Vietnam Journal of Mathematics and Youth

Translated by Hà Duy Hưng & Hà Thu Thảo
Dept. Speci. in Math. and Info.,Hanoi Univer. of Edu.
Vietnam.

Volume35,No.247,January,1998

Note: Each number has a half of the number of problems for under secondery school which you have to prove them by very elementary method ,and this causes the fact that alot of them are not interesting or too simple or tecnique(but not all).That's why i do not post these problems in here.So the problems you will get are kinds of problems for upper secondery school.You can use them to improve your skills in solving mathematics problems or preparing for math competitions,Olympiads,...etc... or just enjoy in the free time!

Problem T6/247: The sequence u_n is difined by

$$u_1 = 1, u_2 = 2, u_n = u_{n-1} + u_{n-2}, \forall n \geq 3$$

Prove that the sequence x_n difined by

$$x_n = \sum_{k=1}^{n} \frac{1}{u_k} \qquad n \geq 1$$

is convergent.

Problem T7/247: Let be given two continous functions f,g:[0;1] $\rightarrow$[0;1] satisfying the condition f(g(x))=g(f(x)) for every $x \in [0;1]$. Suppose that f is an increasing function . Prove that there exist $a \in [0;1]$ such that f(a)=g(a)=a.

Problem T8/247: Prove that in every triangle ABC,we have :

$$\sum_{cyclic}\frac{1}{\sin^2 A}\geq\frac{1}{2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

Problem T9/247: Let be given a rectangle ABCD and a point M , moving on the side BC . The angle-isector of $\angle DAM$ cuts the side BC at N . Determine the position of M so that $\frac{AN}{NB}$ attains its least value.

Problem T10/247: Let be given a tetrahedron ABCD.The planes besecting the dihedral angles at the edges CD,DA,AB,and BC cuts respectively the edges AB,BC,CD and DA at M,N,P,Q.Prove that

$$\frac{MA}{MB}+\frac{NB}{NC}+\frac{PC}{PD}+\frac{QD}{QA}\geq 4$$

Volume35(248),February,1998

Problem T6/248: For each n is natural , let T(n) be the sum of the digits of n writen in decimal system.

1) Prove that

aT(b)+bT(a)-2T(ab) is divisble by 9 for all a,b are natural numbers

2) Calculate Q= $\sum_{k=1}^{1998}T(k)$

Problem T7/248: Prove the equality :

$$(1-C_n^2+C_n^4-...)^2+(C_n^1-C_n^3+C_n^5-...)^2=2^n$$

Problem T8/248: Find all positive integers x,y,z satisfying the equation

$$\sqrt{x+2\sqrt{3}} = \sqrt{y} + \sqrt{z}$$

Problem T9/248: Let M be the point inside a given triangle ABC such that $\angle AMC = 90^0, \angle AMB = 150^0, \angle BMC = 120^0$. Let P , Q , R be respectively the circumcenters of the triangles AMC, AMB , BMC . Compare the areas of the triangle PQR and ABC .

Problem T10/248: Let be given a triangle ABC (BC=a , CA=b , AB=c) and let A_1, B_1, C_1 be respectively the points of intersection of the inbesectors of the angles A , B , C parallel respectively B_1C_1, C_1A_1, A_1B_1 by cutting eah other ,form a triangle $A_2B_2C_2$. Prove that

$$S_{A_2B_2C_2} = \frac{(a+b)(b+c)(c+a)}{8R}$$

where R is the radius of the circumcircle of the triangle ABC

Volume35(249),March,1998

Problem T6/249: Solve the follwing equation, where m is a parameter:

$$\text{arctg}\frac{2x}{1-x^2} = \text{m}\sqrt{1-x^2} + 2\text{arctg}x \quad , \quad \text{m} \in R$$

Problem T7/249: Let be given integers m≥ 2,n$\in N*$.Prove that

Problem T8/249: Let be given an interger n≥ 8 and a convex n-polygon

Suppose that this n-polygon is partitioned in convex octagons so that each side of the n-polygon is a side of a such octagon.Prove that there exist five consecutive vertices of the n-polygon which are five vertices of a such octagon

Problem T9/249: Consider the proposition: "the point of intersection of two inner angled-bisectors of the angles formed by the two extended opposite sides of a convex quadrilateral lies on the line joining the midpoints of its two dagonals"
(if a pair of oposite sides are parallel,their midline is considered as its inner angled-besector).Is this proposition true when the considered quadrilateral is:
1) a trapezium,
2) an circumscibable quadrilateral,
3)an isscribable quardrilateral,
4)an arbitrary quadrilateral? Justify your answer.

Problem T10/249: Let r and R be respectively the radii of the inscribed and circumscribed spheres of a tetrahedron $A_1A_2A_3A_4$.Let (O_i, p) (i=1,2,3,4) be four equal spheres lying inside the tetrahedron,passing through a common pont O' and so that the sphere (O_i) is tangent to three faces passing though vertex S_i of the tetrahedron $A_1A_2A_3A_4$.
Calculate p in terms of R and r and deduce from it,the construction of these four spheres.

Volume35(250),April,1998

Problem T6/250: m is a given positive integer.Prove that there exits integers a,b satisfying simultaneously the conditions:

$$|a| \le m \text{ , } |b| \le m \text{ and } 0 < a+b\sqrt{2} \le \frac{1+\sqrt{2}}{m+\sqrt{2}}$$

Problem T7/250: Find all polynomials with real coefficients f(x) such that cosf(x) ($x \in R$) is a periodic function.

Problem T8/250: Let be given three positive real number a,b,c.Prove the inequalities

$$\frac{1}{2}+\frac{a^2+b^2+c^2}{ab+bc+ca}\geq\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq 2-\frac{1}{2}\frac{ab+bc+ca}{a^2+b^2+c^2}$$

<u>ProblemT9/250</u>: Suppose that the triangle ABC satisfies the condition that cotgA, cotgB, cotgC (in this order) form an arithmetic progression.Prove that GAC=GBA,where G is the center of gravity of the triangle ABC.
[Note : In some countries cotg means cotan or cot , tg means tan]

<u>ProblemT10/250</u>: Two tetrahedra ABCD and $A'B'C'C'$ are in a position in space such that:

$$AB'^2+BC'^2+CD'^2+DA'^2=A'B^2+B'C^2+C'D^2+D'A^2.$$

Prove that if the planes $\alpha,\beta,\gamma,\delta$ passing respectively through A,B,C,D,and perpendicular respectively to the lines $A'B', B'C'.C'D', D'A'$ are concurrent then the planes $\alpha', \beta', \gamma\,\delta'$ passing respectively through A', B', C', D' and perpendicular respectively to the lines AB,BC,CD,DA are also concurrent.

Volume35(250),May,1998

<u>ProblemT6/251</u>: The sequence $\{x_n\}_{n=1}^{\infty}$ is defined by

$$x_1=7, x_2=50, x_{n+1}=4x_n+5x_{n-1}-1975, \forall n\geq 2$$

Prove that x_{1996}M1997.

<u>ProblemT7/251</u>: Prove that if the equation of third degree

$$x^3-px^2+qx-p=0$$

where p,q$\in R$,p>0,q>0,has three real roots, then $p\geq(\frac{1}{4}+\frac{\sqrt{2}}{8})(q+3)$

Problem T8/251: Find all continuous functions f, defined on the segment $\left[-\frac{1}{12};\frac{1}{6}\right]$,satisfying the condition:

$$1996\mathrm{f}(x)\frac{1997}{1998}\mathrm{f}\left(\frac{x}{1999}\right)=1996x^{2000}\quad\forall x\in\left[\frac{1}{12};\frac{1}{6}\right].$$

Problem T9/251: Inside a circle $(\varepsilon)$with radius 1, has been placed a finite number of small circles, the sum of the lengths of the radii of which is equal to 3995.MN is and arbitrary given chord of (ε).Prove that we can construct a line parallel to MN, cutting at least 1998 of these small circles

Problem T10/251: Let G$_i$ be the center of gravity of the face opposite to the vertex A_1 of the tetrahedron $A_1A_2A_3A_4$ $i=\overline{1,4}$ .Prove that if the lines passing G_j,perpendiculer to the face of $A_1A_2A_3A_4$ contaitug G_j,are concurent then the tetrahedron is an orthoentric one(ie.. its altitudes are concurent).

Volume35(252),June,1998

ProblemT6/252:Find all positive integers x,y,z satisfying the condition $x^2+y^2=z^2$ such that z is an odd number, and each x and y is a power of prime number

Problem T7/252:Let be given a real number α >2 and a sequence of positive real numbers $\{a_n\}\frac{\alpha}{n=1}$ satisfying the condition $a\frac{\alpha}{n}=a_1+a_2+...+a_{n-1}$ for all $n\geq 2$

Prove that the sequence $\left\{\frac{a_n}{n}\right\}_{n=1}^{\alpha}$ has a finite limit when $n\to\propto$ and find this limit.

Problem T8/252:Let be given the following 6×5- square grid

(each sque is marked by one of the first 29 positive integers, a square is left blank)
Each time,one can take the number in a square adjacent to the blank square and put in the latter
Can we obtain, after a finite number of such movings, the following grid from the given grid?

1	2	3	4	5
6	7	8	9	10
11	12		13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	29

29	2	3	4	5
6	7	8	9	10
11	12		13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	1

Problem T9/252:Let $A_iH_{i,} A_iD_{i,} A_iM_i$ be respectively the altitude, the inner angled-bisector and the median issued from the vertex A_1 of a triangle $A_1A_2A_3 (i=1,2,3)$. Dose there exist a scalene triangle $A_1A_2A_3$ such that $D_iH_i = D_iM_i, \forall i = 1,2,3.$

.

Problem T10/252: Let be given a tetrahedron ABCD and a point M satisfying the condition:

$$\alpha \overrightarrow{MA} + \beta \overrightarrow{MB} + \gamma \overrightarrow{MC} + \delta \overrightarrow{MD} = \overrightarrow{O}$$

An arbitrary line Δ, passing through M, cuts the planes (BCD) , (CDA) , (DAB) (ABC) respectively at $A_{1,} B_1, C_1, D_1$.Prove that

$$\frac{\alpha}{\overline{MA_1}} + \frac{\beta}{\overline{MB_1}} + \frac{\gamma}{\overline{MC_1}} + \frac{\delta}{\overline{MD_1}} = 0$$

Volume 35(253),July, 1998

Problem T6/253:Let two real numbers x,y satisfy

$$\begin{cases} x^2 + y^2 > 4 \\ x^2 + y^2 - 2x - 2y \le 0 \end{cases}$$

Find the greatest and the least values of the expression

F=2x+y

Problem T7/253: The sequence (x_n) with $n \ge 0$ is defined by:
$x_0 = 1, x_1 = 5, x_{n+1} = 6x_n - x_{n-1}, n = 1,2,...$ Find $\lim_{n\to\infty} \mathrm{x}_n \{x_n\sqrt{2}\}$($\{\alpha\} = \alpha - [\alpha]$denotes the fractional part of α)

Problem T8/253: Prove that the function $y = \cos(x^3)$ is unperiodic.

Problem T9/253: Let ABCD be a square.E is the midpoint of AB,F is a point on BC such that FB>FC. G is the point of intersection of the ray EF with the ray DC. I is the center of the circle which is tangent to BC and to the line DC at the point G.T he second tangent to of the circle (I) passing through F cuts the line DC at a point K. Prove that AEGK is a parallelogram.

Problem 10/253: Let ABCD be a tetrahedron with center of gravity G and center of circumscribed sphere O. Let I be the symmetric point of O with respect to G. Prove that if O is inside the tetrahedron,thenso is I.

Volume35 (254), Agust, 1998

Problem T6/254:Consider the real number x,y,z,t satisfying the condition:

$$(x+y)(z+t)+xy+88=0$$

find the least value of the expression

$$A=x^2+9y^2+6z^2+24t^2.$$

Problem T7/254:A box contains 100 cards labelled by the numbers from 1 to 100.Choose at random 3 cards and take the sum of the numbers written on these 3cards.Calculate the probability that this sum is divisibleby 3.

Problem T8/254: Calculate the angles of a triangle ABC knowing that:

$$\begin{cases} \angle A > \angle B > \angle C \\ \cos 3A + \cos 3B + \cos 3C = 1 \\ \sin 5A + \sin 5B + \sin 5C = 0 \end{cases}$$

Problem T9/245:Let be given a triangle with area S, with sides a,b,c and let n be a positive integer. Prove that

$$a^{2n}+b^{2n}+c^{2n} \geq 3\left(\frac{4}{\sqrt{3}}\right)^n S^n+(a-b)^{2n}+(b-c)^{2n}+(c-a)^{2n}$$

Problem T10/245:Let be given a tetrahedron OABC, right at O (i.e.OA,OB,OC are orthogonal each to the others),OA=a,OB=b,OC=c.Let r be the radius of its inscribed sphere. Prove the inequality

$$\frac{1}{r} \geq \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\left(\frac{3\sqrt{3}}{a+b+c}\right)$$

when dose equality occur?

Volume 35(255),September,1998

Problem T6/255:The sequence of numbers $\{a_n\}_{n=1}^{\alpha}$ is defined by:

$$a_1 = 1964, a_2 = 96, a_{n+2} = 30a_{n+1}^2 - 75a_{n+1}a_n - 1944a_n (n \geq 1)$$

Prove that every term of the sequence $\{a_n\}_{n=1}^{\alpha}$ is not the sum of the seventh powers of thee integers.

Problem T7/255:Let be given positive integers l, m and a polynomial

$$P(x) = a_0 x^{m+1} + a_1 x^m + \ldots + a_m x, a_0 \neq 0$$

Consider the sequence

$$\left\{ v_n = \sum_{k=0}^{\ln} P\left(\frac{1}{n+k}\right) \right\}_{n=1}^{\infty}$$

Find $\lim\limits_{n \to \infty} v_n$

Problem T8/255: Let the polynomial

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e$$

satisfy the condition

$$| f(x) | \leq 1 \text{ for } |x| \leq 1$$

Prove that for every M>1,

$$| f(x) | \leq \frac{32}{3} M^4 - \frac{32}{3} M^2 + 1$$

for |x|<M.

Problem T9/255: In a Cartesian orthogonal system of coordinates Oxy in the plane, let be given two fixed points A,B on Ox with abscisses respectively a,b such that 0<a<b. A point P moves on the line Oy. The perpendicular to Ox at A

cuts the line BP at M. The tangents issued from B to the circle $(P;PM)$ touch the circle at Q,Q'.

Problem 10/255:Let be given a regular tetrahedron ABCD. Find on the planes (BCD),(CDA),(DAB),(ABC) the points X,Y,X,T such that the sum of the lengths of the sides of the tetrahedron XYZT attains its least value.

Volume 35(256),November,1998

Problem T6/256:Let be given a positive integer s , s distinct prime numbers $p_1, p_2,..., p_s$ and s positive integers $k_1, k_2,..., k_s$ .Put $m = p_1^{k_1} p_2^{k_2} ... p_s^{k_s}$ .Prove that

$$\frac{1-p_1^{k_1+1}}{1-p_1}\frac{1-p_2^{k_2+1}}{1-p_2}\cdots\frac{1-p_s^{k_s+1}}{1-p_s} < m(\ln m + 1)$$

Problem T7/256:Let be given a sequence of functions $\{f_n(x)\}_{n=0}^{\alpha}$

$$\begin{aligned} &i) f_n(x) : (0,+\infty), \forall n \in N, \\ &ii) f_0(x) = x, \forall x \in (o,+\infty), \\ &iii) f_{n+1}(x) = \sqrt{x^2 + 6f_n(x)}, \forall x \in (0,+\infty), \forall n \in N. \end{aligned}$$

a)Prove that for every $n = 1,2,3,...$ there exists a unique number $x_n \in (0,+\infty)$ such that $f_n(x_n) = 2x_n$

b) Prove that $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence of numbers and $\lim_{n\to+\infty} x_n = 4$

Problem T8/256:Consider the triangle ABC.Find the least value of the expression:

$$T = \cot gA + \cot gB + \cot gC + tg\frac{A}{2} + tg\frac{B}{2} + tg\frac{C}{2}$$

Problem T9/256:Let be given a triangle ABC and let M,N,P be respectively the midpoints of its sides BC,CA,AB.Draw on the outside of te triangle ABC the rays $Px \perp AB$ and $Ny \perp AC$ and take on Px the point E such that $NE = AC$. Draw

$-$the ray D_Z parallel to and with same direction with the ray BC, and take on the point F such that $DF = \frac{BC}{2}$. Prove that :

$$a)EF \perp AM,$$
$$b)BC^2 + EF^2 = 2\left(AB^2 + AC^2\right).$$

Problem T10/256 Let be given a tetrahedron OABC, the trihedral angle at 0 of which is right. Find the locus of points M in space such that

$$MA^2 + MB^2 + MC^2 = 3MO^2$$

Volume 35(257),November,1998

Problem T6/257:Let be given n positive numbers $a_1, a_2 ..., a_n (n \geq 2)$. Prove that for every $k \in N^*$, we have

$$\sum_{k=1}^{n} \frac{a_k^{2^k}}{\prod_{j=0}^{k-1} (a_k^j + a_{k+1}^j)} \geq \frac{\sum_{k=1}^{n} a_k}{2^k}$$

When does the equality hold?

Problem T7/257: Let be a real a >1 .The sequence $\{a_n\}_{n=1}^{\infty}$ is difined by:

$$a_1 = a, a_{n+1} = a_n^2 - a_n (\forall n \in N^*)$$

Find the limite of the following

$$\lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{a_k} \right)$$

Problem T8/257: Consider the two euquations:

cos2x + acosx +2=0
cos2x+ bcosx+2=0

Suppose that each equation has 4 distinc roots in the interval $(0,2\pi)$ .Prove that the equation:

cos2x+(a+b)cosx+5=0 has no root

Problem T9/257: Prove that in a right triangle the cosinus of the acute angle formed by the medians corresponding to the legs is not less than 0,8(ie..4/5)

Problem T10/257: Let be given a tetrahedron ABCD and let G be its center of gravity .M is an arbitrary point in space ,let N be the point defined by $\overrightarrow{MN} = 4\overrightarrow{MG}$ .Prove that

NA+NB+NC+ND $\leq$ 2MN+MA+MB+MC+MD

When does the equality holds?

Volume 35 (258),December,1998

Problem T6/258:The sequence of non negative numbers $a_0, a_1, \ldots$ satisfies the condition

$$a_{m+n} + a_{m-n} = \frac{1}{2}\left(a_{2m} + a_{2n}\right)$$

for every pair of indexes $m, n(m \geq n)$. Calculate a_{1998}, knowing that $a_1 = 1$

Problem T7/258:Find all functions $f(x)$ satisfying simultaneously the following conditions:

$i) f(x)$ is determined and continuous on R

$ii)$ for every family of 1997 numbers $x_1, x_2, x_3, ..., x_{1997}$, such that $x_1 < x_2 < x_3 < ... < x_{1997}$, we have

$$f(x_{999}) \geq \frac{1}{1996}\left(f(x_1) + f(x_2) + ... + f(x_{998}) + f(x_{1000}) + f(x_{1001}) + ... + f(x_{1997})\right)$$

$iii) f(x) > 0$ for every $x \in Z$ anf $f(1997) = \log_{1998} 1997$.

Problem T8/258:Consider a triangle ABC.Prove that a necessary and sufficient condition for the existence of a point D on the side BC such that AD=BC is $\sin A \geq \sin B \sin C$.

Problem T9/258:Let be given a triangle ABC, the center of gravity of which lies inside the inscribed circle (I).Prove that

$$\max\{a^2, b^2, c^2\} < 4\min\{bc, ca, ab\}.$$

Problem T10/258:Let be given $\alpha \in \left(0, \frac{\pi}{2}\right)$.

Put
$$n = \left[\frac{4}{\sin^2 \alpha \cos^2 \frac{\alpha}{2}}\right] + 1$$

Can we construct in space n lines passing through a common point such that the angle formed by any two of these lines among them has a measure not less than α ? Justify the answer.