A6 Identities

A6.1 The Difference Between an Equation and an Identity

Consider the equation

$$2x + 4 = x + 6 \ \ldots\ldots \ (A)$$

We can solve the equation and find that $x = 2$ is the solution of the equation.

If we substitute $x = 2$ into (A),

$$\text{L.H.S.} = 2(2) + 4 = 8$$
$$\text{R.H.S.} = (2) + 6 = 8$$
$$\therefore \text{L.H.S.} = \text{R.H.S}$$

If we substitute $x = 0$ into (A),

$$\text{L.H.S.} = 2(0) + 4 = 4$$
$$\text{R.H.S.} = (0) + 6 = 6$$
$$\therefore \text{L.H.S.} \neq \text{R.H.S}$$

How about other values of x?

In fact, equation (A) can only be satisfied when $x = 2$, it is not satisfied when any other value of x is used.

Consider another equation

$$6x - 3 = 3(2x - 1) \ \ldots\ldots \ (B)$$

If we solve the equation, we have

$$6x - 3 = 3(2x - 1)$$
$$6x - 3 + 3 - 6x = 6x - 3 + 3 - 6x$$
$$0 = 0$$

We cannot find the value of x in an usual way.

If we substitute $x = 2$ into (B),

$$\text{L.H.S.} = 6(2) - 3 = 9$$
$$\text{R.H.S.} = 3[2(2) - 1] = 9$$
$$\therefore \text{L.H.S.} = \text{R.H.S}$$

This shows that $x = 2$ is a solution of equation (B).

If we substitute $x = 0$ into (B),

$$\text{L.H.S.} = 6(0) - 3 = -3$$
$$\text{R.H.S.} = 3[2(0) - 1] = -3$$
$$\therefore \text{L.H.S.} = \text{R.H.S}$$

This shows that $x = 0$ is a solution of equation (B).

How about other values of x?

In fact, equation (B) is satisfied by ALL values of x. Such an equation is called an **identity**.

Definition:

An identity in x is an equation in x which is satisfied by *ALL* values of x.

We write equation (B) as

$$6x - 3 \equiv 3(2x - 1)$$

The symbol '≡' means "identical to" or "identically equal to".

To determine whether an equation is an identity, we can simplify both sides of the equation to see whether they are the same.

Example 1

For each of the following equations, determine whether it is an identity.

(a) $3x(2x-4) = 2(3x^2 - 6x)$

(b) $(x+1)(3x-2) + 2 = 3x^2 - x$

Solution

(a)
$$\begin{aligned} \text{LHS} &= 3x(2x-4) \\ &= 6x^2 - 12x \\ &= 2(3x^2 - 6x) \\ &= \text{RHS} \end{aligned}$$

$\therefore \quad 3x(2x-4) \equiv 2(3x^2 - 6x)$

(b)
$$\begin{aligned} \text{LHS} &= (x+1)(3x-2) + 2 \\ &= (3x^2 + x - 2) + 2 \\ &= 3x^2 + x \\ &\neq 3x^2 - x \\ &= RHS \end{aligned}$$

$\therefore \quad (x+1)(3x-2) + 2 = 3x^2 - x$ is not an identity.

Alternative Solution

(a)

$$\begin{aligned} \text{LHS} &= 3x(2x-4) \\ &= 6x^2 - 12x \\ \text{RHS} &= 2(3x^2-6x) \\ &= 6x^2 - 12x \end{aligned}$$

$\therefore$ LHS = RHS

$\therefore$ $3x(2x-4) \equiv 2(3x^2-6x)$

(b)

$$\begin{aligned} \text{LHS} &= (x+1)(3x-2)+2 \\ &= (3x^2+x-2)+2 \\ &= 3x^2+x \end{aligned}$$

$\therefore$ $\text{RHS} = 3x^2 - x$

$\therefore$ LHS $\neq$ RHS

Checkpoint 1

For each of the following equations, determine whether it is an identity.

(a) $3(2x-4) = 2(3x-6)$

(b) $(2x-1)(x+1) = 2x^2 + x - 1$

Checkpoint 2

Determine whether $(x+2)^2 = x^2 + 2x + 2$ is an identity.

When an identity holds, the left hand side of the identity is the same as the right hand side. So if

$$2x - 1 = Ax + B,$$

then $A = 2$ and $B = -1$.

Example 2

In each of the following, find the values of the unknowns.

(a) $Ax + 3 \equiv 3(2x + B)$

(b) $(x-3)(2x-A) \equiv Bx^2 + Cx + 3$

Solution

(a)
$$\begin{aligned} Ax + 3 &\equiv 3(2x+B) \\ Ax + 3 &\equiv 6x + 3B \\ \therefore\quad A &= 6 \\ 3B &= 3 \\ \therefore\quad B &= 1 \end{aligned}$$

(b)
$$\begin{aligned} (x-3)(2x-A) &\equiv Bx^2 + Cx + 3 \\ 2x^2 + (-A-6)x + 3A &\equiv Bx^2 + Cx + 3 \\ \therefore\quad B &= 2 \\ 3A &= 3 \\ \therefore\quad A &= 1 \\ \therefore\quad C &= -A - 6 \\ &= -1 - 6 \\ &= -7 \end{aligned}$$

Checkpoint 3

In each of the following, find the value(s) of the unknown(s).

(a) $6x - 3 \equiv 3(Ax + B)$

(b) $(x - 3)(2x + 3) = Ax^2 + Bx + C$

(c) $Py^2 - Qxy - Rx^2 \equiv (2x + y)(x - 2y)$

(d) $(3x + A)(x + 1)^2 \equiv Bx^4 + Cx^3 + (A + 6)x^2 + 5x + D$

Solution

A6.2 Difference of Two Squares

When we expand the expression $(a+b)(a-b)$, we have

$$\begin{aligned}(a+b)(a-b) &= a^2 - ab + ba - b^2 \\ &= a^2 - b^2\end{aligned}$$

Thus, for all values of *a* and *b*,

$$\boldsymbol{a^2 - b^2 \equiv (a+b)(a-b)}$$

Example 3

Expand $(2x+3)(2x-3)$.

Solution

$(2x+3)(2x-3)$

$=(2x)^2 - 3^2$

$=4x^2 - 9$

Checkpoint 4

Expand

(a) $(3a+bc)(bc-3a)$.

(b) $(x+y)(x-y)(x^2+y^2)$

Example 4

Factorize

(a) $a^2 - 16$

(b) $169a^2 - 64b^2$

(c) $121a^2b^2 - 25l^2m^2$

(d) $9(h+k)^2 - 16k^2$

(e) $8b^2 - 2a^2$

(f) $b^4 - a^4$

Solution

(a)
$$\begin{aligned} &a^2 - 16 \\ &= a^2 - 4^2 \\ &= (a+4)(a-4) \end{aligned}$$

(b)
$$\begin{aligned} &169a^2 - 64b^2 \\ &= (13a)^2 - (8b)^2 \\ &= (13a+8b)(13a-8b) \end{aligned}$$

(c)
$$\begin{aligned} &121a^2b^2 - 25l^2m^2 \\ &= (11ab)^2 - (5lm)^2 \\ &= (11ab+5lm)(11ab-5lm) \end{aligned}$$

(d)
$$\begin{aligned} &9(h+k)^2 - 16k^2 \\ &= [3(h+k)]^2 - (4k)^2 \\ &= [3(h+k)+4k][3(h+k)-4k] \\ &= (3h+3k+4k)(3h+3k-4k) \\ &= (3h+7k)(3h-k) \end{aligned}$$

(e)
$$\begin{aligned} &8b^2 - 2a^2 \\ &= 2(4b^2 - a^2) \\ &= 2[(2b)^2 - a^2] \\ &= 2(2b+a)(2b-a) \end{aligned}$$

(f)
$$\begin{aligned} &b^4 - a^4 \\ &= (b^2)^2 - (a^2)^2 \\ &= (b^2+a^2)(b^2-a^2) \\ &= (b^2+a^2)(b+a)(b-a) \end{aligned}$$

Checkpoint 5

Factorize

(a) $x^2 - 4$

(b) $9x^2 - 4y^2$

(c) $25 - (a+b)^2$

(d) $(a-b)^2 - (c+d)^2$

(e) $(2p^2 + q^2)^2 - (p^2 + 2q^2)^2$

(f) $12x^2 - 75y^2$

A6.3 The Perfect Square Expression

When we expand the expression $(a+b)^2$, we have

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a^2+ab+ba+b^2\\ &= a^2+2ab+b^2\end{aligned}$$

Thus, for all values of a and b,

$$\boldsymbol{(a+b)^2 \equiv a^2+2ab+b^2}$$

Also when we expand the expression $(a-b)^2$, we have

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a^2-ab-ba+b^2\\ &= a^2-2ab+b^2\end{aligned}$$

Thus, for all values of a and b,

$$\boldsymbol{(a-b)^2 \equiv a^2-2ab+b^2}$$

Example 5

Expand

(a) $(3x-5y)^2$

(b) $(a+b+c)^2$

Solution

(a) $(3x-5y)^2$

$= (3x)^2-2(3x)(5y)+(5y)^2$

$= 9x^2-30xy+25y^2$

(b) $(a+b+c)^2$

$= [(a+b)+c]^2$

$= (a+b)^2+2(a+b)c+c^2$

$= (a^2+2ab+b^2)+2ac+2bc+c^2$

$= a^2+b^2+c^2+2ab+2bc+2ac$

Checkpoint 6

Expand

(a) $(2x+3y)^2$

(b) $(a-b+c)^2$

Example 6

Factorize

(a) $y^2-8y+16$

(b) $4+12b+9b^2$

(c) $8m^2-8m+2$

Solution

(a) $y^2-8y+16$

$=y^2-2(y)(4)+4^2$

$=(y-4)^2$

(b) $4+12b+9b^2$

$=2^2+2(2)(3b)+(3b)^2$

$=(2+3b)^2$

(c) $8m^2-8m+2$

$=2(4m^2-4m+1)$

$=2[(2m)^2-2(2m)(1)+1^2]$

$=2(2m+1)^2$

Example 7

Factorize

(a) $2a^3 - 28a^2b + 98ab^2$

(b) $(a-b)^2 - 2(a-b)b + b^2$

(c) $x^4 + 4x^2y + 4y^2 - (x^2 - y)^2$

Solution

(a) $2a^3 - 28a^2b + 98ab^2$

$= 2a(a^2 - 14ab + 49b^2)$

$= 2a[a^2 - 2(a)(7b) + (7b)^2]$

$= 2a(a - 7b)^2$

(b) $(a-b)^2 - 2(a-b)b + b^2$

$= [(a-b) + b]^2$

$= a^2$

(c) $x^4 + 4x^2y + 4y^2 - (x^2 - y)^2$

$= (x^2)^2 + 2(x^2)(2y) + (2y)^2 - (x^2 - y)^2$

$= (x^2 + 2y)^2 - (x^2 - y)^2$

$= [(x^2 + 2y) + (x^2 - y)][(x^2 + 2y) - (x^2 - y)]$

$= 3y(2x^2 + y)$

Checkpoint 7

Factorize

(a) $a^2 - 4ab + 4b^2$

(b) $49x^2 + 14xy + y^2$

Checkpoint 8

Factorize

(a) $12k^2 - 12km + 3m^2$

(b) $(a-b)^2 + 6(b-a) + 9$

(c) $x^4 - 2x^2y^2 + y^4$

(d) $(3m+1)^2 - 2(3m+1)(3m-2) + (2-3m)^2$

A6.4 Sum and Difference of Two Cubes

Below are two important algebraic identities:

$$\boldsymbol{(a^3 + b^3) \equiv (a + b)(a^2 - ab + b^2)}$$

and

$$\boldsymbol{(a^3 - b^3) \equiv (a - b)(a^2 + ab - b^2)}$$

These two identities are very useful and should be committed in memory.

Example 8

Factorize

(a) $x^3 + 8a^3$

(b) $2x^6 - 128y^6$

Solution

(a) $x^3 + 8a^3$

$= x^3 + (2a)^3$

$= (x + 2a)[x^2 - x(2a) + (2a)^2]$

$= (x + 2a)(x^2 - 2ax + 4a^2)$

(b) $2x^6 - 128y^6$

$= 2(x^6 - 64y^6)$

$= 2[(x^2)^3 - (4y^2)^3]$

$= 2(x^2 - 4y^2)[(x^2)^2 + (x^2)(4y^2) + (4y^2)^2]$

$= 2[x^2 - (2y)^2](x^4 + 4x^2y^2 + 16y^4)$

$= 2(x + 2y)(x - 2y)(x^4 + 4x^2y^2 + 16y^4)$

Alternatively,

$2x^6 - 128y^6$

$= 2(x^6 - 64y^6)$

$= 2[(x^3)^2 - (8y^3)^2]$

$= 2(x^3 - 8y^3)(x^3 + 8y^3)$

$= 2[x^3 - (2y)^3][x^3 + (2y)^3]$

$= 2(x - 2y)[x^2 + 2xy + (2y)^2](x + 2y)[x^2 - 2xy + (2y)^2]$

$= 2(x - 2y)(x + 2y)(x^2 + 2xy + 4y^2)(x^2 - 2xy + 4y^2)$

Checkpoint 9

Factorize

(a) $64+s^3$

(b) $ab-a^4b$

(c) $\frac{1}{2}-4x^3$

(d) a^3+b^3-a-b

Exercise A6
Identities

A6.1

1. For each of the following, state whether it is an identity.

(a) $6\left(x-\frac{1}{2}\right)+3=6x$ (b) $2\left(3x-\frac{1}{4}\right)=4x-\frac{1}{2}$

(c) $4y-2(1-y)=2(3y-1)$ (d) $(s+3)(s-2)=s^2-s-6$

(e) $(u-4)^2=u^2-16$ (f) $a(b-c)+b(c-a)+c(a-b)=0$

2. Prove the following identities:

(a) $x^2+(x+1)^2 \equiv 2x(x+1)+1$ (b) $\frac{3r-5}{4}-\frac{7r+9}{16}+\frac{8r+19}{8} \equiv \frac{21r+9}{16}$

A6.2

3. Expand the following products:

(a) $(a+5)(a-5)$ (b) $(7x+y)(7x-y)$

(c) $(8+3x)(8-3x)$ (d) $(x^2+5)(x^2-5)$

(e) $\left(\frac{x}{2}-2\right)\left(\frac{x}{2}+2\right)$ (f) $(-3c-1)(1-3c)$

4. Factorize the following expressions:

(a) y^2-1 (b) $25y^2-16$

(c) $121-4m^2$ (d) $25m^2-36n^2$

(e) x^2y^2-36 (f) $81a^2-b^2c^2$

(g) $100x^2y^2-a^2b^2$ (h) $(p+q)^2-144$

(i) $(1+x)^2-9$ (j) $4(m+n)^2-n^2$

(k) $1-4(x-y)^2$ (l) $(x+1)^2-(x-1)^2$

(m) $(3a-2)^2-(2a-3)^2$ (n) $(2a-b)^2-16(3c-d)^2$

(g) $3x^2-75$ (h) $3m^2n^2-27k^2$

(i) a^2b-bc^2 (j) x^2-y^2-x-y

(k) $a^2-b^2+bc-ac$ (l) $2s^2-9t^2-3st$

(m) $9m^2-16n^2+4kn-3km$ (n) $2xy+6x^2-72x^3y-x$

(o) $-81+16x^4$ (p) $\pi R^2-\pi r^2+ar+aR$

A6.3

5. Expand the following expressions:

(a) $(x+4)^2$ (b) $(2p+3)^2$

(c) $(5y-1)^2$ (d) $(3x+y)^2$

(e) $(6p-5q)^2$ (f) $(3x+7y)^2$

(g) $(x-2ab)^2$ (h) $(4xy-3a)^2$

(i) $4(2a-3b)^2$ (j) $(a^2-p)^2$

(k) $(1-2x^2)^2$ (l) $\left(x+\frac{1}{4}\right)^2$

(m) $\left(2y-\frac{1}{2}\right)^2$ (n) $\left(\frac{a}{3}+b\right)^2$

6. Factorize the following expressions:

(a) $g^2+10g+25$ (b) $36-12a+a^2$

(c) $4x^2-12x+9$ (d) $16+24n+9n^2$

(e) $27y^2+18y+3$ (f) $9y^2-24yz+16z^2$

(g) $25m^2-40mn+16n^2$ (h) $80r^2+40rs+5s^2$

(i) $9p^2-6pqr+q^2r^2$ (j) $(x+2y)^2+8(x+2y)+16$

(k) $9(a+3)^2+6(a+3)+1$ (l) $36(c-2d)^2+60(c-2d)+25$

(m) $16(2x-3y)^2+24(3y-2x)+3^2$ (n) $(a^2+8a+16)-b^2$

(o) $e^2-(h^2+2h+1)$ (p) (a^5-2a^3+a)

A6.4

7. Factorize the following expressions.

(a) a^3-27 (b) x^3y^3-1

(c) $5y^3-135z^3$ (d) $3+\frac{3x^3}{8}$

(e) $\frac{1}{2}-4x^3$ (f) $a^3-(a-4)^3$

(g) $25m^2-40mn+16n^2$ (h) $(a+b)^3+(a-b)^3$

(i) x^6+y^6 (j) $(x-1)^6-(1-x)^3$