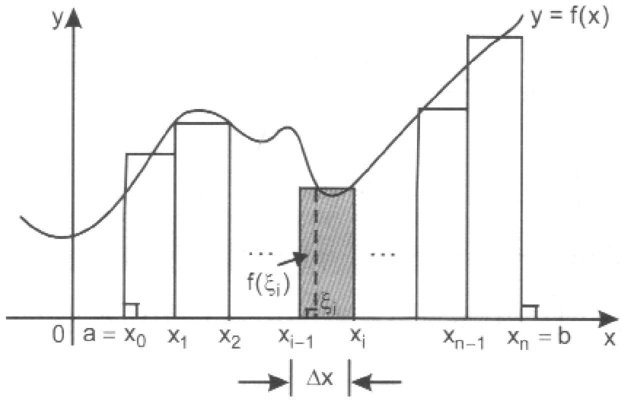
Chapter 20 Definite Integrals

20.1 The Riemann Definition of Definite Integral

Let the graph $y = f(x)$ be continuous and $f(x) \geq 0$ on the interval [a, b]. Divide [a, b] into n subintervals by the points $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$. Let Δx denote the width of the subinterval $[x_{i-1}, x_i]$ and ξ_i be any arbitrary point in $[x_{i-1}, x_i]$. The area, A, bounded by the curve, $x = a$, $x = b$ and the x-axis can be approximated by $\sum_{i=1}^{n} f(\xi_i)\Delta x$.

When n increases and $\Delta x \to 0$, A can be found by limiting the sum and is defined as the *definite integral* of $f(x)$ from $x = a$ to $x = b$ and it is denoted by

$$A = \int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x$$



In the definite integral $\int_a^b f(x)\,dx$, a and b are called the lower limit and the upper limit respectively.

20.2 Fundamental Theorem of Calculus

If $y = f(x)$ is continuous on the interval [a, b] and $F(x)$ is any primitive function of $f(x)$, then

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

Example 20.1

Evaluate $\int_{\frac{1}{8}}^{1} \frac{1}{\sqrt[3]{x}}\, dx$.

Solution

$$
\begin{aligned}
\int_{\frac{1}{8}}^{1} \frac{1}{\sqrt[3]{x}}\, dx &= \int_{\frac{1}{8}}^{1} x^{-\frac{1}{3}}\, dx \\
&= \left[\frac{3}{2}x^{\frac{2}{3}}\right]_{\frac{1}{8}}^{1} \\
&= \frac{3}{2}(1)^{\frac{2}{3}} - \frac{3}{2}\left(\frac{1}{8}\right)^{\frac{2}{3}} \\
&= \frac{9}{8}
\end{aligned}
$$

Example 20.2

Evaluate $\int_{0}^{\pi} \sin x\, dx$.

Solution

$$
\begin{aligned}
\int_{0}^{\pi} \sin x\, dx &= [-\cos x]_{0}^{\pi} \\
&= -\cos\pi + \cos 0 \\
&= 2
\end{aligned}
$$

Checkpoint 20.1

Evaluate the following definite integrals.

(a) $\int_{0}^{2} 2x\, dx$

(b) $\int_{-1}^{3} x^{3}\, dx$

Checkpoint 20.2

Evaluate the following definite integrals.

(a) $\int_0^{\frac{\pi}{2}} \cos x \, dx$

(b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2 x \, dx$

20.3 Properties of Definite Integrals

Let $f(x)$ and $g(x)$ be continuous on the interval $[a, b]$. Then

(1) $\int_a^a f(x) \, dx = 0$

(2) $\int_a^c f(x) \, dx + \int_c^b f(x) \, dx$, where c is a constant.

(3) $\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$

(4) $\int_a^b k \cdot f(x) \, dx = k \int_a^b f(x) \, dx$, where k is a constant.

(5) $\int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$

(6) $\int_a^b f(x) \, dx = \int_a^b f(u) \, du$

Example 20.3

Evaluate $\int_1^3 (4x+5)\,dx + \int_3^1 5\,dx - 4\int_1^3 u\,du$.

Solution

$$\int_1^3 (4x+5)\,dx + \int_3^1 5\,dx - 4\int_1^3 u\,du$$

$$= \int_1^3 4x\,dx + 5\int_1^3 x\,dx - \int_1^3 5\,dx - \int_1^3 4x\,dx$$

$$= 0$$

Example 20.4

Evaluate $\int_0^{\frac{\pi}{3}} \frac{1-\cos^3 y}{\cos^2 y}\,dy - \int_0^{\frac{\pi}{6}} \frac{1-\cos^3 y}{\cos^2 y}\,dy$.

Solution

$$\int_0^{\frac{\pi}{3}} \frac{1-\cos^3 y}{\cos^2 y}\,dy - \int_0^{\frac{\pi}{6}} \frac{1-\cos^3 y}{\cos^2 y}\,dy$$

$$= \int_0^{\frac{\pi}{6}} \frac{1-\cos^3 y}{\cos^2 y}\,dy + \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\cos^3 y}{\cos^2 y}\,dy - \int_0^{\frac{\pi}{6}} \frac{1-\cos^3 y}{\cos^2 y}\,dy$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\cos^3 y}{\cos^2 y}\,dy$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sec^2 y - \cos y)\,dy$$

$$= [\tan y - \sin y]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \left(\tan\frac{\pi}{3} - \sin\frac{\pi}{3}\right) - \left(\tan\frac{\pi}{6} - \sin\frac{\pi}{6}\right)$$

$$= \left(\sqrt{3} - \frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{3}} - \frac{1}{2}\right)$$

$$= \frac{3+\sqrt{3}}{6}$$

Checkpoint 20.3

Evaluate the following definite integrals.

(a) $\int_0^{\pi} \frac{1}{3} \sin x \, dx$

(b) $\int_{-\pi}^{\pi} (\cos x - 2\sin x) \, dx$

(c) $\int_0^{\frac{\pi}{4}} \sec^2 x \, dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (1 + \tan^2 u) \, du$

(d) $\int_0^{\pi} \sec x \, dx + \int_{\pi}^{0} \sec x \, dx$

20.4 More Examples on Definite Integrals

Example 20.5

Evaluate $\int_1^4 \frac{1}{\sqrt{x}}(1+\sqrt{x})^2\, dx$.

Solution

$$\int_1^4 \frac{1}{\sqrt{x}}(1+\sqrt{x})^2\, dx = \int_1^4 \frac{1}{\sqrt{x}}(1+2\sqrt{x}+x)\, dx$$

$$= \int_1^4 (x^{-\frac{1}{2}}+2+x^{\frac{1}{2}})\, dx$$

$$= \left[2x^{\frac{1}{2}} + 2x + \frac{2x^{\frac{3}{2}}}{3} \right]_1^4$$

$$= \left[2(4)^{\frac{1}{2}} + 2(4) + \frac{2(4)^{\frac{3}{2}}}{3} \right] - \left[2(1)^{\frac{1}{2}} + 2(1) + \frac{2(1)^{\frac{3}{2}}}{3} \right]$$

$$= \frac{38}{3}$$

Example 20.6

Evaluate $\int_{-2}^1 (y+1)\sqrt{y+3}\, dy$.

Solution

$$\int_{-2}^1 (y+1)\sqrt{y+3}\, dy = \int_{-2}^1 [(y+3)-2]\sqrt{y+3}\, dy$$

$$= \int_{-2}^1 (y+3)^{\frac{3}{2}}\, dy - 2\int_{-2}^1 (y+3)^{\frac{1}{2}}\, dy$$

$$= \left[\frac{2}{5}(y+3)^{\frac{5}{2}} \right]_{-2}^1 - 2\left[\frac{2}{3}(y+3)^{\frac{3}{2}} \right]_{-2}^1$$

$$= \frac{2}{5}(32-1) - \frac{4}{3}(8-1)$$

$$= \frac{46}{15}$$

Checkpoint 20.4

Evaluate the following definite integrals.

(a) $\int_1^4 \frac{(2x-\sqrt{x})^2}{x}\,dx$

(b) $\int_0^8 (2x+1)\sqrt{x+1}\,dx$

(c) $\int_{-4}^1 (x+3)(x+4)(x+5)\,dx$

Example 20.7

Evaluate $\int_0^1 \frac{x^3 + x^2 - 6x}{x-2}\,dx$.

Solution

$$\begin{aligned}\int_0^1 \frac{x^3 + x^2 - 6x}{x-2}\,dx &= \int_0^1 \frac{x(x+3)(x-2)}{x-2}\,dx \\ &= \int_0^1 (x^2 + 3x)\,dx \\ &= \left[\frac{1}{3}x^3 + \frac{3}{2}x^2\right]_0^1 \\ &= \frac{1}{3} + \frac{3}{2} \\ &= \frac{11}{6}\end{aligned}$$

Checkpoint 20.5

Evaluate $\int_2^5 \frac{x^3 - x^2 + x - 1}{x-1}\,dx$

Example 20.8

Evaluate $\int_0^1 \frac{x^3 - 2x^2 - 4x + 13}{(x-2)^2}\, dx$.

Solution

$$\int_0^1 \frac{x^3 - 2x^2 - 4x + 13}{(x-2)^2}\, dx = \int_0^1 \left[(x+2) + \frac{5}{(x-2)^2} \right] dx$$

$$= \int_0^1 [x + 2 + 5(x-2)^{-2}]\, dx$$

$$= \left[\frac{1}{2}x^2 + 2x - \frac{5}{(x-2)} \right]_0^1$$

$$= \left[\frac{1}{2}(1)^2 + 2(1) - \frac{5}{(1-2)} \right] - \left[\frac{1}{2}(0)^2 + 2(0) - \frac{5}{(0-2)} \right]$$

$$= \left(7 + \frac{1}{2} \right) - \frac{5}{2}$$

$$= 5$$

Checkpoint 20.6

Evaluate $\int_1^2 \frac{x^3 + 1}{(x+1)^5}\, dx$

Checkpoint 20.7

Evaluate $\int_1^3 \frac{4x^4 + 4x^3 - 3x^2 - 4x}{(2x+1)^2}\,dx$

Example 20.9

Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{2}{\cot\frac{x}{2} + \tan\frac{x}{2}}\,dx$.

Solution

$$
\begin{aligned}
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{2}{\cot\frac{x}{2} + \tan\frac{x}{2}}\,dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{2}{\dfrac{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}}}\,dx \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\sin\frac{x}{2}\cos\frac{x}{2}\,dx \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x\,dx \\
&= [-\cos x]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\
&= -\cos\frac{\pi}{2} - \left(-\cos\frac{\pi}{4}\right) \\
&= \frac{\sqrt{2}}{2}
\end{aligned}
$$

Checkpoint 20.8

Evaluate the following definite integrals.

(a) $\int_0^{\frac{\pi}{2}} [(\sin x + \cos x)^2 - \sin 2x]\, dx$

(b) $\int_{0.1}^{0.5} \frac{\cos 2x + 1}{\cos 2x - 1}\, dx$

Example 20.10

Evaluate $\int_{0}^{\frac{\pi}{4}} \frac{2\cos^3 x}{1-\sin x}\, dx$.

Solution

$$\begin{aligned}
\int_{0}^{\frac{\pi}{4}} \frac{2\cos^3 x}{1-\sin x}\, dx &= \int_{0}^{\frac{\pi}{4}} \frac{2\cos x(1-\sin^2 x)}{1-\sin x}\, dx \\
&= \int_{0}^{\frac{\pi}{4}} \frac{2\cos x(1+\sin x)(1-\sin x)}{1-\sin x}\, dx \\
&= \int_{0}^{\frac{\pi}{4}} 2\cos x(1+\sin x)\, dx \\
&= \int_{0}^{\frac{\pi}{4}} (2\cos x+\sin 2x)\, dx \\
&= \left[2\sin x-\frac{1}{2}\cos 2x\right]_{0}^{\frac{\pi}{4}} \\
&= \left[2\sin\frac{\pi}{4}-\frac{1}{2}\cos 2\left(\frac{\pi}{4}\right)\right]-\left[2\sin 0-\frac{1}{2}\cos 2(0)\right] \\
&= \sqrt{2}+\frac{1}{2}
\end{aligned}$$

Checkpoint 20.9

Evaluate $\int_{0}^{\frac{\pi}{6}} \left(\frac{1}{1-\tan x}-\frac{1}{1+\tan x}\right) dx$.

Example 20.11

Evaluate $\int_{0}^{\frac{\pi}{6}} \cos^2 2x \cos 4x \, dx$.

Solution

$$\int_{0}^{\frac{\pi}{6}} \cos^2 2x \cos 4x \, dx$$

$$= \int_{0}^{\frac{\pi}{6}} \frac{1}{2}(1+\cos 4x)\cos 4x \, dx$$

$$= \frac{1}{2}\int_{0}^{\frac{\pi}{6}} (\cos 4x + \cos^2 4x) \, dx$$

$$= \frac{1}{2}\int_{0}^{\frac{\pi}{6}} \left[\cos 4x + \frac{1}{2}(1+\cos 8x)\right] dx$$

$$= \frac{1}{2}\left[\frac{1}{4}\sin 4x + \frac{1}{2}x + \frac{1}{2}\left(\frac{1}{8}\sin 8x\right)\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left[\frac{1}{4}\sin 4\left(\frac{\pi}{6}\right) + \frac{1}{2}\left(\frac{\pi}{6}\right) + \frac{1}{16}\sin 8\left(\frac{\pi}{6}\right)\right] - \frac{1}{2}\left[\frac{1}{4}\sin 4(0) + \frac{1}{2}(0) + \frac{1}{16}\sin 8(0)\right]$$

$$= \frac{1}{2}\left(\frac{1}{4}\times\frac{\sqrt{3}}{2} + \frac{\pi}{12} - \frac{1}{16}\times\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{24} + \frac{\sqrt{3}}{32}$$

Checkpoint 20.10

Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x \sin \frac{3x}{2}}{\sin \frac{x}{2}} \, dx$.

Example 20.12

(a) Show that $\frac{d}{dx}(x^{n-1}\sqrt{1+x^2}) = \frac{(n-1)x^{n-2}}{\sqrt{1+x^2}} + \frac{nx^n}{\sqrt{1+x^2}}$ for any positive integer $n \geq 3$.

(b) Using (a), show that

$$\int \frac{x^n}{\sqrt{1+x^2}}dx = \frac{x^{n-1}\sqrt{1+x^2}}{n} - \frac{n-1}{n}\int \frac{x^{n-2}}{\sqrt{1+x^2}}dx \text{ for } n \geq 3.$$

(c) Let $I_n = \int_0^1 \frac{x^n}{\sqrt{1+x^2}}dx$, where n is a positive integer.

Using (b), show that $I_n = \frac{\sqrt{2}}{n} - \frac{n-1}{n}I_{n-2}$ for $n \geq 3$.

(d) (i) Find $\frac{d}{dx}\sqrt{1+x^2}$. Hence evaluate I_1.

(ii) Using (c) and (d)(i), evaluate $\int_0^1 \frac{x^3}{\sqrt{1+x^2}}dx$.

Solution

(a)
$$\begin{aligned}\frac{d}{dx}(x^{n-1}\sqrt{1+x^2}) &= (n-1)x^{n-2}(1+x^2)^{\frac{1}{2}} + x^{n-1}\cdot\frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x)\\ &= (n-1)x^{n-2}\sqrt{1+x^2} + \frac{x^n}{\sqrt{1+x^2}}\\ &= \frac{(n-1)x^{n-2}(1+x^2)}{\sqrt{1+x^2}} + \frac{x^n}{\sqrt{1+x^2}}\\ &= \frac{(n-1)x^{n-2}}{\sqrt{1+x^2}} + \frac{nx^n}{\sqrt{1+x^2}}\end{aligned}$$

(b) Since $\frac{d}{dx}(x^{n-1}\sqrt{1+x^2}) = \frac{(n-1)x^{n-2}}{\sqrt{1+x^2}} + \frac{nx^n}{\sqrt{1+x^2}}$, we have

$$\begin{aligned}x^{n-1}\sqrt{1+x^2} &= \int\left[\frac{(n-1)x^{n-2}}{\sqrt{1+x^2}} + \frac{nx^n}{\sqrt{1+x^2}}\right]dx\\ &= (n-1)\int\frac{x^{n-2}}{\sqrt{1+x^2}}dx + n\int\frac{x^n}{\sqrt{1+x^2}}dx\end{aligned}$$

$$\int \frac{x^n}{\sqrt{1+x^2}}dx = \frac{x^{n-1}\sqrt{1+x^2}}{n} - \frac{n-1}{n}\int \frac{x^{n-2}}{\sqrt{1+x^2}}dx$$

(c) $$I_n = \left[\frac{x^{n-1}\sqrt{1+x^2}}{n}\right]_0^1 - \frac{n-1}{n}\int_0^1 \frac{x^{n-2}}{\sqrt{1+x^2}}\,dx$$

$$= \frac{\sqrt{2}}{n} - \frac{n-1}{n} I_{n-2}$$

(d) (i) $$\frac{d}{dx}\sqrt{1+x^2} = \frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x)$$

$$= \frac{x}{\sqrt{1+x^2}}$$

$$I_1 = \int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx = [(1+x^2)^{\frac{1}{2}}]_0^1$$

$$= \sqrt{2} - 1$$

(ii) $$\int_0^1 \frac{x^3}{\sqrt{1+x^2}}\,dx = I_3$$

$$= \frac{\sqrt{2}}{3} - \frac{3-1}{3} I_1$$

$$= \frac{\sqrt{2}}{3} - \frac{2}{3}(\sqrt{2}-1)$$

$$= \frac{2-\sqrt{2}}{3}$$

Checkpoint 20.11

(a) Evaluate $\int_0^1 (1-x^2)\,dx$.

(b) Show that $\frac{d}{dx} x(1-x^2)^n = (2n+1)(n-x^2)^n - 2n(1-x^2)^{n-1}$.

(c) Let $I_n = \int_0^1 (1-x^2)^n\,dx$. Using the result of (b), show that $I_n = \frac{2n}{2n+1} I_{n-1}$ for any positive integer n.

(d) Hence find $\int_0^1 (1-x^2)^6\,dx$. (*Give the answer correct to 4 decimal places.*)

Exercise 20 Definite Integrals

20.4

1. Evaluate the following definite integrals.

(a) $\int_1^4 3x(x^2+2)\,dx$ (b) $\int_1^4 x\sqrt{1+x}\,dx$

(c) $\int_0^5 \frac{x}{\sqrt{4+x}}\,dx$ (d) $\int_{-2}^{-1} (x+2)^3(x+3)^2\,dx$

2. Evaluate the following definite integrals.

(a) $\int_0^2 \frac{x^2+3x+2}{x+1}\,dx$ (b) $\int_0^1 \frac{(x+2)^2}{(x+1)^4}\,dx$

(c) $\int_2^4 \frac{x^2-2x+6}{x^2-2x+1}\,dx$ (d) $\int_{-\frac{1}{3}}^0 \frac{9x^3-30x^2+28x-10}{(3x-2)^2}\,dx$

3. Evaluate the following definite integrals.

(a) $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$ (b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin x\cos 2x\,dx$

(c) $\int_0^{\frac{\pi}{4}} \sin 2x\cos^3 2x\,dx$ (d) $\int_0^{\pi} (\cos 3x\sin x-\cos x\sin 3x)^2\,dx$

4. (a) Let $y=\sin^7 x$. Find $\frac{dy}{dx}$.

(b) Hence evaluate $\int_0^{\frac{\pi}{2}} \sin^6 x\cos x\,dx$.

5. Evaluate $\int_0^{2\pi} |\sin x|\,dx$.

6. (a) Let $y = \tan^{m-1} x \sec^n x$ where m and n are positive integers and $m \geq 2$.

Show that $\dfrac{dy}{dx} = (m-1)\tan^{m-2} x \sec^n x + (m+n-1)\tan^m x \sec^n x$.

(b) Using (a), show that

$$\int \tan^m x \sec^n x \, dx = \frac{\tan^{m-1} x \sec^n x}{m+n-1} - \frac{m-1}{m+n-1}\int \tan^{m-2} x \sec^n x \, dx$$

for $m \geq 2$ and $n \geq 1$.

(c) Let $I_{m,n} = \int_0^{\frac{\pi}{4}} \tan^m x \sec^n x \, dx$ where m and n are non-negative integers.

Using (b), show that

$$I_{m,n} = \frac{2^{\frac{n}{2}}}{m+n-1} - \frac{m-1}{m+n-1} I_{m-2,n} \text{ for } m \geq 2 \text{ and } n \geq 1.$$

(d) (i) Evaluate $I_{0,2}$.

(ii) Hence evaluate $\int_0^{\frac{\pi}{4}} \tan^4 x \sec^2 x \, dx$

Exercise 20 Answers

(*Definite Integrals*)

1.	(a)	$\frac{945}{4}$	(b)	$\frac{116}{15}$	(c)	$\frac{14}{3}$	(d)	$\frac{49}{60}$
2.	(a)	6	(b)	$\frac{37}{24}$	(c)	$\frac{16}{3}$	(d)	$-\frac{5}{6}$
3.	(a)	$\frac{\pi}{4}$	(b)	$\frac{\sqrt{2}}{3}-\frac{\sqrt{3}}{4}$	(c)	$\frac{1}{8}$	(d)	$\frac{\pi}{2}$
4.	(a)	$7\sin^6 x\cos x$		(b)	$\frac{1}{7}$			
5.	4							
6.	(a)	–	(b)	–	(c)	–		
	(d)	(i) 1	(ii)	$\frac{1}{5}$				