S.5 Additional Mathematics
1^{st} Term Pre-exam

Time allowed: 2.5 hours

Total Mark: 110

- *Answer ALL questions in Sections A and B.*
- *All workings must be clearly shown.*
- *Unless otherwise specified in a question, numerical answers must be* ***exact****.*
- *The diagrams in the paper are not necessarily drawn to scale.*

FORMULAS FOR REFERENCE

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$2\sin A \cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

SECTION A (62 marks)

1. Find the coefficient of x^5 in the expansion of $(1-2x+3x^3)^7$.

(5 marks)

2. Prove, by mathematical induction, that

$$\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+...+\frac{1}{n(n+1)(n+2)}=\frac{1}{2}\left[\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right]$$

for all positive integers *n*.

(5 marks)

3. Given that $y=\dfrac{x+2}{(x+1)^2}$ …… (*).

(a) Express (*) in the form $ax^2+bx+c=0$.

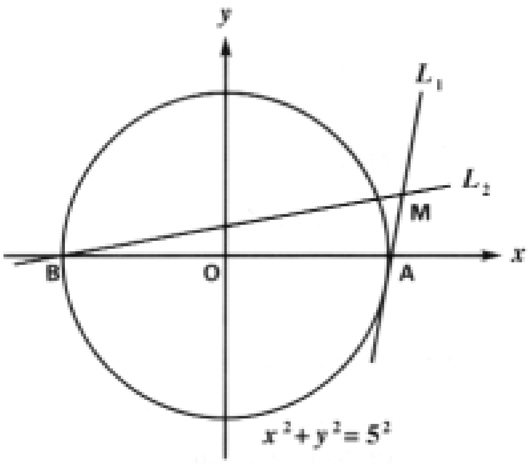
(b) Hence find the range of values of y for real values of x.

(5 marks)

4. Find $\dfrac{d}{dx}(\sqrt{1-x})$, $0\le x\le 1$, from first principles.

(5 marks)

5. In the figure, the circle $x^2+y^2=5^2$ cuts the x-axis at A and B. The line L_1 with slope m passes through A. The line L_2 with slope $\dfrac{1}{m}$ passes through B. M is the point of intersection of L_1 and L_2. Find the equation of the locus of M as m varies.



(5 marks)

6. The equations $x^2-a|x|+b=0$ and $x^2+bx-a=0$, $b>a>0$, have a common root α. Find the value(s) of α.

(6 marks)

7. (a) Prove that $\cos^2 x+\cos^2 2x=1+\cos x\cos 3x$.

(b) Hence, or otherwise, find the general solution of the equation

$$\cos^2 x+\cos^2 x+\cos^2 3x=1.$$

(6 marks)

8. Given the curve $C: y=\tan(x+y^2)-1$.

(a) Find $\frac{dy}{dx}$.

(b) Find the equation of the tangent to C at the point $\left(\frac{\pi}{4}, 0\right)$.

(6 marks)

9. A straight line $L_1: y=mx+c$, where m and c are constants, makes an angle of $45°$ with the line $L_2: x+7y-3=0$.

(a) Find the two values of m.

(b) If the distance from the point (2, 0) to L_1 is 4 and $m > 0$, find the two values of c.

(6 marks)

10. Given two circles $C_1: x^2+y^2=r^2$ $(r>0)$ and $C_2: x^2+y^2-8x+6y+21=0$. If C_1 and C_2 touch externally at P, find the coordinates of P, the value of r and the equation of the common tangent at P.

(6 marks)

11. Let $x=a\cos\theta$ and $y=a\sin\theta$ where a is a constant and $0\le\theta<2\pi$.

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x and y.

(b) Hence find the value of $\left(\frac{dy}{dx}\right)^2+y\left(\frac{d^2y}{dx^2}\right)$.

(7 marks)

SECTION B (48 marks)

12. Given that C_1 is the curve $y=\frac{\sin 2x}{2-\cos 2x}$, where $0\le x\le\pi$.

(a) Find the x- and y-intercepts of C_1.

(2 marks)

(b) Find the turning point(s) of C_1.

(7 marks)

(c) (i) Sketch the curve C_1.

(ii) Sketch, on another graph, the curve $C_2: y=\frac{|\sin 2x|}{2-\cos 2x}$.

(3 marks)

13. Two circles $C_1 : x^2 + y^2 + 6x - 4y + 5 = 0$ and $C_2 : x^2 + y^2 + 8x - 3y + 4 = 0$ intersect at A and B. C_3 is another circle passing through A and B such that the area of C_3 is a minimum.

(a) Find the common chord of the circles C_1 and C_2.

(2 marks)

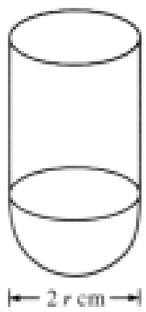
(b) Hence find the equation of C_3.

(5 marks)

(c) Two tangents are drawn from the origin to C_3 with slope m_1 and m_2.
Show that m_1 and m_2 are the roots of $m^2 - 3m - 3 = 0$.

(5 marks)

14. A metal sheet is used to make a vessel in the shape of a cylinder with a hemisphere in the bottom (see the figure). The capacity of the vessel is $144\pi\ \text{cm}^3$. Let r cm be the radius of the hemisphere and the circular base of the cylinder Let $A\,\text{cm}^2$ be the area of the metal sheet used.



(a) Show that $A = \left(\dfrac{288}{r} + \dfrac{2r^2}{3}\right)\pi$.

(4 marks)

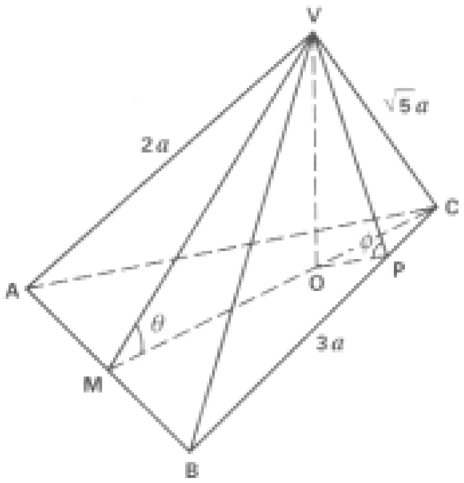
(b) For the convenience of transportation, the radius of the hemisphere cannot be greater than 3 cm. Find the dimensions of the vessel if minimum amount of metal is used.

(5 marks)

(c) Water is flowing at the rate of $\pi\ \text{cm}^3\text{s}^{-1}$ into the vessel with the dimensions found in (b) and the depth of water is greater than 3 cm.
How fast is the water level rising?

(3 marks)

15. The figure shows a tetrahedron such that $VA = VB = AB = 2a$, $VC = \sqrt{5}a$ and $CA = CB = 3a$. O is the foot of the perpendicular from V to the base ABC. M is the mid-point of AB. P is a point on BC such that $BP = ra$ where $0 \le r \le 3$. $\angle VMC = \theta$ and $\angle VPO = \phi$.



(a) Express CM^2 and VM^2 in terms of a.
Hence find the value of $\cos\theta$ in surd form.

(5 marks)

(b) Using the results of (a), find $\sin\theta$ and OV. (*Give your answers in surd form.*)

(3 marks)

(c) Express VP^2 in terms of a and r.

Hence show that $\sin\phi = \sqrt{\dfrac{45}{8(3r^2 - 8r + 12)}}$.

(4 marks)

END OF PAPER