N6 Rational Numbers and Irrational Numbers

N6.1 Rational Numbers and Irrational Numbers

A. Rational Numbers

Any number that can be written as an integer divided by another integer, i.e. $\frac{\text{integer}}{\text{integer}}$, is called a ***rational number***.

e.g. $\frac{3}{7}$, $\frac{14}{5}$, $\frac{-3}{17}$, $\frac{21}{-8}$, $0\left(=\frac{0}{1}\right)$, $0.013\left(=\frac{13}{1000}\right)$.

Note that the denominator (i.e. the lower integer) cannot be 0 because the division by 0 is meaningless.

B. Irrational Numbers

Any number that cannot be written as a fraction of two integers is called an ***irrational number***. Therefore, it may be said that any number which is not a rational number is an irrational number.

e.g. π, $\sqrt{2}$, $-\sqrt{6}$

C. Decimal Representation of Rational and Irrational Numbers

A rational number can be distinguished from an irrational number when both are expressed in decimals.

A rational number always gives either a terminating decimal or a recurring (repeating) decimal.

e.g. $\left.\begin{array}{r} \frac{5}{2}=2.5 \\ -\frac{3}{4}=-0.75 \end{array}\right\}$ Terminating decimals

$\left.\begin{array}{r} \frac{1}{3}=0.\dot{3}=0.333333... \\ \frac{2}{11}=0.\dot{1}\dot{8}=0.181818... \end{array}\right\}$ Recurring decimals

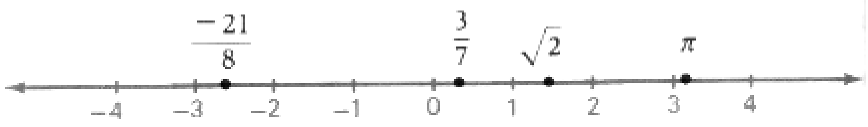
An irrational number does not behave this manner, and when expressed as a decimal, it always gives an infinite non-repeating decimal.

e.g. π = 3.141592653589793238462643383279502884…

$\sqrt{2}$ = 1.414213562419339166281…

D. Rational Numbers and Irrational Numbers in the Number Line

The set of rational numbers $\left(\text{e.g.} -\frac{21}{8}, \frac{3}{7}\right)$ and irrational numbers $\left(\text{e.g.} \sqrt{2}, \pi\right)$ can be represented in the number line.



Checkpoint 1

Identify each of the following numbers as rational or irrational by putting a tick in the correct column.

Numbers	Rational	Irrational
$-\pi$		
$\frac{22}{7}$		
$\sqrt{8}$		
$3.1\dot{4}$		
$0.\dot{2}8571\dot{4}$		

N6.2 Surds

Surds are irrational numbers involving the radical sign. There are three important properties of surds:

IMPORTANT PROPERTIES OF SURDS

For $a > 0$, $b > 0$,

$$(\sqrt{a})^2 = a$$

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

By using the above properties, a surd may be expressed as the product of two or more surds or as the product of a rational number and a surd. For example,

$$\sqrt{12} = \sqrt{4\times3} = \sqrt{4}\times\sqrt{3} = 2\sqrt{3}$$
$$\sqrt{30} = \sqrt{2\times3\times5} = \sqrt{2}\times\sqrt{3}\times\sqrt{5}$$
$$\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{3}$$

Note that $\frac{1}{2}$ is called the coefficient of $\sqrt{3}$.

When a surd is expressed so that the integer under the radical sign is as small as possible, the surd is said to be in its *simplest form*.

e.g. When expressed in their simplest forms,

$$\sqrt{12} = 2\sqrt{3}$$
$$\sqrt{30} = \sqrt{2}\times\sqrt{3}\times\sqrt{5}$$
$$\sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$$

Conversely, a coefficient of a surd may be brought under the radical sign. For example,

$$5\sqrt{2} = \sqrt{25}\times\sqrt{2} = \sqrt{25\times2} = \sqrt{50}$$
$$4\sqrt{5} = \sqrt{4}\times\sqrt{5} = \sqrt{4\times5} = \sqrt{20}$$

Checkpoint 2

Express the following surds in their simplest forms:

(a) $\sqrt{8} =$ ____________ (b) $\sqrt{24} =$ ____________

(c) $-\sqrt{96} =$ ____________ (d) $\frac{1}{2}\sqrt{32} =$ ____________

(e) $-5\sqrt{50} =$ ____________ (f) $2\sqrt{98} =$ ____________

(g) $-\sqrt{\frac{7}{16}} =$ ____________ (h) $\sqrt{\frac{28}{25}} =$ ____________

Checkpoint 3

Express the following surds into entire surds:

(a) $2\sqrt{5} =$ ___________ (b) $5\sqrt{2} =$ ___________

(c) $6\sqrt{2} =$ ___________ (d) $-10\sqrt{5} =$ ___________

(e) $2\sqrt{\frac{1}{2}} =$ ___________ (f) $5\sqrt{\frac{2}{5}} =$ ___________

(g) $-3\sqrt{a} =$ ___________ (h) $a\sqrt{5} =$ ___________

N6.3 Addition, Subtraction and Multiplication of Surds

Surds such as $2\sqrt{3}$, $3\sqrt{3}$, $5\sqrt{3}$ are called like surds because they all contain the same surd $\sqrt{3}$. Surds such as $2\sqrt{5}$ and $\sqrt{6}$ are called unlike surds.

Example 1

Simplify

(a) $\sqrt{20}+\sqrt{45}+\sqrt{180}$

(b) $\sqrt{28}+\sqrt{54}-\sqrt{63}$

Solution

(a) $$\begin{aligned}\sqrt{20}+\sqrt{45}+\sqrt{180} &= 2\sqrt{5}+3\sqrt{5}+6\sqrt{5}\\ &= 11\sqrt{5}\end{aligned}$$

(b) $$\begin{aligned}\sqrt{28}+\sqrt{54}-\sqrt{63} &= 2\sqrt{7}+3\sqrt{6}-3\sqrt{7}\\ &= 3\sqrt{6}-\sqrt{7}\end{aligned}$$

Note: Like surds can be reduced to a single surd; unlike surds cannot be collected and reduced to a single surd.

Checkpoint 4

Simplify

(a) $\sqrt{18}+\sqrt{8}$

(b) $2\sqrt{80}-3\sqrt{45}+\sqrt{5}$

(c) $\sqrt{27x^3}+x\sqrt{12x}$

Surds can be multiplied and combined together.

Example 2

Simplify $\sqrt{32}\cdot\sqrt{125}\cdot\sqrt{24}$.

Solution

$$\begin{aligned}\sqrt{32}\cdot\sqrt{125}\cdot\sqrt{24} &= 4\sqrt{2}\cdot 5\sqrt{5}\cdot 2\sqrt{6}\\ &= 40\cdot\sqrt{2}\cdot\sqrt{5}\cdot\sqrt{2}\cdot\sqrt{3}\\ &= 40\times 2\times\sqrt{5}\times\sqrt{3}\\ &= 80\sqrt{15}\end{aligned}$$

Example 3

Simplify

(a) $(3+\sqrt{2})(2-\sqrt{2})$ (b) $(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})$

Solution

(a)
$$\begin{aligned}(3+\sqrt{2})(2-\sqrt{2}) &= 6-3\sqrt{2}+2\sqrt{2}-(\sqrt{2})(\sqrt{2}) \\ &= 6-\sqrt{2}-2 \\ &= 4-\sqrt{2}\end{aligned}$$

(b)
$$\begin{aligned}(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2}) &= (\sqrt{5})^2-(\sqrt{2})^2 \\ &= 5-2 \\ &= 3\end{aligned}$$

Checkpoint 5

Find each product and express the result in the simplest form.

(a) $\sqrt{24}\times\sqrt{54}\times\sqrt{18}$ (b) $4\sqrt{8}\times3\sqrt{10}\times7\sqrt{3}$

(c) $3\sqrt{2}\times5\sqrt{8}\times2\sqrt{20}$

Checkpoint 6

Express $\sqrt{5}(4\sqrt{5}-3\sqrt{2})$ in the simplest form.

Checkpoint 7

Simplify

(a) $(2\sqrt{3}+5)(\sqrt{3}-2)$

(b) $(2\sqrt{2}+5)^2$

(c) $(\sqrt{3}-\sqrt{2})(3\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$

Checkpoint 8

Simplify $2(3\sqrt{6}-2\sqrt{2})^2-3(\sqrt{6}-2\sqrt{2})^2$.

N6.4 Rationalization

The process of making the denominator rational is called rationalizing the denominator or rationalization.

Example 4

Express $\dfrac{5}{2\sqrt{2}}$ with rational denominator.

Solution

$$\begin{aligned}\frac{5}{2\sqrt{2}} &= \frac{5}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{5\sqrt{2}}{2\cdot 2}\\ &= \frac{5\sqrt{2}}{4}\end{aligned}$$

Checkpoint 9

Rationalize $\dfrac{4}{3\sqrt{12}}$.

Example 5

Rationalize $\dfrac{1}{\sqrt{3}+\sqrt{2}}$.

Solution

(In order to make the denominator rational, we may use the identity $(a+b)(a-b) \equiv a^2 - b^2$.)

$$\begin{aligned}\frac{1}{\sqrt{3}+\sqrt{2}} &= \frac{1}{\sqrt{3}+\sqrt{2}} \times \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} \\ &= \frac{\sqrt{3}-\sqrt{2}}{3-2} \\ &= \sqrt{3}-\sqrt{2}\end{aligned}$$

Example 6

Rationalize $\dfrac{5\sqrt{2}+2\sqrt{5}}{5\sqrt{2}-2\sqrt{5}}$.

Solution

$$\begin{aligned}\frac{5\sqrt{2}+2\sqrt{5}}{5\sqrt{2}-2\sqrt{5}} &= \frac{5\sqrt{2}+2\sqrt{5}}{5\sqrt{2}-2\sqrt{5}} \times \frac{5\sqrt{2}+2\sqrt{5}}{5\sqrt{2}+2\sqrt{5}} \\ &= \frac{50+20\sqrt{10}+20}{50-20} \\ &= \frac{70+20\sqrt{10}}{30} \\ &= \frac{7+2\sqrt{10}}{3}\end{aligned}$$

Checkpoint 10

Rationalize the following:

(a) $\dfrac{1}{\sqrt{2}-1}$

(b) $\dfrac{35}{6\sqrt{5}-5\sqrt{3}}$

(c) $\dfrac{\sqrt{2}+1}{\sqrt{3}+\sqrt{2}}$

(d) $\dfrac{1}{\sqrt{2}-1}+\dfrac{2}{\sqrt{3}+1}$

Exercise N6
Rational and Irrational Numbers

N6.3

1. Express the following in their simplest forms:

(a) $3\sqrt{3}-\sqrt{3}$
(b) $3\sqrt{8}-\sqrt{50}$
(c) $\sqrt{75}+\sqrt{108}$
(d) $\sqrt{\frac{9}{4}}+\sqrt{\frac{25}{4}}$
(e) $2\sqrt{2}-\sqrt{8}+\sqrt{32}$
(f) $\sqrt{27}+4\sqrt{12}-5\sqrt{75}$
(g) $4\sqrt{27}-4\sqrt{98}+\sqrt{147}+28\sqrt{2}$
(h) $42\sqrt{\frac{1}{6}}-12\sqrt{\frac{1}{24}}$
(i) $\sqrt{ax^2}+2\sqrt{a^2x}-\sqrt{4ax^2}-a\sqrt{4x}$
(j) $3\sqrt{a^4b}-2a^2\sqrt{b}+6\sqrt{a^4b^5}$

2. Find each product and express the result in the simplest form.

(a) $\sqrt{27}\times\sqrt{12}\times\sqrt{45}\times\sqrt{80}$
(b) $2\sqrt{8}\times3\sqrt{6}\times\sqrt{12}$
(c) $\sqrt{3}(\sqrt{12}-\sqrt{18})$
(d) $(\sqrt{2}-\sqrt{3})(\sqrt{2}-\sqrt{6})$
(e) $(2\sqrt{a}-\sqrt{b})(\sqrt{a}+2\sqrt{b})$
(f) $(2\sqrt{3}-1)(2\sqrt{3}+1)$
(g) $(\sqrt{7x}+\sqrt{2x})(\sqrt{7x}-\sqrt{2x})$
(h) $(\sqrt{5}-3+\sqrt{2})(\sqrt{5}-3-\sqrt{2})$
(i) $(4+3\sqrt{2})(\sqrt{2}-2)$
(j) $(3+2\sqrt{2}-\sqrt{5})(3+\sqrt{5}+2\sqrt{2})$

3. Simplify:

(a) $(2\sqrt{3}-3)(\sqrt{2}+\sqrt{3})-3(\sqrt{3}+\sqrt{2})+\sqrt{6}(\sqrt{6}-2)$

(b) $(2\sqrt{5}-\sqrt{3})(\sqrt{5}+2\sqrt{3})-(3\sqrt{5}-2\sqrt{3})^2$

(c) $3(\sqrt{12}-\sqrt{27})^2+(\sqrt{48}-\sqrt{75})(-3\sqrt{48}-2\sqrt{75})$

4. If $x=2\sqrt{5}+4\sqrt{3}$ and $y=4\sqrt{5}-2\sqrt{3}$, evaluate each of the following without using a calculator.

(a) xy;
(b) x^2+y^2
(c) x^2-y^2

(*Leave the answers in surd form if necessary.*)

N6.4

5. Rationalize the denominator of each expression and simplify the result.

(a) $\dfrac{10}{3\sqrt{5}}$

(b) $\dfrac{\sqrt{15}-3}{\sqrt{3}}$

(c) $\dfrac{1}{1-\sqrt{6}}$

(d) $\dfrac{\sqrt{2}}{1+\sqrt{2}}$

(e) $\dfrac{26}{5-2\sqrt{3}}$

(f) $\dfrac{2}{\sqrt{12}-\sqrt{18}}$

(g) $\dfrac{2+\sqrt{6}}{1-\sqrt{6}}$

(h) $\dfrac{3\sqrt{2}-\sqrt{3}}{2\sqrt{3}-7\sqrt{2}}$

(i) $\dfrac{\sqrt{2}}{\sqrt{10}-2\sqrt{2}}$

(j) $\dfrac{\sqrt{2x}}{\sqrt{6x}-\sqrt{3x}}$

6. Simplify:

(a) $\dfrac{2\sqrt{10}}{\sqrt{10}-\sqrt{5}}-\dfrac{5}{2\sqrt{2}-\sqrt{3}}$

(b) $\dfrac{2\sqrt{7}}{\sqrt{7}-\sqrt{5}}-\dfrac{\sqrt{5}}{\sqrt{5}-\sqrt{3}}$

(c) $\dfrac{\sqrt{2}-\sqrt{\dfrac{25}{18}}}{\sqrt{3}\left(\sqrt{2}+\dfrac{1}{\sqrt{2}}\right)}$

7. If $x=\dfrac{\sqrt{5}+1}{2}$, find the value of $x^2-\dfrac{1}{x^2}$.

(*Give the answer in surd form.*)

8. Rationalize $\dfrac{x-y-\sqrt{x^2-y^2}}{x-y+\sqrt{x^2-y^2}}$, where $x > y > 0$.