A3 Laws of Integral Indices

A3.1 Positive Integral Indices

In a^n, a is called the base, n is called the index (also called the exponent). a^n means $\underbrace{a \times a \times a \times \ldots\ldots \times a}_{n \text{ times}}$.

Example 1

Simplify

(a) $a^3 \times a^5$ (b) $a^6 \div a^4$ (c) $(a^2)^3$

Solution

(a) $a^3 \times a^5 = (a \times a \times a) \times (a \times a \times a \times a \times a)$
$= a \times a \times a \times a \times a \times a \times a \times a$
$= a^8$

(b) $a^6 \div a^4 = \dfrac{a \times a \times a \times a \times a \times a}{a \times a \times a \times a}$
$= a \times a$
$= a^2$

(c) $(a^2)^3 = (a^2) \times (a^2) \times (a^2)$
$= (a \times a) \times (a \times a) \times (a \times a)$
$= a^6$

Example 2

Simplify

(a) $(ab)^3$ (b) $\left(\dfrac{a}{b}\right)^3$

Solution

(a) $(ab)^3 = (a \times b) \times (a \times b) \times (a \times b)$
$= (a \times a \times a) \times (b \times b \times b)$
$= a^3 b^3$

(b) $\left(\dfrac{a}{b}\right)^3 = \left(\dfrac{a}{b}\right) \times \left(\dfrac{a}{b}\right) \times \left(\dfrac{a}{b}\right)$
$= \dfrac{a \times a \times a}{b \times b \times b}$
$= \dfrac{a^3}{b^3}$

The results in **Example 1** and **Example 2** illustrate the following laws of indices:

If *m* and *n* are positive integers, then

(i) $a^m \times a^n = a^{m+n}$

(ii) $a^m \div a^n = a^{m-n}$, $a \neq 0$ **and** $m > n$

(iii) $(a^m)^n = a^{mn}$

(iv) $(ab)^n = a^n b^n$

(v) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, $b \neq 0$

Example 3

Simplify

(a) $\frac{a^2 \times a^3}{a^4 \times a}$

(b) $\left(\frac{2a}{b}\right)^4 \times \left(\frac{b^2}{a}\right)^3$

(c) $(ab^2)^3 \div (4a^2b^3)^2$

(d) $(-ab^3)^3 \times \left(\frac{a}{3b^2}\right)^4 \times \left(-\frac{9}{a^2}\right)^2$

Solution

(a) $\frac{a^2 \times a^4}{a^3 \times a} = \frac{a^{2+4}}{a^{3+1}} = \frac{a^6}{a^4} = a^{6-4} = a^2$

(b) $\left(\frac{2a}{b}\right)^4 \times \left(\frac{b^2}{a}\right)^3 = \frac{(2a)^4}{b^4} \times \frac{(b^2)^3}{a^3} = \frac{2^4 a^4}{b^4} \times \frac{b^{2\times 3}}{a^3} = 16a^{4-3}b^{6-4} = 16ab^2$

(c)
$$(ab^2)^3 \div (4a^2b^3)^2 = a^3(b^2)^3 \times \frac{1}{4^2(a^2)^2(b^3)^2}$$
$$= a^3b^6 \times \frac{1}{16a^4b^6}$$
$$= \frac{1}{16a}$$

(d)
$$(-ab^3)^3 \times \left(\frac{a}{3b^2}\right)^4 \times \left(-\frac{9}{a^2}\right)^2 = (-1)^3 a^3 (b^3)^3 \times \frac{a^4}{3^4(b^2)^4} \times (-1)^2 \times \frac{9^2}{(a^2)^2}$$
$$= -a^3b^9 \times \frac{a^4}{81b^8} \times \frac{81}{a^4}$$
$$= -a^3b$$

Checkpoint 1

Simplify

(a) $\dfrac{p^5q^3r^2}{p^3q^3r}$

(b) $(b^3)^4 \times b^4$

(c) $(4p^2)^2 \div (2p^3)^3 \times (2p^3)^4$

(d) $-p^2q^3 \times \left(\dfrac{p}{2q^2}\right)^3 \times \left(-\dfrac{4q}{p^2}\right)^2$

A3.2 Zero and Negative Integral Indices

We are now going to extend the idea of exponents to zero and negative integral indices in such a way that the laws of indices hold for all integral indices.
Consider the following patterns:

10^3	=	1000
10^2	=	100
10^1	=	10
10^0	=	1
10^{-1}	=	?
10^{-2}	=	?
10^{-3}	=	?

The indices of 10 are decreasing each step by 1.

These numbers are decreasing each step to $\frac{1}{10}$ of the previous number.

2^3	=	8
2^2	=	4
2^1	=	2
2^0	=	1
2^{-1}	=	?
2^{-2}	=	?
2^{-3}	=	?

The indices of 2 are decreasing each step by 1.

These numbers are decreasing each step to $\frac{1}{2}$ of the previous number.

From the patterns, we observe that

$$10^0 = 1 \qquad 2^0 = 1$$

$$10^{-1} = \frac{1}{10} \qquad 2^{-1} = \frac{1}{2}$$

$$10^{-2} = \frac{1}{10^2} \qquad 2^{-2} = \frac{1}{2^2}$$

$$10^{-3} = \frac{1}{10^3} \qquad 2^{-3} = \frac{1}{2^3}$$

By the law $\boldsymbol{a^m \times a^n = a^{m+n}}$, we have if

(1) $a^n \cdot a^0 = a^{n+0} = a^n$

$\therefore a^0 = \dfrac{a^n}{a^n} = 1 \qquad$ if $a \neq 0$

(2) $a^n \cdot a^{-n} = a^{n+(-n)} = a^0 = 1$

$\therefore a^{-n} = \dfrac{1}{a^n} \qquad$ if $a \neq 0$

Thus, if $a \neq 0$ and n is an integer, we define

(1) $a^0 = 1$

(2) $a^{-n} = \dfrac{1}{a^n}$

Now we have the following laws of indices:

If m and n are ~~positive~~ integers, then
(i) $a^m \times a^n = a^{m+n}$
(ii) $a^m \div a^n = a^{m-n}$, $a \neq 0$ and ~~$m > n$~~
(iii) $(a^m)^n = a^{mn}$
(iv) $(ab)^n = a^n b^n$
(v) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, $b \neq 0$

Example 4

Find the values of

(a) $3^0 \cdot 5^{-3}$

(b) $5^2 \div 5^4$

Solution

(a) $3^0 \cdot 5^{-3} = 1 \cdot \frac{1}{5^3} = \frac{1}{125}$

(b) $5^2 \div 5^4 = 5^{2-4} = 5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

Example 5

Simplify

(a) $c^0 \cdot m^{-3}$

(b) $(a^{-m})^n$

(c) $(3^2 a^{-4})(3a^2)$

(d) $\frac{(a^{-1}b)^2}{ab}$

(*Give the answers with positive indices.*)

Solution

(a) $c^0 \cdot m^{-3} = 1 \cdot \frac{1}{m^3} = \frac{1}{m^3}$

(b) $(a^{-m})^n = a^{(-m)n} = a^{-mn} = \frac{1}{a^{mn}}$

(c) $(3^2 a^{-4})(3a^2) = 3^{2+1} a^{-4+2} = 3^3 a^{-2} = \frac{27}{a^2}$

(d) $\frac{(a^{-1}b)^2}{ab} = \frac{(a^{-1})^2 b^2}{ab} = \frac{a^{-2}b^2}{ab}$

$= a^{-2-1}b^{2-1} = a^{-3}b = \frac{b}{a^3}$

Checkpoint 2

Find the values of

(a) $\left(\dfrac{9999}{8874214}\right)^{0}$

(b) $(-4)^{3}$

(c) $\left(\dfrac{1}{2}\right)^{-3} \div 4^{-2}$

(d) $2^{-2} \times \left(\dfrac{5}{2}\right)^{-3}$

Checkpoint 3

Simplify the following:

(a) $a^{-2} \times ab^{-3}$

(b) $-(-k)^{-2} \div k^{-1}$

(c) $\left(\dfrac{p}{2s}\right)^{-3}$

(*Give the answers with positive indices.*)

Checkpoint 4

Simplify the following:

(a) $(-2k)^{-3} \times \left(\dfrac{3}{k}\right)^{-2}$

(b) $a^{-2}b + (b^{-1}a^{2})^{-1}$

(c) $(a^{-2} + b^{-2})(a^{-2} - b^{-2})$

(*Give the answers with positive indices.*)

A3.3 Notations for Various Numeral Systems

A. Numeral Systems around Us

In everyday life, we prefer using measuring units in the *metric system* (十進制) to simplify calculation and conversion. The metric system is a decimal (or denary) system, the conversion factors are either 10 or powers of 10.
For example, units for measuring lengths are metre (m), centimetre (cm), millimetre (mm), etc., where 1 m = 100 cm and 1 cm = 10 mm.

As well as metric or decimal system, other numeral systems are used in everyday life. The following table shows some everyday examples of non-metric systems.

Non-metric system	Example	Remark
Hexadecimal (Base 16)	Catties(斤) and taels (兩) or pounds (磅) and ounces (安士) are used in measuring weights. 	1 catty = 16 taels 1 pound = 16 ounces
Duodecimal (Base 12)	Feet (尺) and inches (寸) are used in measuring heights. 	1 foot = 12 inches
Binary (Base 2)	The numeral system is used in internal operations of computers.	Computers store information in the form of combi-nations of '0' and '1'.

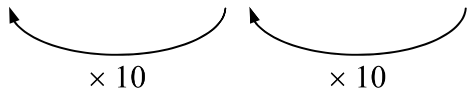
B. Place Value

Place Value of Decimal Numbers

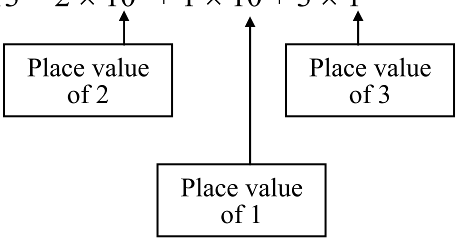
In the decimal system of numeration, each digit (數字) can take any one of the following Arabic numerals: 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9.
For example, we can write 'two hundred and thirteen' as '213'.
In a decimal number, the position that each numeral takes has a certain value, which is called the ***place value*** (位值).
For example, in the number '213', the place value of each digit from the right to left is as follows:
The place value of the first digit, 3, is 'one' (1);
the place value of the second digit, 1, is 'ten' (10);
the place value of the third digit, 2, is 'hundred' (100).

Moreover, place values increase in powers of 10 from right to left. The place value of each digit is 10 times that of the number to its right.

	Hundreds	**Tens**	**Ones**
Digit	2	1	3
Place value	10^2	10	1



We can use a numerical expression which shows the place value of each digit to represent any number.
e.g. $213 = 2 \times 10^2 + 1 \times 10 + 3 \times 1$



We call the numerical expression that shows the place value of each digit the ***expanded form*** (展開式). By using the expanded form, we can show clearly the relation between the place values of different digits.

For the decimal number 210, its expanded form is $2 \times 10^2 + 1 \times 10 + 0 \times 1$; '0' is called the ***place holder*** (補位數字). Although '0' means 'nothing in quantity', it plays a very important role here. It is because we can determine the place values of other digits only after '0' has taken its position.

Example 6

Consider the number 3004.

(a) What are the place values of 3 and 4?

(b) Write 3004 in the expanded form.

Solution

(a) Place value of 3 = 10^3

Place value of 4 = 1

(b) $3004 = 3 \times 10^3 + 0 \times 10^2 + 0 \times 10 + 4 \times 1$

Checkpoint 5

Write the following decimal numbers in the expanded form.

(a) 23_{10} (b) 167_{10} (c) 485_{10}

Checkpoint 6

Represent each of the following expressions as a decimal number.

(a) $2 \times 10^2 + 5 \times 10 + 6 \times 1$

(b) $9 \times 10^3 + 0 \times 10^2 + 7 \times 10 + 8 \times 1$

(c) $7 \times 1000 + 8 \times 100000 + 9 \times 10 + 6$

Place value of Binary Numbers and Hexadecimal Numbers

Other than the decimal system, the binary and the hexadecimal systems are two common numeral systems.

The binary system has two numerals: 0 and 1.

The hexadecimal system has sixteen numerals: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E and F. The table shows the decimal numerals corresponding to the numerals A to F in the hexadecimal system.

Hexadecimal	**Decimal**
A	10
B	11
C	12
D	13
E	14
F	15

In order to distinguish numbers in different systems, we indicate the base of the number at its bottom right corner.

e.g. 26_{10} is a decimal number.
11010_2 is a binary number.
$1A_{16}$ is a hexadecimal number.

Like the decimal numbers, place values of binary numbers and hexadecimal numbers increase from right to left.

For example:

(1) Place values of binary number 11010_2:

Binary	1	1	0	1	0
Place value	2^4	2^3	2^2	2	1

The expanded form of $11010_2 = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2 + 0 \times 1$

(2) Place values of binary number $1A_{16}$:

Hexadecimal	1	A
Place value	16	1

The expanded form of $1A_{16} = 1 \times 16 + 10 \times 1$

Note: All binary numbers and hexadecimal numbers expressed in the expanded form (i.e. the right-hand side of the equal sign) are written as decimal numbers.

Checkpoint 7

Write down the place value of the digit '0' in each of the following numbers.

(a) 7086_{10}

(b) 10111_2

(c) $5E06_{16}$

Checkpoint 8

Write the following binary numbers in the expanded form.

(a) 11_2 (b) 101_2 (c) 1111_2

Checkpoint 9

Write the following hexadecimal numbers in the expanded form.

(a) 35_{16} (b) $27A_{16}$ (c) $F69_{16}$

Example 7

(a) Represent the expression $2^3 + 1$ as a binary number.

(b) Represent the expression $8 \times 16^2 + 15 \times 16 + 0 \times 1$ as a hexadecimal number.

Solution

(a) $$\begin{aligned} 2^3 + 1 &= 1 \times 2^3 + 0 \times 2^2 + 0 \times 2 + 1 \times 1 \\ &= 1001_2 \end{aligned}$$

(b) $$8 \times 16^2 + 15 \times 16 + 0 \times 1 = 8F0_{16}$$

Checkpoint 10

Represent each of the following expressions as a binary number.

(a) $1 \times 2^2 + 0 \times 2 + 1 \times 1$

(b) $1 \times 2^3 + 1 \times 2^2 + 1 \times 2 + 0 \times 1$

(c) $1 \times 16 + 0 \times 8 + 1 \times 4 + 0 \times 2 + 0$

Checkpoint 11

Represent each of the following expressions as a hexadecimal number.

(a) $6 \times 16^2 + 7 \times 16 + 9 \times 1$

(b) $15 \times 16^2 + 2 \times 16 + 13 \times 1$

(c) $5 \times 16^2 + 10 \times 16 + 4 + 13 \times 16^3$

A3.4 Conversions Between Numbers of Different Numeral Systems

A. Converting Binary Numbers or Hexadecimal Numbers into Decimal Numbers

We can convert the original binary or hexadecimal number into a decimal number by evaluating the expanded form of that number.

Example 8

Convert 11101_2 into a decimal number.

Solution

$$\begin{aligned} 11101_2 &= 1\times 2^4 + 1\times 2^3 + 1\times 2^2 + 0\times 2 + 1\times 1 \\ &= 16 + 8 + 4 + 0 + 1 \\ &= 29_{10} \end{aligned}$$

Example 9

Convert $21D_{16}$ into a decimal number.

Solution

$$\begin{aligned} 21D_{16} &= 2\times 16^2 + 1\times 16 + 13\times 1 \\ &= 512 + 16 + 13 \\ &= 541_{10} \end{aligned}$$

Checkpoint 12

Convert each of the following binary numbers into a decimal number.

(a) 1101_2

(b) 101101_2

Checkpoint 13

Convert each of the following hexadecimal numbers into a decimal number.

(a) $3A_2$

(b) 508_2

B. Converting Binary Numbers or Hexadecimal Numbers into Decimal Numbers

We can form a binary number if we know its expanded form. For example,

$$1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2 + 0 \times 1 = 11010_2$$

When we convert a decimal number, such as 13, into a binary number, how do we find the expanded form of binary number from the decimal number? The method is shown as follows:

(1) Divide the decimal number continuously by 2 until the quotient is 1, i.e. less than 2.

(2) Read the final quotient (i.e. 1) first, then all the remainders from bottom to top successively.

For example, to convert 13_{10} into its expanded form of binary number,

(1)

```
2 | 13
2 | 6    ...... 1   <- Remainder
2 | 3    ...... 0
    1    ...... 1    (Quotient less than 2)
```

(2)

```
2 | 13
2 | 6    ...... 1   ^
2 | 3    ...... 0   |
    1    ...... 1   |
   -----------------
  (Read the binary number)
```

Thus, the expanded form of 13_{10} of binary number is $1 \times 2^3 + 1 \times 2^2 + 0 \times 2 + 1 \times 1$, and $13_{10} = 1101_2$.

Example 10

Convert 23_{10} into a binary number.

Solution

$$
\begin{array}{r|ll}
2 & 23 & \\ \hline
2 & 11 & \ldots\ldots\ 1 \\ \hline
2 & 5 & \ldots\ldots\ 1 \\ \hline
2 & 2 & \ldots\ldots\ 1 \\ \hline
 & 1 & \ldots\ldots\ 0
\end{array}
$$

$\therefore \quad 23_{10} = 10111_2$

Similarly, we can also use the continual division to convert a decimal number into a hexadecimal number. The method is to divide the decimal number continually by 16 until the quotient is less than 16.

Example 11

Convert 2001_{10} into a hexadecimal number.

Solution

$$
\begin{array}{r|ll}
16 & 2001 & \\ \hline
16 & 125 & \ldots\ldots\ 1 \\ \hline
 & 7 & \ldots\ldots\ 13
\end{array}
$$

$\therefore \quad 2001_{10} = 7D1_{16}$

Checkpoint 14

In each of the following,

(i) convert the decimal number into a binary number;

(ii) convert the decimal number into a hexadecimal number.

(a) 27_{10} (b) 273_{10} (c) 161_{10}

Exercise A3
Laws of Integral Indices

A3.1

1. Simplify the following:

(a) $(k^6 \div k^3) \div k^2$ (b) $p^8 \div (p^4 \div p^2)$

(c) $\dfrac{c^5 \times c^6}{c^2 \times c^3}$ (d) $\dfrac{a^4 \times b^5 \times a^6}{b^3 \times a^2 \times b}$

(e) $(-c^2)^3$ (f) $(-b^3)^4 \times b^2$

(g) $(2q^2r^3)^4$ (h) $(-2x^3y^2z)^2$

(i) $\dfrac{(-a^2b)^3}{(-ab)^2}$ (j) $\dfrac{(h^2)^2 \times 9h^5}{(3h^2)^2}$

(k) $\dfrac{(ab^3)^4 \times (-c)^{13}}{(-ac^6)^2}$ (l) $(-3p)^3 \times (2p^6)^2 \div (6p^2)^3$

A3.2

2. Find the values of the following and give the answers in fractions.

(a) $3^2 \times 3^{-2}$ (b) $8^{-2} \div 2^{-8}$

(c) $(-9)^0 \times \left(\dfrac{1}{3}\right)^{-2}$ (d) $2^{-2} \times (4^3 \div 8^0)$

(e) $2^{-2} \times (3^{-3} \div 6^{-5})$ (f) $(5^3 \times 15^{-2}) \div 3^{-2}$ *(5)*

(g) $\left(\dfrac{-2}{3}\right) + \left(\dfrac{3}{-2}\right)^{-1} + \dfrac{2}{3}$ (h) $\dfrac{1}{(-2)^{-2}} - 2$

3. In each of the following, simplify the expression and express the answer with positive indices. All letters given represent non-zero numbers.

(a) $(a^n)^{-m}$ (b) $(b^m)^m$

(c) $a^{-4} \times a^3$ (d) $a^{-1} \times a^0 \times a \times a^2$

(e) $-(-c)^{-1} \div c^{-2}$ (f) $(-2a^0)^{-1}$

(g) $(-b^{-2})^{-3}$ (h) $\left(\dfrac{-b^2}{c}\right)^{-3}$

(i) $(4^{-2}b^{-5})(2^{-3}b^2)^{-2}$ (j) $\dfrac{(a^2b^{-1})^3}{a^2b^{-3}}$

(k) $(-a^2b^{-1})^{-1} \div (-b^2a^{-1})^{-1}$ (l) $\dfrac{(rs^{-2})^{-2}}{(r^2s^{-3})^{-1}}$ *(s)*

A3.3

4. Represent each of the following expressions as a decimal number.
   (a) $2 \times 10^2 + 6 \times 10 + 3 \times 1$
   (b) $7 \times 10^3 + 0 \times 10^2 + 2 \times 10$
   (c) $3 \times 10^4 + 1 \times 10^3 + 1 \times 10^2 + 9 \times 10 + 5 \times 1$
   (d) $2 \times 10^3 + 1 \times 10 + 4 \times 1 + 5 \times 10^5$

5. Represent each of the following expressions as a binary number.
   (a) $1 \times 2^2 + 0 \times 2 + 0 \times 1$
   (b) $1 \times 2^3 + 0 \times 2^2 + 0 \times 2 + 1 \times 1$
   (c) $1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2 + 0 \times 1$
   (d) $1 \times 2^5 + 1 \times 2^4 + 1 \times 1$

6. Represent each of the following expressions as a hexadecimal number.
   (a) $2 \times 16 + 3 \times 1$
   (b) $5 \times 16^2 + 1 \times 16 + 0 \times 1$
   (c) $7 \times 16^2 + 14 \times 16 + 10 \times 1$
   (d) $1 \times 16^2 + 9 \times 16 + 4 \times 1 + 12 \times 16^3$

7. Write the following numbers of different systems in the expanded form.
   (a) 576_{10} (b) 980_{10}
   (c) 1234_{10} (d) 1110_2
   (e) 1100_2 (f) 10001_2
   (g) 54_{16} (h) 101_{16}
   (i) $AD7_{16}$ (j) BEA_{16}

A3.4

8. Convert the following binary numbers into decimal numbers.
   (a) 101010_2 (b) 111111_2
   (c) 1000101_2 (d) 1100111_2

9. Convert the following hexadecimal numbers into decimal numbers.
   (a) AB_{16} (b) $10E_{16}$
   (c) 369_{16} (d) $DC8_{16}$

10. Convert the following decimal numbers into binary numbers.

(a) 17_{10} (b) 89_{10}

(c) 146_{10} (d) 235_{10}

11. Convert the following decimal numbers into hexadecimal numbers.

(a) 97_{10} (b) 184_{10}

(c) 506_{10} (d) 4321_{10}

12. Mr Yung bought a safe to keep some precious property at home. The password of the safe was a 5-digit number in the decimal system. He told his daughter, Debby, the password and asked her to find a way to remember it. Debby thought for a while and used her name as a hint to the password. She considered the first four alphabets of hers name as a 4-digit hexadecimal number. The password of the safe would be the number obtained by converting this hexadecimal number into a decimal number. Find the password.