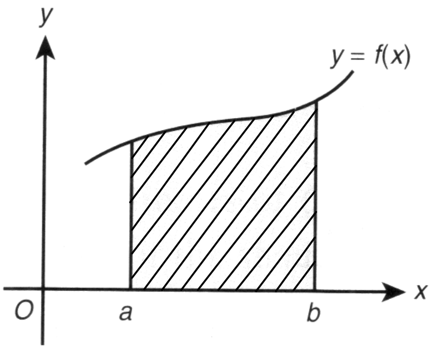
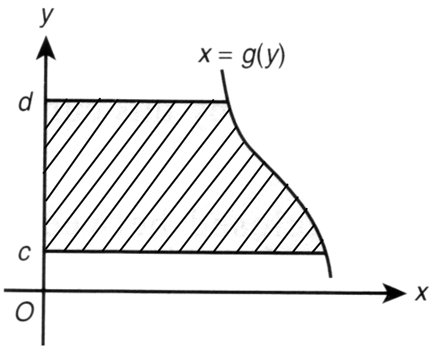
Chapter 21 Applications of Definite Integrals

21.1 Areas of Region Enclosed in a Rectangular Coordinate Plane

21.1.1 Areas Between the Curve $y = f(x)$ and the Coordinate Axes



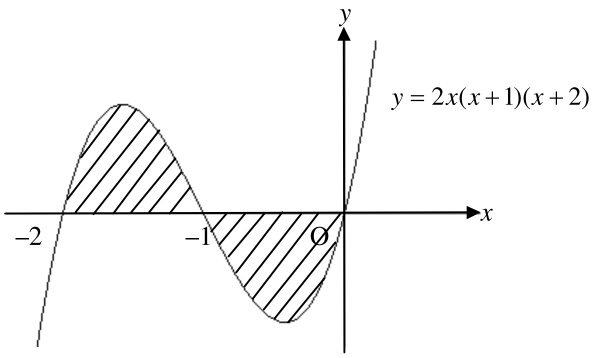
Area $= \int_a^b y\,dx$



Area $= \int_c^d x\,dy$

Example 21.1

The figure shows the curve $y = 2x(x+1)(x+2)$. Find the area enclosed by the curve and the x-axis.



Solution

The required area

$$= \left|\int_{-2}^{-1} 2x(x+1)(x+2)\,dx\right| + \left|\int_{-1}^{0} 2x(x+1)(x+2)\,dx\right|$$

$$= \left|\int_{-2}^{-1} (2x^3 + 6x^2 + 4x)\,dx\right| + \left|\int_{-1}^{0} (2x^3 + 6x^2 + 4x)\,dx\right|$$

$$= \left|\left[\frac{1}{2}x^4 + 2x^3 + 2x^2\right]_{-2}^{-1}\right| + \left|\left[\frac{1}{2}x^4 + 2x^3 + 2x^2\right]_{-1}^{0}\right|$$

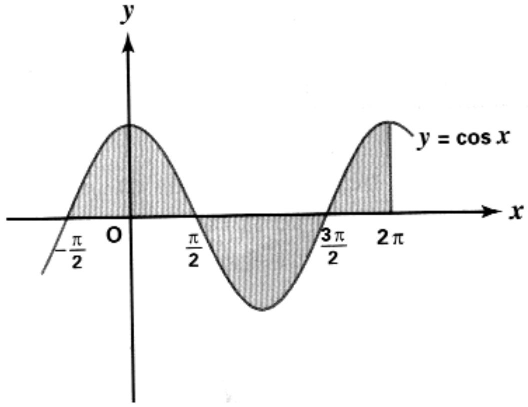
$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1$$

> $\left|\int_{-2}^{-1} 2x(x+1)(x+2)\,dx\right|$ is the area of the region above the x-axis. and $\left|\int_{-1}^{0} 2x(x+1)(x+2)\,dx\right|$ is the area of the region below the x-axis.

Example 21.2

The figure shows the curve $y = \cos x$. Find the area enclosed by the curve and the x-axis for $-\frac{\pi}{2} \le x \le 2\pi$.



Solution

The required area

$$= \left| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx \right| + \left| \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x \, dx \right| + \left| \int_{\frac{3\pi}{2}}^{2\pi} \cos x \, dx \right|$$

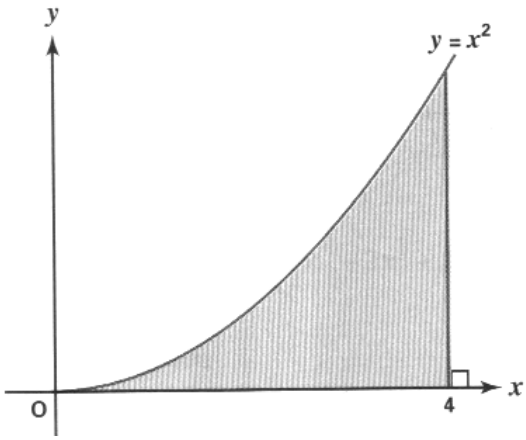
$$= \left| [\sin x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \right| + \left| [\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \right| + \left| [\sin x]_{\frac{3\pi}{2}}^{2\pi} \right|$$

$$= 2 + 2 + 1$$

$$= 5$$

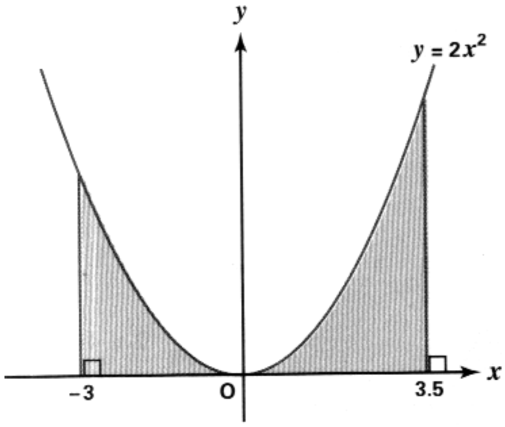
Checkpoint 21.1

Find the area enclosed by the curve $y = x^2$ and the x-axis for $0 \le x \le 4$.



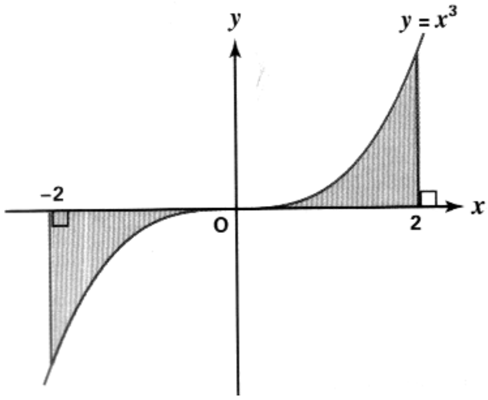
Checkpoint 21.2

Find the area enclosed by the curve $y = 2x^2$ and the x-axis for $-3 \le x \le 3.5$.



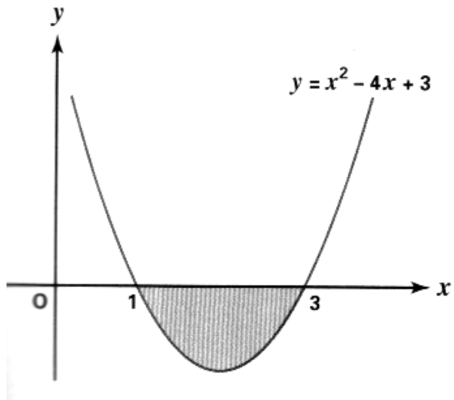
Checkpoint 21.3

Find the area enclosed by the curve $y = x^3$ and the x-axis for $-2 \le x \le 2$.



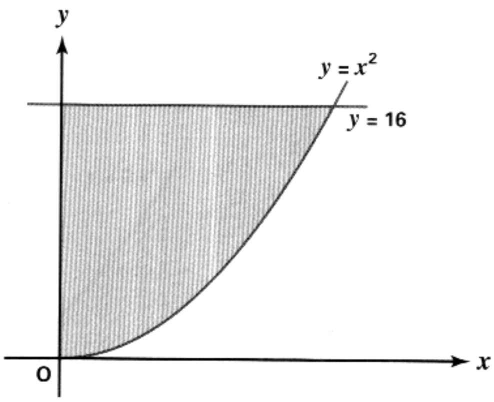
Checkpoint 21.4

Find the area enclosed by the curve $y = x^2 - 4x + 3$ and the x-axis for $1 \le x \le 3$.



Example 21.3

Find the area bounded by the curve $y = x^2$, where $x \ge 0$, the y-axis and the line $y = 16$



Solution

$y = x^2$

$x = \sqrt{y}$ for $x \ge 0$

$\therefore$ The required area $= \int_0^{16} \sqrt{y}\, dy$

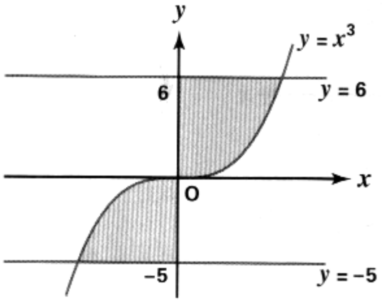
$$= \left[\frac{2}{3} y^{\frac{3}{2}} \right]_0^{16}$$

$$= \frac{2}{3}(16)^{\frac{3}{2}}$$

$$= \frac{128}{3}$$

Example 21.4

Find the area of the region enclosed by the curve $y = x^3$, the y-axis and the lines $y = -5$ and $y = 6$.
(*Give the answer correct to the nearest integer.*)



Solution

$$y = x^3$$
$$x = y^{\frac{1}{3}}$$

$\therefore$ The required area $= \left|\int_0^6 y^{\frac{1}{3}}\, dy\right| + \left|\int_{-5}^0 y^{\frac{1}{3}}\, dy\right|$

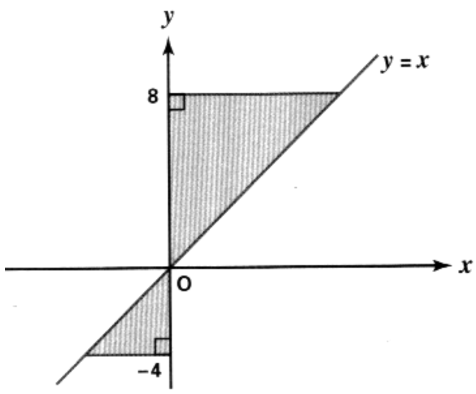
$$= \left[\frac{3}{4}y^{\frac{4}{3}}\right]_0^6 + \left|\left[\frac{3}{4}y^{\frac{4}{3}}\right]_{-5}^0\right|$$

$$= \frac{3}{4}(6^{\frac{4}{3}}) + \left|\frac{3}{4}(-5)^{\frac{4}{3}}\right|$$

$$= 15 \quad \text{(corr. to the nearest integer)}$$

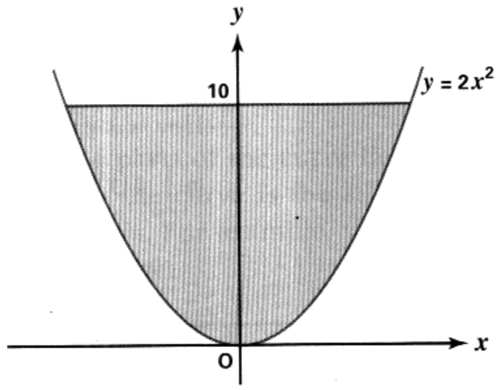
Checkpoint 21.5

Find the area of the shaded region.



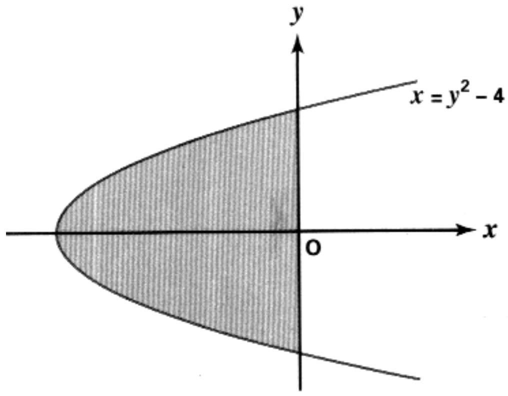
Checkpoint 21.6

Find the area of the region bounded by the curve $y = 2x^2$ and the line $y = 10$.



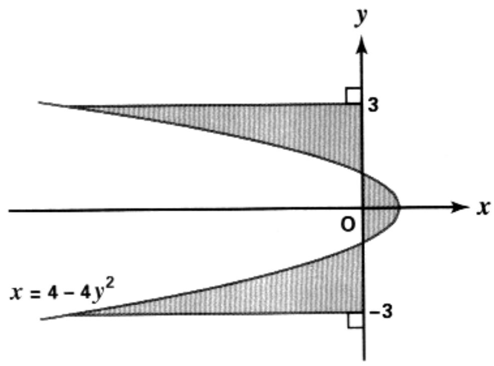
Checkpoint 21.7

Find the area of the region enclosed by the curve $x = y^2 - 4$ and the y-axis.

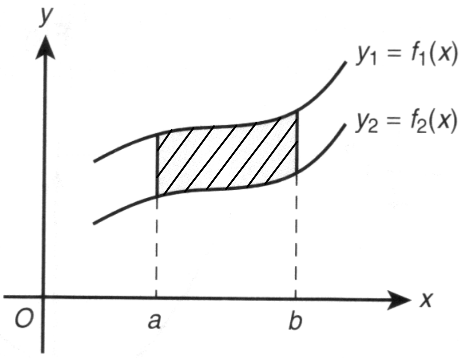


Checkpoint 21.8

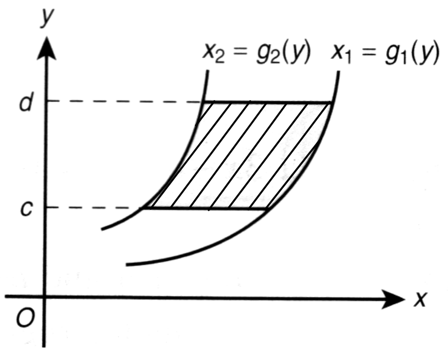
Find the area of the shaded region.



21.1.2 Areas Between Two Curves



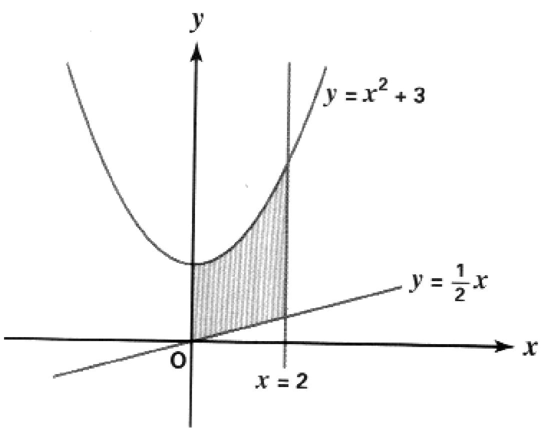
$$\text{Area} = \int_a^b (y_1 - y_2)\, dx$$



$$\text{Area} = \int_c^d (x_1 - x_2)\, dy$$

Example 21.5

Find the area of the region enclosed by the curve $y = x^2 + 3$, the lines $y = \frac{1}{2}x$, $x = 2$ and the y-axis.



Solution

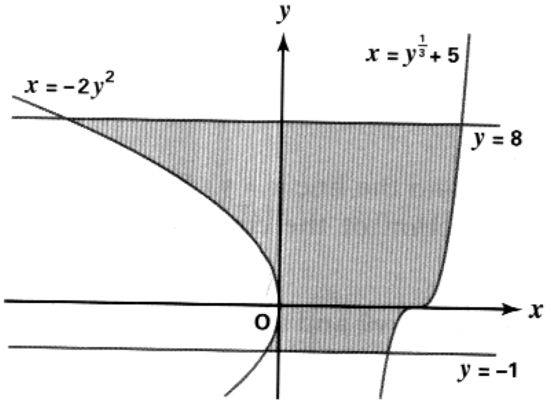
$$\text{The required area} = \int_0^2 \left[(x^2 + 3) - \frac{1}{2}x \right] dx$$

$$= \left[\frac{1}{3}x^3 + 3x - \frac{1}{4}x^2 \right]_0^2$$

$$= 7\frac{2}{3}$$

Example 21.6

Find the area of the region bounded by the curves $x = y^{\frac{1}{3}} + 5$, $x = -2y^2$, the lines $y = -1$ and $y = 8$.



Solution

$$\text{The required area} = \int_{-1}^{8} \left[(y^{\frac{1}{3}} + 5) - (-2y^2) \right] dy$$

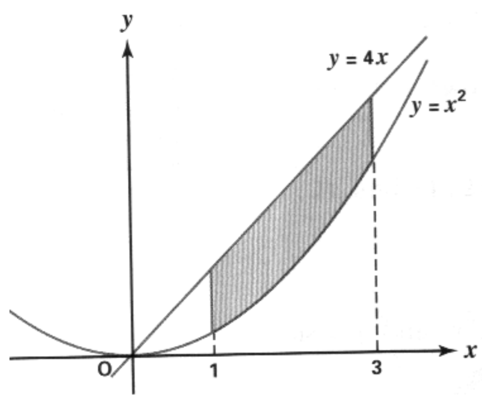
$$= \int_{-1}^{8} \left[y^{\frac{1}{3}} + 5 + 2y^2 \right] dy$$

$$= \left[\frac{3}{4} y^{\frac{4}{3}} + 5y + \frac{2}{3} y^3 \right]_{-1}^{8}$$

$$= 398\frac{1}{4}$$

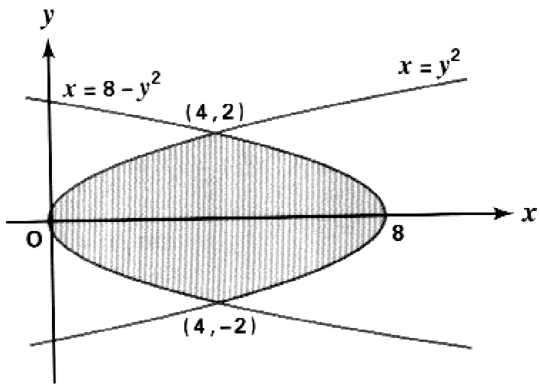
Checkpoint 21.9

Find the area of the region bounded by the curves $y = 4x$ and $y = x^2$ for $1 \le x \le 3$.



Checkpoint 21.10

Find the area of the shaded region.

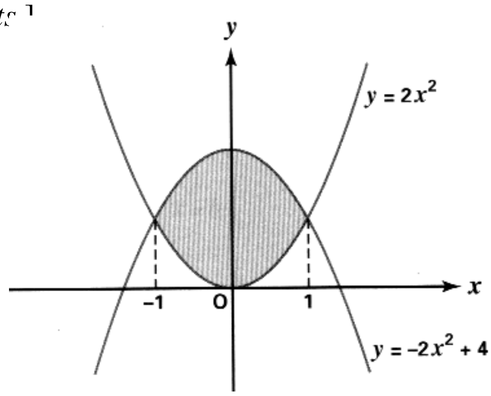


Example 21.7

Find the area o bounded by the curves $y = 2x^2$ and $y = -2x^2 + 4$.

Solution

[*Sketch the curve and solve for the intersection points*]



$$\begin{cases} y = 2x^2 & \text{......(1)} \\ y = -2x^2 + 4 & \text{......(2)} \end{cases}$$

$(1) - (2):\quad 0 = 4x^2 - 4$

$$x^2 = 1$$

$$x = \pm 1$$

$\therefore$ The points of intersection are (–1, 2) and (1, 2).

The required area $= \int_{-1}^{1} \left[(-2x^2 + 4) - (-2x^2)\right] dx$

$$= \int_{-1}^{1} \left[4 - 4x^2\right] dy$$

$$= \left[4x - \frac{4}{3}x^3\right]_{-1}^{1}$$

$$= 5\frac{1}{3}$$

Checkpoint 21.11

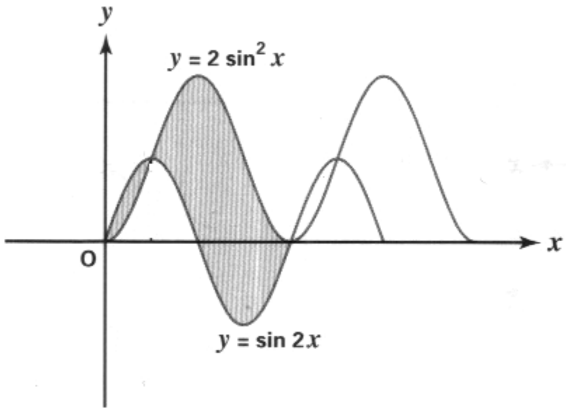
Find the area of the region bounded by the curves $y = x^2$ and $x = y^2$.

Checkpoint 21.12

Find the area of the region enclosed by the curves $y = 2x^2$ and $y = 1$ and $y = x$.

Example 21.8

Find the area o bounded by the curves $y = 2\sin^2 x$ and $y = \sin 2x$ for $0 \le x \le \pi$.



Solution

$$2\sin^2 x = \sin 2x$$

$$2\sin^2 x = 2\sin x\cos x$$

$$\sin^2 x - \sin x\cos x = 0$$

$$\sin x(\sin x - \cos x) = 0$$

$$\sin x = 0 \quad \text{or} \quad \tan x = 1$$

$$x = 0, \quad \pi, \quad \frac{\pi}{4}$$

$\therefore$ The intersection points are (0, 0), $\left(\frac{\pi}{4}, 1\right)$ and (π, 0).

$\therefore$ The required area $= \int_0^{\frac{\pi}{4}} (\sin 2x - 2\sin^2 x)\, dx + \int_{\frac{\pi}{4}}^{\pi} (2\sin^2 x - \sin 2x)\, dx$

$$= \int_0^{\frac{\pi}{4}} [\sin 2x - (1 - \cos 2x)]\, dx + \int_{\frac{\pi}{4}}^{\pi} [(1 - \cos 2x) - \sin 2x]\, dx$$

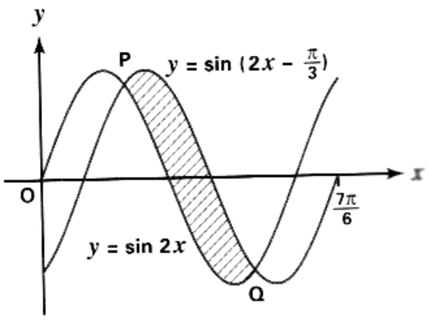
$$= \int_0^{\frac{\pi}{4}} (\sin 2x - 1 + \cos 2x)\, dx + \int_{\frac{\pi}{4}}^{\pi} (1 - \cos 2x - \sin 2x)\, dx$$

$$= \left[-\frac{1}{2}\cos 2x - x + \frac{1}{2}\sin 2x \right]_0^{\frac{\pi}{4}} + \left[x - \frac{1}{2}\sin 2x + \frac{1}{2}\cos 2x \right]_{\frac{\pi}{4}}^{\pi}$$

$$= \frac{\pi + 4}{2}$$

Checkpoint 21.13

The figure shows the curves $y = \sin 2x$ and $y = \sin\left(2x - \frac{\pi}{3}\right)$ over the interval $0 \le x \le \frac{7\pi}{6}$.

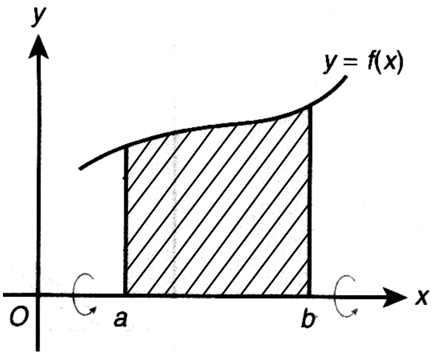


(a) Find the coordinates of P and Q.

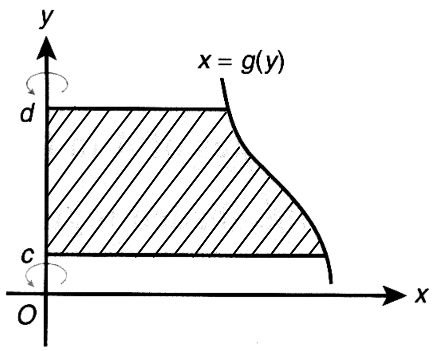
(b) Find the shaded area.

21.2 Volumes of Solids of Revolution

21.2.1 Volumes of Solids of Revolution Revolved about the Coordinate Axes



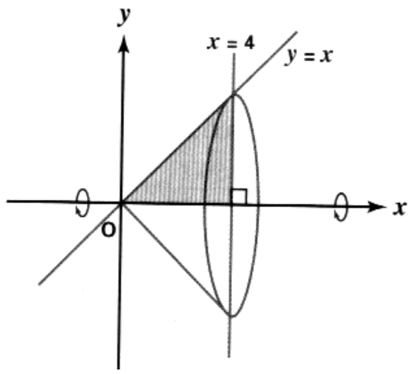
$$\text{Volume} = \int_a^b \pi\, y^2\, dx$$



$$\text{Volume} = \int_a^b \pi\, x^2\, dy$$

Example 21.9

The region bounded by the curve $y = x$, the x-axis and the line $x = 4$ is revolved about the x-axis to form a solid of revolution. Find the volume of the solid.

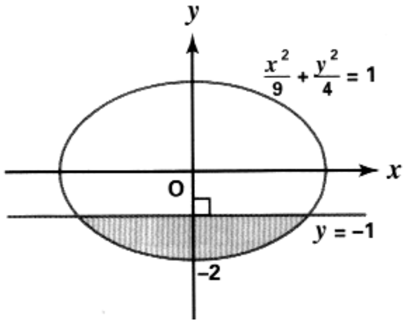


Solution

$$\begin{aligned} \text{The required volume} &= \pi \int_0^4 x^2\, dx \\ &= \pi \left[\frac{1}{3} x^3 \right]_0^4 \\ &= \frac{64\pi}{3} \end{aligned}$$

Example 21.10

In the figure, the shaded region is enclosed by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the line $y=-1$. Find the volume of the solid of revolution formed by revolving this region about the y-axis.



Solution

$$\because \quad \frac{x^2}{9}+\frac{y^2}{4}=1$$

$$x^2 = 9\left(1-\frac{y^2}{4}\right)$$

$$\therefore \quad x = \pm\frac{3}{2}\sqrt{4-y^2}$$

$$\text{The required volume} = \pi\int_{-2}^{-1}\left(\frac{3}{2}\sqrt{4-y^2}\right)^2 dy$$

$$= \frac{9\pi}{4}\int_{-2}^{-1}(4-y^2)\,dy$$

$$= \frac{9\pi}{4}\left[4y-\frac{1}{3}y^3\right]_{-2}^{-1}$$

$$= \frac{9\pi}{4}\left(\frac{5}{3}\right)$$

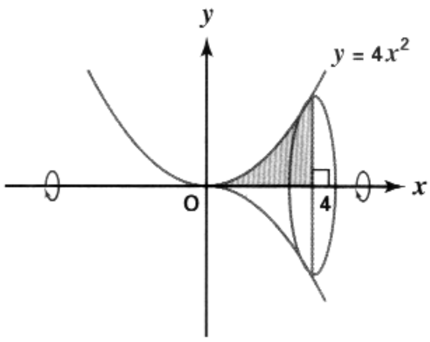
$$= \frac{15\pi}{4}$$

Checkpoint 21.14

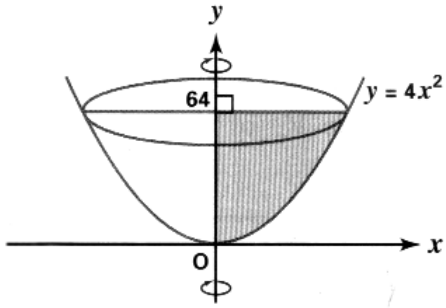
Find the volume of the solid of revolution formed by revolving

(a) the region bounded by the curve $y = 4x^2$, the x-axis and the line $x = 4$ about the x-axis;

(b) the region bounded by the curve $y = 4x^2$, the y-axis and the line $y = 64$ about the y-axis.

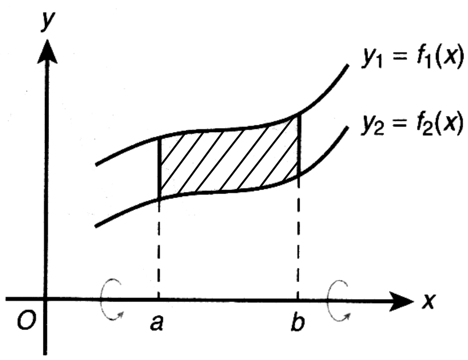


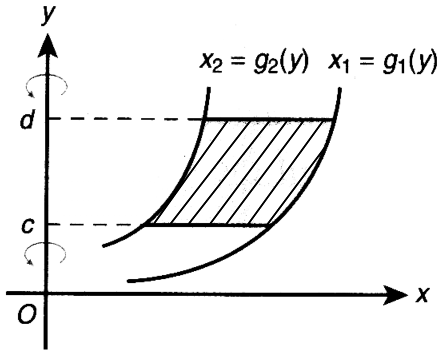
(a)



(b)

21.2.2 Volumes of Hollow Solids of Revolution Revolved about the Coordinate Axes



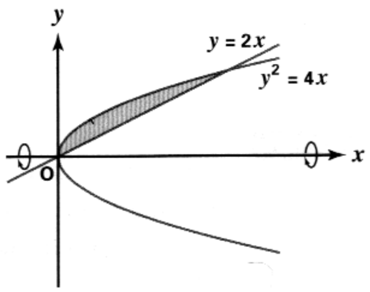


Volume $= \int_a^b \pi\,(y_1^{\,2} - y_2^{\,2})\,dx$

Volume$= \int_a^b \pi\,(x_1 - x_2^{\,2})\,dy$

Example 21.11

(a) Find the point(s) of intersection of the curves $y^2 = 4x$ and $y = 2x$ as shown in the figure.



(b) Hence find the volume of the solid of revolution formed by revolving the region bounded by $y^2 = 4x$ and $y = 2x$ about the x-axis.

Solution

(a) $y^2 = 4x$ ......(1)

$y = 2x$ ......(2)

Substitute (2) into (1),

$$(2x)^2 = 4x$$
$$4x^2 - 4x = 0$$
$$x = 0 \quad \text{or} \quad 1$$

When $x = 0$, $y = 2(0) = 0$.

When $x = 1$, $y = 2(1) = 2$.

$\therefore$ The curves intersect at (0, 0) and (1, 2).

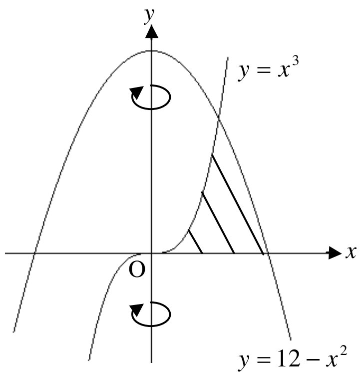
(b) The required volume $= \pi \int_0^1 [4x - (2x)^2]\, dx$

$$= \pi \int_0^1 [4x - 4x^2]\, dx$$

$$= \pi \left[2x^2 - \frac{4}{3}x^3 \right]_0^1$$

$$= \frac{2\pi}{3}$$

Example 21.12



The figure shows the curves $y = x^3$ and $y = 12 - x^2$, which intersect at (2, 8). Find the volume of the solid of revolution formed by revolving the region bounded by the curves and the x-axis about y-axis.

Solution

For $y = x^3$, $x = y^{\frac{1}{3}}$.

For $y = 12 - x^2$, $x = \sqrt{12 - y}$ ($y \le 12$).

The required volume $= \pi \int_0^8 \left[(\sqrt{12 - y})^2 - (y^{\frac{1}{3}})^2 \right] dy$

$$= \pi \int_0^8 [12 - y - y^{\frac{2}{3}}]\, dy$$

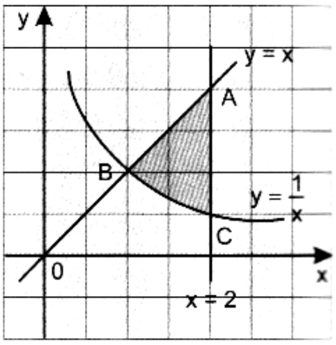
$$= \pi \left[12y - \frac{1}{2}y^2 - \frac{3}{5}y^{\frac{5}{3}} \right]_0^8$$

$$= \frac{224\pi}{5}$$

-

Checkpoint 21.15

The curve $y = \frac{1}{x}$ intersects the lines $y = x$ and $x = 2$ at the points B and C respectively. The lines $y = x$ and $x = 2$ meet at the point A.



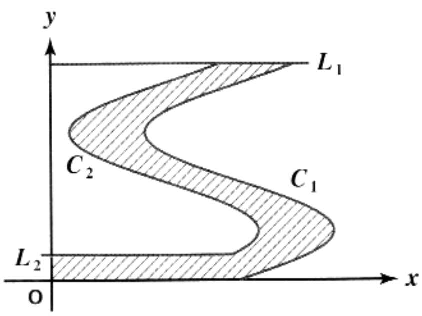
(a) Find the coordinates of the points A, B and C.

(b) The shaded region is bounded by the curve $y = \frac{1}{x}$ and the lines $y = x$ and $x = 2$.

(i) The shaded region is rotated about the y-axis. Find the volume of the solid generated.

(ii) The shaded region is rotated about the x-axis. Find the volume of the solid generated.

Checkpoint 21.16



In the figure, the shaded region is bounded by the curves C_1, C_2, the lines L_1, L_2 and the axes where

$$C_1: \quad x = 2 + \sin y$$
$$C_2: \quad x = 1.8 + \sin y$$
$$L_1: \quad y = \frac{11\pi}{5}$$
$$L_2: \quad y = 0.2$$

A vase is formed by revolving the shaded region about the y-axis.

(a) Find the capacity of the vase. (*Give the answer correct to 3 significant figures.*)

(b) Show that the volume of porcelain required to make such a vase is approximately 19.0 cubic units.

Exercise 20 Definite Integrals

20.4

1. Evaluate the following definite integrals.

(a) $\int_1^4 3x(x^2+2)\,dx$ (b) $\int_1^4 x\sqrt{1+x}\,dx$

(c) $\int_0^5 \frac{x}{\sqrt{4+x}}\,dx$ (d) $\int_{-2}^{-1} (x+2)^3(x+3)^2\,dx$

2. Evaluate the following definite integrals.

(a) $\int_0^2 \frac{x^2+3x+2}{x+1}\,dx$ (b) $\int_0^1 \frac{(x+2)^2}{(x+1)^4}\,dx$

(c) $\int_2^4 \frac{x^2-2x+6}{x^2-2x+1}\,dx$ (d) $\int_{-\frac{1}{3}}^0 \frac{9x^3-30x^2+28x-10}{(3x-2)^2}\,dx$

3. Evaluate the following definite integrals.

(a) $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$ (b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin x\cos 2x\,dx$

(c) $\int_0^{\frac{\pi}{4}} \sin 2x\cos^3 2x\,dx$ (d) $\int_0^{\pi} (\cos 3x\sin x - \cos x\sin 3x)^2\,dx$

4. (a) Let $y=\sin^7 x$. Find $\frac{dy}{dx}$.

(b) Hence evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x\cos x\,dx$.

5. Evaluate $\int_0^{2\pi} |\sin x|\,dx$.

6. (a) Let $y = \tan^{m-1} x \sec^n x$ where m and n are positive integers and $m \geq 2$.

Show that $\dfrac{dy}{dx} = (m-1)\tan^{m-2} x \sec^n x + (m+n-1)\tan^m x \sec^n x$.

(b) Using (a), show that

$$\int \tan^m x \sec^n x \, dx = \frac{\tan^{m-1} x \sec^n x}{m+n-1} - \frac{m-1}{m+n-1} \int \tan^{m-2} x \sec^n x \, dx$$

for $m \geq 2$ and $n \geq 1$.

(c) Let $I_{m,n} = \int_0^{\frac{\pi}{4}} \tan^m x \sec^n x \, dx$ where m and n are non-negative integers.

Using (b), show that

$$I_{m,n} = \frac{2^{\frac{n}{2}}}{m+n-1} - \frac{m-1}{m+n-1} I_{m-2,n} \text{ for } m \geq 2 \text{ and } n \geq 1.$$

(d) (i) Evaluate $I_{0,2}$.

(ii) Hence evaluate $\int_0^{\frac{\pi}{4}} \tan^4 x \sec^2 x \, dx$

Exercise 20 Answers

(*Definite Integrals*)

1. (a) $\frac{945}{4}$ (b) $\frac{116}{15}$ (c) $\frac{14}{3}$ (d) $\frac{49}{60}$

2. (a) 6 (b) $\frac{37}{24}$ (c) (d)

3. (a) $\frac{\pi}{4}$ (b) $\frac{\sqrt{2}}{3}-\frac{\sqrt{3}}{4}$ (c) $\frac{1}{8}$ (d) $\frac{\pi}{2}$

4. (a) $7\sin^6 x\cos x$ (b) $\frac{1}{7}$

5. 4

6. (a) – (b) – (c) –

   (d) (i) 1 (ii) $\frac{1}{5}$

7.