G8 Square Roots and Pythagoras' Theorem

G8.1 Square and Square Roots

Definitions:

If $a \times a = n$, then

(1) n is the square of a, i.e. $n = a^2$.

(2) a is the square root of n.

For example, since 2 × 2 = 4 and (–2) × (–2) = 4,

(1) The square of 2 is 4 and the square of –2 is also 4.

(2) 2 and –2 are the square roots of 4.

The symbol for square root is $\sqrt{}$, called the radical sign. We may write $\sqrt{4} = 2$.

In fact, for any positive number n, there are two square roots, the positive square root $\sqrt{n}$ and the negative square root $-\sqrt{n}$. So if $a^2 = 9$, then $a = \sqrt{9} = 3$ or $a = -\sqrt{9} = -3$. Note that $\sqrt{9} = 3$ but $\sqrt{9} \neq -3$.

Checkpoint 1

By the use of a calculator, evaluate the following: (Give the answers correct to 1 decimal place where necessary.)

(a) $(-29)^2$

(b) $(1.21)^2$

(c) $\sqrt{361}$

(d) $-\sqrt{289}$

(e) $\sqrt{0.05 + 1.64}$

(f) $\dfrac{\sqrt{6} + 2}{3}$

(g) $\dfrac{\sqrt{3} - 1}{\sqrt{3} + 1}$

(h) $\dfrac{2\sqrt{3} - 3\sqrt{2}}{\sqrt{5}}$

Facts:

For positive numbers a and b,

(1) $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$;

(2) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

We may use these results to evaluate square roots and expressions involving square roots.

Example 1

Evaluate the following without using a calculator:

(a) $\sqrt{484}$

(b) $\sqrt{1764}$

(c) $\sqrt{1600}$

(d) $\sqrt{0.04}$

(e) $\sqrt{\frac{36}{49}}$

(f) $\sqrt{2\frac{1}{4}}$

Solution

(a) $\sqrt{484} = \sqrt{2 \times 2 \times 11 \times 11} = \sqrt{2^2 \times 11^2}$

$= \sqrt{2^2} \times \sqrt{11^2}$

$= 2 \times 11 = 22$

(b) $\sqrt{1764} = \sqrt{2 \times 2 \times 3 \times 3 \times 7 \times 7} = \sqrt{2^2 \times 3^2 \times 7^2}$

$= \sqrt{2^2} \times \sqrt{3^2} \times \sqrt{7^2}$

$= 2 \times 3 \times 7 = 42$

(c) $\sqrt{1600} = \sqrt{16 \times 100}$

$= \sqrt{16} \times \sqrt{100}$

$= 4 \times 10 = 40$

(d) $\sqrt{0.04} = \sqrt{4 \times 0.01}$

$= \sqrt{4} \times \sqrt{0.01}$

$= 2 \times 0.1 = 0.2$

(e) $\sqrt{\frac{36}{49}} = \frac{\sqrt{36}}{\sqrt{49}} = \frac{6}{7}$

(f) $\sqrt{2\frac{1}{4}} = \sqrt{\frac{9}{4}} = \frac{\sqrt{9}}{\sqrt{4}} = \frac{3}{2}$

Checkpoint 2

Evaluate the following without using a calculator:

(a) $\sqrt{1369}$

(b) $\sqrt{6400}$

(c) $\sqrt{2.56}$

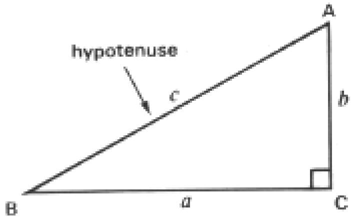
(d) $\sqrt{15^2 \times 9}$

(e) $\sqrt{\dfrac{324}{1156}}$

(f) $\sqrt{4\dfrac{76}{81}}$

G8.2 Pythagoras' Theorem

The figure shows a right-angled triangle ABC.

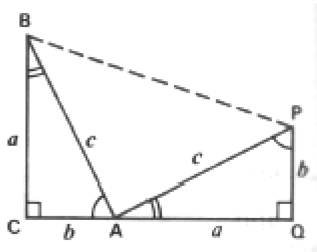


The longest side, which is opposite to the right angle, is called the hypotenuse.

Pythagoras' Theorem:
In a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.

i.e. If ΔABC is right-angled at C, then $a^2 + b^2 = c^2$.
[*Reference: Pyth. Theorem*]

Proof:
Consider two congruent right-angled triangles ABC and PAQ placed as shown in the figure so that QAC is a straight line. We are going to find the area of the trapezium BCQP in two different ways.



(1) Area of trapezium BCQP

$$= \text{Area of } \Delta ABC + \text{Area of } \Delta PAQ + \text{Area of } \Delta PBA$$

$$= \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}c^2$$

$$= \frac{1}{2}(2ab + c^2)$$

(2) Area of trapezium BCQP

$$= \frac{1}{2}(BC + PQ) \times QC$$

$$= \frac{1}{2}(a+b)(a+b)$$

$$= \frac{1}{2}(a^2 + 2ab + b^2)$$

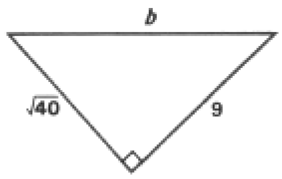
$$\therefore \quad \frac{1}{2}(2ab + c^2) = \frac{1}{2}(a^2 + 2ab + b^2)$$

$$2ab + c^2 = a^2 + 2ab + b^2$$

i.e. $$a^2 + b^2 = c^2$$

Example 2

In the figure, find the value of *b*.



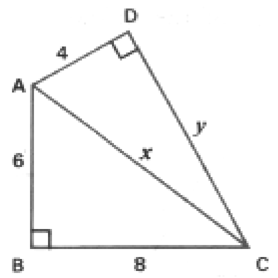
Solution

$9^2 + (\sqrt{40})^2 = b^2$ (Pyth. Theorem)

$$\begin{aligned} b^2 &= 81 + 40 \\ &= 121 \\ b &= \sqrt{121} \\ &= 11 \end{aligned}$$

Example 3

In the figure, find the values of *x* and *y*. (Give the answers correct to 2 decimal places where necessary.)



Solution

(a) In ΔABC,

$x^2 = 6^2 + 8^2$ (Pyth. Theorem)

$$\begin{aligned} &= 100 \\ x &= \sqrt{100} = 10 \end{aligned}$$

(b) In ΔACD,

$4^2 + y^2 = x^2$ (Pyth. Theorem)

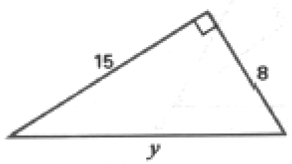
$$\begin{aligned} 16 + y^2 &= 100 \\ y^2 &= 100 - 16 = 84 \\ y &= \sqrt{84} \end{aligned}$$

$= 9.17$ (corr. to 2 decimal places)

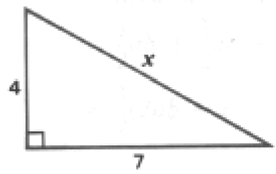
Checkpoint 3

In each of the following figures, find the values of the unknowns. (Correct the answers to 2 decimal places where necessary.)

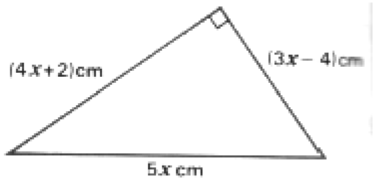
(a)



(b)

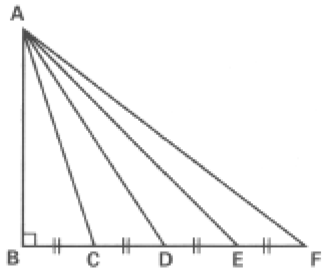


(c)



Example 4

In the figure, $\angle B = 90^o$. BC = CD = DE = EF. Prove that $7AC^2 + 3AF^2 = 7AD^2 + 3AE^2$



Solution

In ΔABC,

$AC^2 = AB^2 + BC^2$ (Pyth. Theorem)

In ΔABD,

$$\begin{aligned} AD^2 &= AB^2 + BD^2 \quad \text{(Pyth. Theorem)} \\ &= AB^2 + (2BC)^2 \\ &= AB^2 + 4BC^2 \end{aligned}$$

In ΔABE,

$$\begin{aligned} AE^2 &= AB^2 + BE^2 \quad \text{(Pyth. Theorem)} \\ &= AB^2 + (3BC)^2 \\ &= AB^2 + 9BC^2 \end{aligned}$$

In ΔABF,

$$\begin{aligned} AF^2 &= AB^2 + BF^2 \quad \text{(Pyth. Theorem)} \\ &= AB^2 + (4BC)^2 \\ &= AB^2 + 16BC^2 \end{aligned}$$

$7AC^2 + 3AF^2$

$= 7(AB^2 + BC^2) + 3(AB^2 + 16BC^2)$

$= 10AB^2 + 55BC^2$

$7AD^2 + 3AE^2$

$= 7(AB^2 + 4BC^2) + 3(AB^2 + 9BC^2)$

$= 10AB^2 + 55BC^2$

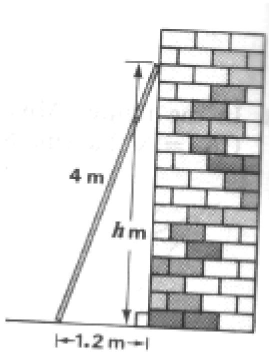
$\therefore 7AC^2 + 3AF^2 = 7AD^2 + 3AE^2$

G8.3 Applications of Pythagoras' Theorem

There are many problems that involve right-angled triangles and some times Pythagoras' Theorem can be used to solve them.

Example 5

A ladder 4 m long leans against a vertical wall. Its foot is 1.2 m from the wall. How far up the wall will the ladder reach? (Correct the answers to 2 decimal places.)



Solution

Let h m be the height reached.

$h^2 + (1.2)^2 = 4^2$ (Pyth. Theorem)

$h^2 + 1.44 = 16$

$h^2 = 16 - 1.44 = 14.56$

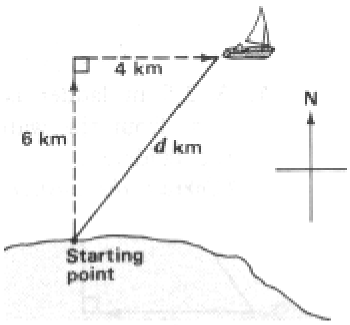
$h = \sqrt{14.56}$

$= 3.82$ (corr. to 2 decimal places)

$\therefore$ The ladder will reach a height of 3.82 m.

Example 6

A boat sails 6 km due north and then 4 km due east. How far is it from its starting point? (Correct the answers to 2 decimal places.)



Solution

Let d km be the required distance.

$d^2 = 6^2 + 4^2$ (Pyth. Theorem)

$d^2 = 52$

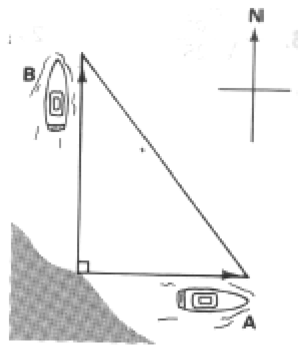
$d = \sqrt{52}$

$= 7.21$ (corr. to 2 decimal places)

$\therefore$ The boat is 7.21 km from its starting point.

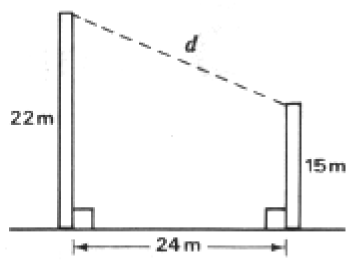
Checkpoint 4

Ship A and Ship B leave a port together. Ship A sails due east at 15 km/h and Ship B sails due north at 20 km/h. What is the distance between Ship A and Ship B after 3 hours?

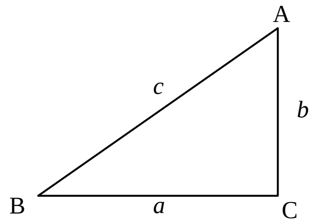


Checkpoint 5

Two vertical walls are 24 m apart. One post is 22 m high and the other is 15 m high. What is the distance (*d*) between the tops of the two posts?



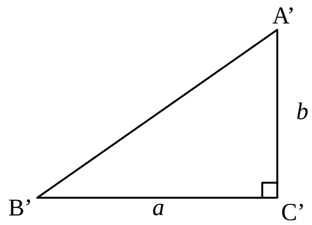
G8.4 Converse of Pythagoras' Theorem



The converse of Pythagoras' Theorem is also true.
i.e. In ΔABC, if $a^2 + b^2 = c^2$, then $\angle$C = 90^o.
[*Reference: Converse of Pyth. Theorem*]

Proof:
Construct a right-angled triangle A'B'C' such that $\angle$C' = 90^o, B'C' = *a* and A'C' = *b*.

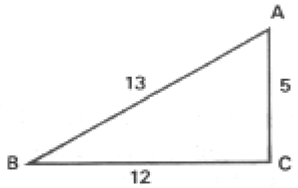
By Pythagoras' Theorem, we have



$$B'C'^2 + A'C'^2 = A'B'^2$$
$$A'B'^2 = a^2 + b^2$$
$$a^2 + b^2 = c^2$$
Since $A'B'^2 = c^2$
$\therefore$ $A'B' = c$

$BC = B'C' = a$ (By construction)
$AC = A'C' = b$ (By construction)
$AB = A'B' = c$ (proved)
$\therefore$ $\Delta ABC \cong \Delta A'B'C'$ (SSS)
We then have $\angle$C = $\angle$C' = 90^o.

Example 7
In the figure, determine whether ΔABC is right-angled.



Solution

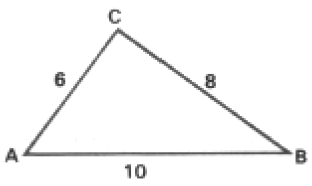
$$AB^2 = 13^2 = 169$$
$$BC^2 = 12^2 = 144$$
$$AC^2 = 5^2 = 25$$
$\because$ $BC + AC = AB$
$\therefore$ $\angle C = 90°$ (Converse of Pyth. Theorem)

Hence ΔABC is right-angled at C.

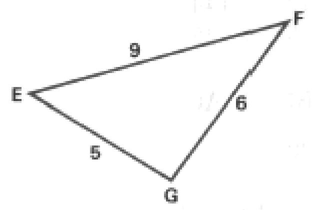
Checkpoint 6

In the following figures, determine whether they are right-angled triangles.

(a)



(b)



G8.5 Rational Numbers and Irrational Numbers

Definitions:

(1) A rational number is a number that can be expressed as a fraction $\frac{m}{n}$, where m and n are integers.

(2) An irrational number is a number that is not a rational number, i.e. a number that cannot be expressed as a fraction $\frac{m}{n}$, where m and n are integers.

Example 8

$5, \frac{2}{3}, 1\frac{1}{5}$, 3.2 are rational numbers.

Proof:

$5 = \frac{5}{1} = \frac{m}{n}$, where $m = 5$ and $n = 1$;

$\frac{2}{3} = \frac{m}{n}$, where $m = 2$ and $n = 3$;

$1\frac{1}{5} = \frac{6}{5}$, where $m = 6$ and $n = 5$;

$3.2 = \frac{16}{5}$, where $m = 16$ and $n = 5$.

$\therefore 5, \frac{2}{3}, 1\frac{1}{5}$, 3.2 are rational numbers.

Example 9

$\sqrt{2}$ is an irrational number.

Proof: (by "Suppose the Converse")

Suppose $\sqrt{2}$ is a rational number.

Then $\sqrt{2} = \frac{m}{n}$ or $\sqrt{2}n = m$, where m and n are integers and they do not have common factors except 1.

We have $m^2 = 2n^2 \ldots(*)$.

Since m and n are integers, 2 is a factor of m^2.

So m^2 is an even number. Also m is an even number and can be expressed as $m = 2k$.

By equation (*), we have $4k^2 = (2k)^2 = m^2 = 2n^2$ or $2k^2 = n^2$.

Since n and k are integers, 2 is a factor of n^2.

Hence n^2 is an even number and so n is also an even number.

But m and n have a common factor 2. This contradicts our supposition.

Thus we can conclude that our supposition is a false statement,

i.e.* $\sqrt{2}$ *is not an irrational number

or* $\sqrt{2}$ *is a rational number.

Usually, radicals like $\sqrt{2}$, $\sqrt{6}$ are irrational numbers. But $\sqrt{4}$ is a rational number since $\sqrt{4} = 2 = \frac{2}{1}$.

There are non-radical irrational numbers like π.

Checkpoint 7

Are the following numbers rational or irrational?

(a) $\sqrt{2} + 5$

(b) $\sqrt{3} - 3$

(c) $\sqrt{2} \times \sqrt{2}$

(d) π^2

(e) $3\sqrt{2} - 2\sqrt{3}$

G8 Exercises
Square Roots and Pythagoras' Theorem

G8.1

1. Evaluate the following with a calculator:

   (a) $\dfrac{\sqrt{3}-\sqrt{2}}{5}$ (b) $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$

   (c) $\dfrac{2\sqrt{3}+\sqrt{2}}{\sqrt{5}}$ (d) $\dfrac{2\sqrt{12}}{\sqrt{2+\sqrt{3}}}-1$

2. Evaluate the following without using a calculator:

   (a) $\sqrt{729}$ (b) $\sqrt{14400}$

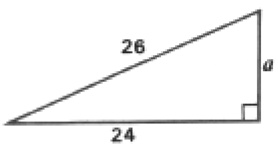
   (c) $-\sqrt{0.000225}$ (d) $\sqrt{\dfrac{196}{256}}$

   (e) $\sqrt{2\dfrac{46}{49}}$ (f) $\dfrac{\sqrt{12}}{3\sqrt{3}}$

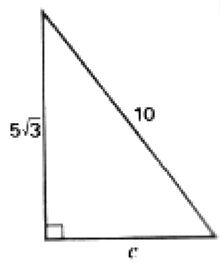
G8.2

3. Find the values of the unknowns in the following figures.

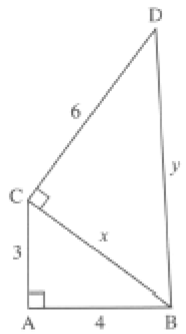
   (a)



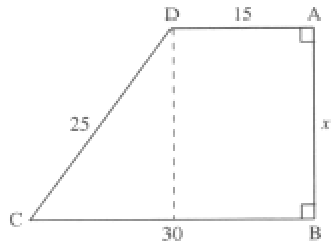
   (b)



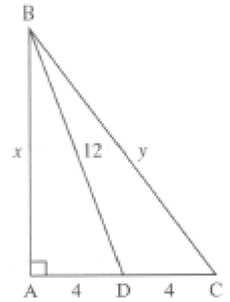
   (c)



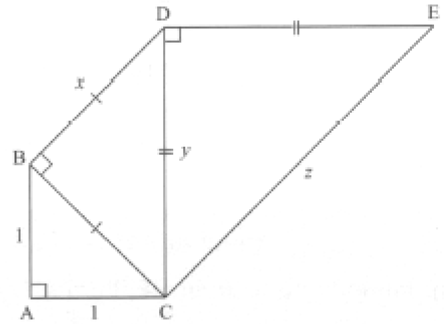
   (d)



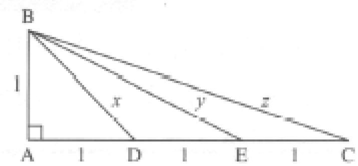
(e)



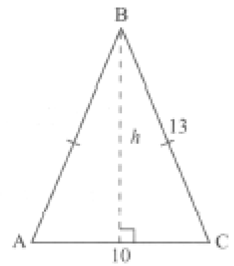
(f)



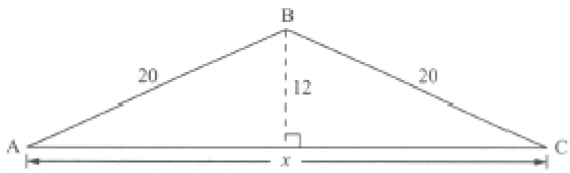
(g)



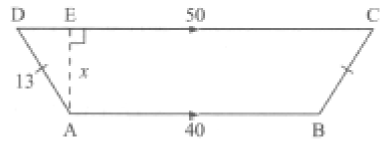
(h)



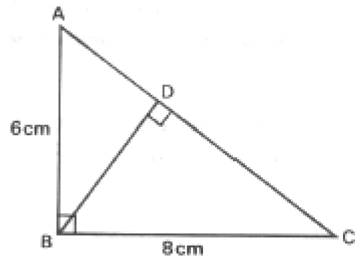
(i)



(j)

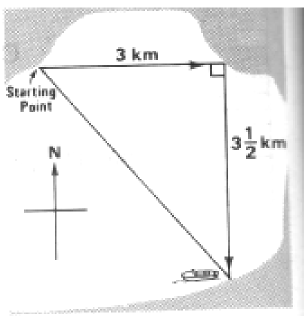


4. In the figure, ABC is a triangle right-angled at B. $BD \perp AC$. AB = 6 cm and BC = 8 cm.
   (a) Find the length of AC.
   (b) Find the area of ΔABC.
   (c) Using the result of (a) and (b), find the length of BD.
   (d) Find the length of AD.

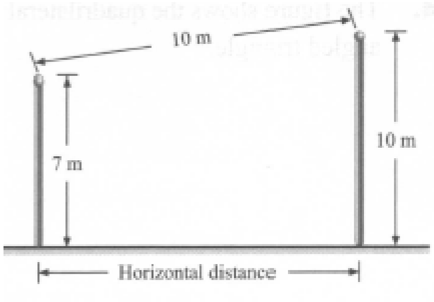


G8.3

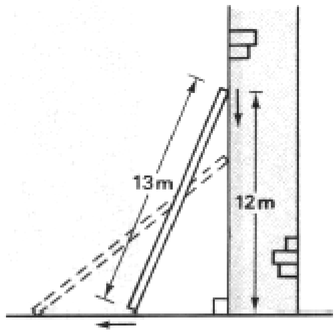
5. A boat sails 3 km due east and then 3.5 km due south. How far is the boat now from its starting point? (Correct the answer to 1 decimal place.)



6. Two flagpoles are 7 m and 10 m tall respectively. If the distance between the tops of the two flagpoles is 10 m, find the horizontal distance between them. (Give the answers correct to 2 decimal places.)



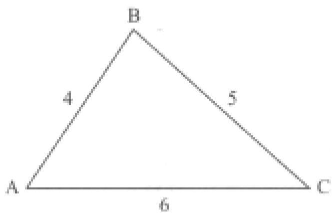
7. A ladder 13 m long is placed against a vertical wall and reaches a height of 12 m. If the top of the ladder slides 4 m down the wall, how far will the foot of the ladder slide? (Correct the answer to 2 decimal places.)



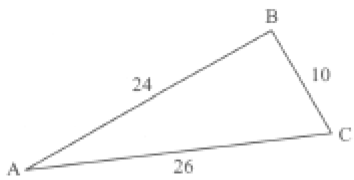
G8.4

8. Determine whether the following triangles are right-angled.

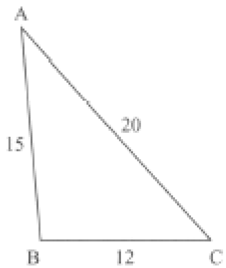
(a)



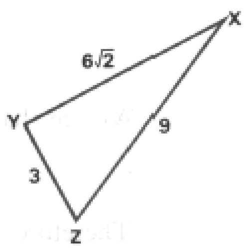
(b)



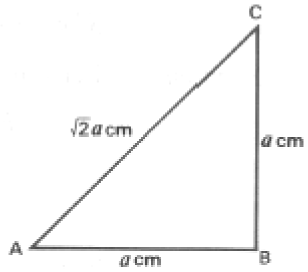
(c)



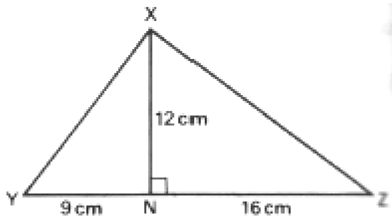
(d)



9. In the figure, ABC is a triangle with AB = BC = a cm an CA = $\sqrt{2}a$ cm. Show that $\angle$B is a right angle.



10. In the figure, XYZ is a triangle. XN is the perpendicular from X to YZ. If YN = 9 cm, NZ = 16 cm and XN = 12 cm, show that $\angle YXZ = 90^{\circ}$.



Supplementary

11. In the figure, ΔABC is a right-angled triangle. D and E are the mid-points of two shorter sides BC and AB respectively. Prove that $4(AD^2 + CE^2) = 5AC^2$.