sum_{n=0}^{K-1} h[l](1-\sigma[n-l]) e^{j\frac{2\pi}{K}n(m-k)} \end{aligned} \tag{9}$$

H_{ICI} is the ICI matrix and H_{ISI} is the ISI matrix. S_i and S_{i-1} are the K point DFT of $s_i[n]$ and $s_{i-1}[n]$ respectively. N is the Gaussian noise vector. In the downlink case that we are considering, we cannot eliminate this ISI because we do not perform a joint demodulation due to computational constraints in a mobile set. Therefore in order to avoid ICI and ISI, sufficient guard interval or cyclic prefix (CP) has to be added. Because of the guard interval of sufficient length, the linear convolution of equation (5) starts to look like a circular convolution. Also, circular convolution in the time domain is equivalent to a point wise multiplication of the DFT's in the frequency domain. In such a case, the ISI matrix turns out to be a zero matrix and ICI matrix is a diagonal matrix with $H_{ICI}(k,k) = DFT\{h[l]_k\}$. Thus the effect of the channel reduces to independent scaling of every sub-carrier. The received signal in this case can be written as

$$Y = HCA + N \tag{10}$$

.
where

$$H = \begin{bmatrix} \rho_0 e^{j\theta_0} & \ldots & 0 & 0 \\ 0 & \rho_1 e^{j\theta_1} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \rho_{K-1} e^{j\theta_{K-1}} \end{bmatrix} \tag{11}$$

$\rho_i' s$ are i.i.d Rayleigh random variables and $\theta_i' s$ are uniformly distributed random variables.

III. Minimum Probability of Error Detector

At the transmitter different users are separated using orthogonal Walsh Hadamard codes but due to independent scaling of each of the sub-carriers, the code orthogonality between the users is destroyed. After the DFT is performed on the received signal, a single tap equalizer has to restore this orthogonality before forming the decision variable. The MMSE receiver tries to minimize the mean square error between the transmitted and the estimated symbol. Since the BER, and not the MSE is the deciding factor in a digital communication system, a detector which tries to minimize the BER is desired. It is seen that a MPOE detector can achieve a better BER than the MMSE detector as it tries to minimize the probability of error. In the following sections we will derive the expression for the probability of error for MC-CDMA and then use this expression to arrive at the MPOE detector.

A. Probability of Error

Let $w_k = [w_0\ w_1\ \ldots\ w_{K-1}]^H$ be the filter vector used to demodulate the bits of the desired user (user k in our case). The filter output x_k can be written as

$$\begin{aligned} x_k &= w_k^H Y \\ x_k &= w_k^H HCA + w_k^H N \end{aligned} \tag{12}$$

The real part of x_k is sufficient and can be used to demodulate the user's bit

$$\Re(x_k) = \Re(w_k^H HCA) + \Re(w_k^H N) \tag{13}$$

Conditioned on the transmitted bit vector A, $\Re(x_k)$ is Gaussian with mean

$$\mu_k = \Re(w_k^H HCA) \tag{14}$$

and variance

$$\sigma_k^2 = \frac{\sigma^2 \|w_k\|^2}{2} \tag{15}$$

where σ^2 is the variance of each component of the noise vector N. Since the demodulated bit is $sgn\left(\Re(x_k)\right)$, $sgn(x)$ being the signum function, an error in demodulation occurs whenever

$$\Re(x_k|a_k = -1) > 0 \text{ or } \Re(x_k|a_k = +1) < 0 \tag{16}$$

Since $\Re(x_k)$ is Gaussian, with mean μ_k and variance σ_k^2, the pdf of $\Re(x_k)$ is

$$f_{\Re(x_k|A)}(u) = \frac{1}{\sigma_k\sqrt{2\pi}} exp\left(-\frac{(u-\mu_k)^2}{2\sigma_k^2}\right) \tag{17}$$

Conditioned on $a_k = +1$ the probability of error in demodulation is

$$P_e(x_k|a_k = +1) = \int_{-\infty}^{0} \frac{1}{\sigma_k\sqrt{2\pi}} exp\left(-\frac{(u-\mu_k)^2}{2\sigma_k^2}\right) du \tag{18}$$

and conditioned on $a_k = -1$, the probability of error in demodulation is

$$P_e(x_k|a_k = -1) = \int_{0}^{-\infty} \frac{1}{\sigma_k\sqrt{2\pi}} exp\left(-\frac{(u-\mu_k)^2}{2\sigma_k^2}\right) du \tag{19}$$

Using the standard Q function, the probability of error can be written as

$$P_e(x_k|A) = Q\left(\frac{a_k \mu_k}{\sigma_k}\right) \tag{20}$$

Since the 2^K transmitted bit vectors are independent and equiprobable, the average probability of error in demodulating the user can be written as

$$P_e = \frac{1}{2^K} \sum_{\forall A} Q\left(\frac{a_k \mu_k}{\sigma_k}\right) \tag{21}$$

The filter vector that minimizes the probability of error is the MPOE detector.

B. MPOE Algorithm

The filter vector that minimizes the probability of error given by expression (21) can be found adaptively by two different approaches. The first approach requires computation of the gradient of the function to be minimized. The gradient may be computed analytically or numerically. Algorithms like conjugate gradient method, variable metric method fall in this category. The other approach requires function and not gradient evaluation. Algorithms that fall in this approach are downhill simplex method and Powell's method.

We have used the method of gradient descent, as the gradient of (21) can be easily calculated. Starting from an initial guess, the filter weights are updated iteratively along the negative gradient of the probability of error surface. During the i^{th} iteration,

$$w_k^{i+1} = w_k^i - \mu \bigtriangledown \left(P_e(w_k)\right) \tag{22}$$

where μ is a suitably chosen step size. To compute the gradient of (21) we note that if $J(w)$ is a real valued cost function, then the gradient $\bigtriangledown J$ is given by [10]

$$\bigtriangledown J = 2\frac{\partial J}{\partial w^*} \tag{23}$$

$\frac{\partial J}{\partial w^*}$ is the conjugate derivative of the function $J(w)$. The conjugate derivative of (21) is given by

$$\frac{\partial P_e(w_k)}{\partial w_k^*} = -\frac{1}{\sqrt{2\pi}2^K} \sum_{\forall A} \left(e^{-\frac{\mu_k^2}{2\sigma_k^2}} \frac{\partial}{\partial w_k^*}\left(\frac{\mu_k a_k}{\sigma_k}\right)\right) \tag{24}$$

The term $\frac{\partial}{\partial w_k^*}\left(\frac{\mu_x a_0}{\sigma_x}\right)$ can be written as

$$\begin{aligned} \frac{\partial}{\partial w_k^*}\left(\frac{\mu_k a_k}{\sigma_k}\right) &= \frac{\sigma_k a_k \frac{\partial}{\partial w_k^*}\mu_k - \mu_k a_k \frac{\partial}{\partial w_k^*}\sigma_k}{\sigma_k^2} \\ &= \frac{a_k}{\sigma_k}\left(\frac{\partial}{\partial w_k^*}\Re(w_k^H HCA) - \frac{\mu_k}{\sigma_k}\frac{\partial}{\partial w_k^*}\sigma\|w_k\|\right) \\ &= \frac{a_k}{\sigma_k}\left(HCA - \frac{\mu_k}{\|w_k\|}\frac{\partial}{\partial w_k^*}(w_k^H w)^{1/2}\right) \\ &= \frac{a_k}{\sigma_k}\left(HCA - \frac{\mu_k w_k}{\|w_k\|^2}\right) \end{aligned} \tag{25}$$

Using the above, the gradient of (21) can be written as

$$\frac{\partial P_e(w_k)}{\partial w_k^*} = -\frac{1}{\sqrt{2\pi}2^T} \sum_{\forall A} \left(a_k e^{-\frac{\mu_k^2}{2\sigma_k^2}} \phi\right) \tag{26}$$

Fig. 1. BER of MPOE v/s MMSE for decreasing SNR for 8 users

Fig. 2. BER of MPOE and MCPOE v/s MMSE for increasing users at 5 dB SNR

where ϕ is

$$\phi = \frac{1}{\sigma_k}\left(HCA - \frac{\mu_k w_k}{\|w_k\|^2}\right) \tag{27}$$

w can be normalized to unit length for computational advantage, but that makes the algorithm more sensitive to the step size μ. The computational complexity of the above algorithm can be reduced by noting that the expression (21) for the probability of error is a sum of 2^K terms, one for each bit vector A. The sum remains unaffected if A is replaced by -A, i.e., all $+1s$ are changed to $-1s$ and vice versa. Therefore we need to sum up only 2^{K-1} terms now instead of 2^K terms. Complexity of the detector can be further reduced by using the Minimum Conditional Probability of Error (MCPOE) approach.

IV. Minimum Conditional Probability of Error Detector

The MPOE based detector proposed in the previous section has a BER superior to MMSE detector, but it has a computational complexity exponential in the number of users. The computational complexity of the MPOE detector can be reduced by minimizing the conditional probability of error instead of the average probability of error. The resulting detector, MCPOE detector has a BER performance marginally lower than the MPOE detector, but at a computational complexity that is linear in the number of users. This is a significant improvement over MPOE, which is $O(2^K)$ and MMSE, which is $O(K^3)$. In the following subsections, we will derive the expression for the conditional probability of error and use this to arrive at the MCPOE detector.

A. Conditional Probability of Error

Let $w_k = [w_0 \ w_1 \ \ldots \ w_{K-1}]^H$ be the filter vector acting on the received signal Y to demodulate the k^{th} user.

$$x_k = w_k^H HCA + w_k^H N \tag{28}$$

$\Re(x_k)$ is sufficient for demodulating the k^{th} user's bit. Conditioned on the transmitted bit vector A, an error in demodulating the k^{th} user occurs whenever,

$$\Re(x_k) < 0 | a_k = 1 \text{ or } \Re(x_k) > 0 | a_k = -1 \tag{29}$$

Assuming the symbols $+1, -1$ to be equally likely, the probability of error for the k^{th} user, conditioned on the transmitted bit vector A can be written as

$$P_{k|A} = \frac{1}{2}Pr[\Re(x_k) < 0 | a_k = 1] + \frac{1}{2}Pr[\Re(x_k) > 0 | a_k = -1] \tag{30}$$

From Section III A we know that conditioned on the transmitted bit vector A, $\Re(x_k)$ is Gaussian with mean μ_k and variance σ_k^2. The pdf of $\Re(x_k)$ is given by equation eq (17). Let us denote the mean of $\Re(x_k)$ as $\mu_{k|1}$ when $a_k = 1$ and as $\mu_{k|-1}$ when $a_k = -1$. ie.,

$$\begin{aligned} \mu_{k|1} &= \Re(w_k^H HCA | a_k = 1) \\ \mu_{k|-1} &= \Re(w_k^H HCA | a_k = -1) \end{aligned} \tag{31}$$

The probability of error expressed in eq. (30) can then be written as

$$P_{k|A} = \frac{1}{2}Q\left(\frac{\mu_{k|1}}{\sigma_k}\right) + \frac{1}{2}Q\left(\frac{-\mu_{k|-1}}{\sigma_k}\right) \tag{32}$$

The filter vector that minimizes the above is the MCPOE detector.

B. MCPOE Algorithm

The conditional probability of error expressed in eq (32) can be minimized by using a training sequence to adapt the filter weights using the gradient descent approach. Again, starting from an initial guess for the filter weights, the weights are adapted iteratively along the negative gradient of the probability of error surface. Therefore, during the i^{th} iteration,

$$w_k^{i+1} = w_k^i - \mu(\bigtriangledown P_{k|A}) \tag{33}$$

Using the results from section III B, the conjugate derivative of eq. (32) can be written as

$$\begin{aligned} \frac{\partial P_{k|A}}{\partial w_k^*} = &- \frac{1}{2\sqrt{2\pi}} exp\left(-\frac{\mu_{k|1}^2}{2\sigma_k^2}\right) \frac{\partial}{\partial w_k^*}\left(\frac{\mu_{k|1}}{\sigma_k}\right) \\ &+ \frac{1}{2\sqrt{2\pi}} exp\left(-\frac{\mu_{k|-1}^2}{2\sigma_k^2}\right) \frac{\partial}{\partial w_k^*}\left(\frac{\mu_{k|-1}}{\sigma_k}\right) \end{aligned} \tag{34}$$

where,

$$\begin{aligned} \frac{\partial}{\partial w_k^*}\left(\frac{\mu_{k|1}}{\sigma_k}\right) &= \frac{1}{\sigma_k}\left(HCA|_{a_k=1} - \frac{\mu_{k|1} w}{\|w\|^2}\right) \\ \frac{\partial}{\partial w_k^*}\left(\frac{\mu_{k|-1}}{\sigma_k}\right) &= \frac{1}{\sigma_k}\left(HCA|_{a_k=-1} - \frac{\mu_{k|-1} w}{\|w\|^2}\right) \end{aligned} \tag{35}$$

V. Simulations

Two sets of simulations were carried out to test the theoretical results. In the first set, the effect of noise on the performance of the MPOE and MCPOE detector was studied for a full load system. In the second set, the effect of increasing the load on the system was studied for MPOE and MCPOE detector.

Fig. 3. BER of MCPOE v/s MMSE for decreasing SNR for 8 users

A. Simulation Parameters

The number of sub-carriers in the simulations was 8 and the number of maximum active users was also 8. Orthogonal Walsh-Hadamard codes of length 8 were used to spread the incoming data stream. All the user's signals were assumed to be transmitted with the same power. Flat Rayleigh fading was assumed and also the fading was assumed to be independent for all the sub-carriers. Estimate of the channel matrix H was assumed to be known at the receiver. Also synchronization was assumed.

B. Results

Figure 1 shows the effect of increasing the noise on the system. It can be seen that the MPOE detector outperforms the MMSE detector at all SNR levels. Figure 2 shows the effect of increasing the load on the system. The SNR is $5dB$. It can be seen that as the load on the system increases, the MPOE detector performs better than the MMSE detector, although for 6 or more users, MMSE and MCPOE perform almost the same. Figure 3 shows the BER of MCPOE detector v/s MMSE detector for various SNR. It can be seen that MCPOE detector performs comparable to the MMSE detector at most SNR levels.

VI. Conclusions

A MPOE detector has been proposed in this paper for MC-CDMA. Performance of this detector has been compared with the MMSE detector. The MPOE detector attains a lower BER compared to the MMSE detector albeit at a higher computational complexity. The computational complexity of the MPOE algorithm can be decreased by using the MCPOE approach. The MCPOE detector has a BER marginally lower than MMSE detector, but its computational complexity is significantly lesser than the MPOE and MMSE detector.

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