A7 Factorization of Simple Polynomials

A7.1 Factorization of Quadratic Polynomials

Consider the following polynomials:

$x^2+8x+16$, $xy+2y^2$, $21x^2+3xy+16y^2$, $2x^2-8x$, $x^2+y^2-3x+4y-6$.

In each of the above polynomials, the degree of the term of the highest degree is two, so these polynomials are called ***quadratic polynomials***.

We have learnt several methods to factorize simple polynomials. The following example will revise the methods of factorization.

Example 1

Factorize the following expressions.

(a) $6ab-3a^2$

(b) $x(a+b)-y(a+b)$

(c) $a^2-ac+ab-bc$

(d) x^2-9

(e) $x^2+8x+16$

(f) $x^2+10x+25$

(g) x^2-4x+4

(h) x^2-6x+9

(i) $49x^2-56xy+16y^2$

(j) $5a(a+c)^2-180a^3$

Solution

(a) $6ab-3a^2=3a(2b-a)$

(b) $x(a+b)-y(a+b)=(a+b)(x-y)$

(c) $a^2-ac+ab-bc=a(a-c)+b(a-c)$
$=(a-c)(a+b)$

(d) $x^2-9=(x+3)(x-3)$

(e) $x^2+8x+16=x^2+2(x)(4)+4^2$
$=(x+4)^2$

(f) $x^2+10x+25=x^2+2(x)(5)+5^2$
$=(x+5)^2$

(g) $x^2-4x+4=x^2-2(x)(2)+2^2$
$=(x-2)^2$

(h) $x^2-6x+9=x^2-2(x)(3)+3^2$
$=(x-3)^2$

(i) $49x^2 - 56xy + 16y^2 = (7x)^2 - 2(7x)(4y) + (4y)^2$

$= (7x - 4y)^2$

(j) $5a(a+c)^2 - 180a^3 = 5a[(a+c)^2 - 36a^2]$

$= 5a[(a+c)^2 - (6a)^2]$

$= 5a[(a+c+6a)(a+c-6a)]$

$= 5a(c+7a)(c-5a)$

Checkpoint 1

Factorize:

(a) $8pr - 10ps + 12qr - 15qs$

(b) $25a^2b^2 - 100c^2$

(c) $48 - 3k^4$

(d) $75m^2 - 30mn + 3n^2$

(e) $4x^2 - 36z^2 - 28xy + 49y^2$

(f) $(3a+2b)^2 - 18(3a+2b) + 81$

(g) $8(c+2d)^2 - 50(3c-d)^2$

(h) $(x-2y)^2 - 9a^2 - 54ay - 81y^2$

We are now going to have more discussion on the factorization of quadratic polynomials.

A. Factorization of Quadratic Polynomials of the Form $x^2 + qx + r$

Consider the product $(x+a)(x+b)$.

By expansion, we have

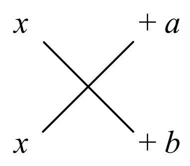
$$
\begin{aligned}
(x+a)(x+b) &= x^2 + ax + bx + ab \\
&= x^2 + (a+b)x + ab
\end{aligned}
$$

Comparing this result with $x^2 + qx + r$, we have

$$
\begin{aligned}
q &= a + b \\
r &= ab
\end{aligned}
$$

Thus the quadratic polynomial $x^2 + qx + r$ can be factorized if r can be expressed as a product of two numbers that their sum is equal to q.

We are going to use the *cross method* to factorize quadratic polynomials easily.

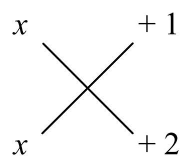


$$qx = \quad +ax \quad +bx \quad = (a+b)x$$

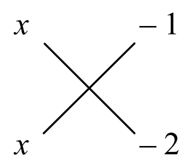
Example 2

Factorize $x^2 + 3x + 2$.

Solution



$$x \quad +2x \quad = 3x$$



$$-x \quad -2x \quad = -3x$$

$\therefore x^2 + 3x + 2 = (x+1)(x+2)$

Example 3

Factorize $x^2 + 5x - 6$.

Solution

x		$+6$
x		-1
$6x$	$-x$	$= 5x$

x		$+1$
x		-6
x	$-6x$	$= -5x$

x		$+2$
x		-3
$2x$	$-3x$	$= -x$

x		$+3$
x		-2
$3x$	$-2x$	$= x$

$\therefore x^2 + 5x - 6 = (x+6)(x-1)$

Checkpoint 2

Factorize $x^2 + 7x + 10$.

x		$+1$
x		____
x	____	= ____

x		-1
x		____
____	____	= ____

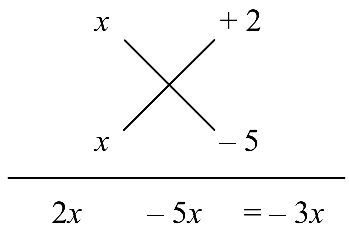
x		$+2$
x		____
____	____	= ____

x		-2
x		____
____	____	= ____

$\therefore x^2 + 7x + 10 =$ ____________________

Example 4

Factorize $x^2 - 3x - 10$.

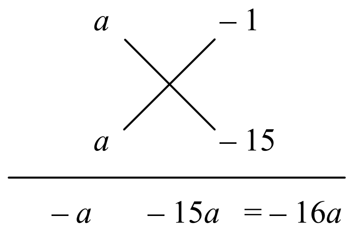


Solution

$x^2 - 3x - 10 = (x+2)(x-5)$

Example 5

Factorize $a^2 - 16a + 15$.



Solution

$a^2 - 16a + 15 = (a-1)(a-15)$

Checkpoint 3

Factorize the following expressions using the cross method.

(a) $x^2 + x - 2$

(b) $p^2 + 4p - 12$

(c) $c^2 + 14c + 33$

(d) $q^2 + 3q - 70$

B. Factorization of Quadratic Polynomials of the Form $px^2 + qx + r$, $p \neq 1$

Example 6

Factorize $4x^2 + 8x + 3$.

Solution

$$\begin{array}{ccc} x & & +1 \\ & \times & \\ 4x & & +3 \\ \hline 4x & +3x & =7x \end{array} \qquad \begin{array}{ccc} x & & -1 \\ & \times & \\ 4x & & -3 \\ \hline -4x & -3x & =-7x \end{array}$$

$$\begin{array}{ccc} x & & +3 \\ & \times & \\ 4x & & +1 \\ \hline 12x & +x & =13x \end{array} \qquad \begin{array}{ccc} x & & -3 \\ & \times & \\ 4x & & -1 \\ \hline -12x & -x & =-13x \end{array}$$

$$\begin{array}{ccc} 2x & & +1 \\ & \times & \\ 2x & & +3 \\ \hline 2x & +6x & =8x \end{array} \qquad \begin{array}{ccc} 2x & & -1 \\ & \times & \\ 2x & & -3 \\ \hline -2x & -6x & =-8x \end{array}$$

$\therefore 4x^2 + 8x + 3 = (2x+1)(2x+3)$

Do we need to think about this?

$$\begin{array}{ccc} -x & & +1 \\ & \times & \\ -4x & & +3 \\ \hline -4x & -3x & =-7x \end{array}$$

Example 7

Factorize $4x^2 - 7x - 2$.

Solution

$4x^2 - 7x - 2 = (x-2)(4x+1)$

$$\begin{array}{ccc} x & & -2 \\ & \times & \\ 4x & & +1 \\ \hline -8x & +x & =-7x \end{array}$$

Example 8

Factorize $6x^2 + x - 1$.

Solution

$6x^2 + x - 1 = (2x + 1)(3x - 1)$

$2x \quad +1$

$3x \quad -1$

$+3x \quad -2x \quad = +x$

Example 9

Factorize $8x^2 - 42x + 45$.

Solution

$8x^2 - 42x + 45 = (2x - 3)(4x - 15)$

$2x \quad -3$

$4x \quad -15$

$-12x \quad -30x \quad = -42x$

Checkpoint 4

Factorize $2x^2 - x - 10$.

$x \quad +1$

$2x$ ____

$2x$ ____ = ____

$x \quad -1$

$2x$ ____

____ ____ = ____

$x \quad +10$

$2x$ ____

____ ____ = ____

$x \quad -10$

$2x$ ____

____ ____ = ____

$x \quad +2$

$2x$ ____

____ ____ = ____

$x \quad -2$

$2x$ ____

____ ____ = ____

$x \quad +5$

$2x$ ____

____ ____ = ____

$x \quad -5$

$2x$ ____

____ ____ = ____

$\therefore 2x^2 - x - 10 =$ ____________________

Checkpoint 5

Factorize the following expressions using the cross method.

(a) $2y^2 + 9y + 4$

(b) $5m^2 + 3m - 2$

(c) $3x^2 + 22x + 7$

(d) $12k^2 + 7k - 49$

(e) $3x^2 + 12x - 63$

(f) $-8d^2 - 2d + 3$

C. Harder Factorization of Quadratic Polynomials

In this section, we consider factorization of quadratic polynomials in two variables.

Example 10

Factorize $x^2 - xy - 6y^2$.

Solution

$x \quad + y$

$x \quad - 6y$

$xy \quad - 6xy \quad = - 5xy$

$x \quad - y$

$x \quad + 6y$

$- xy \quad + 6xy \quad = + 5xy$

$x \quad + 2y$

$x \quad - 3y$

$2xy \quad - 3xy \quad = - xy$

$x \quad - 2y$

$x \quad + 3y$

$- 2xy \quad + 3xy \quad = + xy$

$\therefore x^2 - xy - 6y^2 = (x + 2y)(x - 3y)$

Example 11

Factorize $8x^2 + 10xy - 3y^2$.

Solution

$8x^2 + 10xy - 3y^2 = (4x - y)(2x + 3y)$

$4x \quad - y$

$2x \quad + 3y$

$- 2xy \quad + 12xy = + 10xy$

Checkpoint 6

Factorize the following expressions using the cross method.

(a) $p^2 - 11pq + 28q^2$

(b) $x^2 - 9xy - 36y^2$

(c) $a^2 - 27ab - 160b^2$

Checkpoint 7

Factorize the following expressions using the cross method.

(a) $2c^2 + cd - 21d^2$

(b) $2x^2 - 15xy + 7y^2$

(c) $-19m^2 + 25mn + 126n^2$

Exercise A7
Factorization of Simple Polynomials

A7.1

1. Factorize each of the following expressions. If not possible, say so.

(a) $x^2 + 3x + 2$ (b) $m^2 + 6m + 8$

(c) $x^2 + 8x + 15$ (d) $x^2 - 7x + 10$

(e) $x^2 - x - 6$ (f) $x^2 - x - 42$

(g) $a^2 - 6a - 27$ (h) $b^2 - 19b + 90$

(i) $t^2 - 5t - 36$ (j) $m^2 + 4m - 96$

(k) $2t^2 - 2t - 40$

2. Factorize

(a) $x^2 - 2x + 1$;

(b) $(x^2 - 2x + 1) - y^2$

3. Factorize $(x^2 + 2x)^2 + 2(x^2 + 2x) + 1$.

4. Factorize

(a) $3x^2 - 12x + 12$

(b) $3(x^2 - x)^2 - 12(x^2 - x) + 12$

5. Factorize each of the following expressions. If not possible, say so.

(a) $2x^2 + 3x + 1$ (b) $2x^2 + 7x + 5$

(c) $2x^2 - 3x - 5$ (d) $21x^2 + 11x - 2$

(e) $3x^2 - 16x - 12$ (f) $6t^2 - 11t + 3$

(g) $2a^2 - 3a - 35$ (h) $30x^2 - 23x + 2$

(i) $25t^2 - 10t + 1$ (j) $6m^2 - 2m - 4$

(k) $8k^2 - 22k + 12$ (l) $8r^2 + 7r - 15$

(m) $42x^2 - x - 30$ (n) $4x^3 - 8x^2 - 60x$

6. Factorize each of the following expressions. If not possible, say so.

(a) $a^2 - ab - 6b^2$ (b) $r^2 - 2rs - 48s^2$

(c) $x^2 - 3xy + 2y^2$ (d) $x^2 - 5xy - 24y^2$

(e) $4s^2 + 15rs + 14r^2$ (f) $2a^2 - 11ab + 15b^2$

(g) $21x^2 - 17xy + 2y^2$ (h) $28p^2 - 13pq - 5q^2$