Chapter 19 Techniques of Integration

19.1 More Integration Formulae

Here are three more important integration formulae:

(1) $\int (ax+b)^n \, dx = \dfrac{(ax+b)^{n+1}}{a(n+1)} + C$, where $a \neq 0$ and n is a rational number other than -1.

(2) $\int \sin(ax+b) \, dx = -\dfrac{\cos(ax+b)}{a} + C$, where $a \neq 0$.

(3) $\int \cos(ax+b) \, dx = \dfrac{\sin(ax+b)}{a} + C$, where $a \neq 0$.

Example 19.1

Find $\int (3x+4)^{10} \, dx$.

Solution

$$\begin{aligned}\int (3x+4)^{10} \, dx &= \frac{(3x+4)^{10+1}}{3(10+1)} + C \\ &= \frac{(3x+4)^{11}}{33} + C\end{aligned}$$

Example 19.2

Find $\int \cos(3x+2) \, dx$.

Solution

$$\int \cos(3x+2) \, dx = \frac{\sin(3x+2)}{3} + C$$

Example 19.3

Find $\int \sin(5-2x) \, dx$.

Solution

$$\begin{aligned}\int \sin(5-2x) \, dx &= -\frac{\cos(5-2x)}{-2} + C \\ &= \frac{\cos(5-2x)}{2} + C\end{aligned}$$

Checkpoint 19.1

Find the following indefinite integrals.

(a) $\int (3x+2)\,dx$

(b) $\int \frac{dx}{\sqrt{1-2x}}$

(c) $\int \sin 7x\,dx$

(d) $\int \cos(3x+7)\,dx$

(e) $\int \sin\left(\frac{x}{5}-1\right)dx$

19.2 More About Integration of Algebraic Functions

We may rewrite the algebraic functions into terms of the form $(ax+b)^n$ so that we can use the formula $\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$ to integrate the functions.

Example 19.4

Find $\int x\sqrt{x-1}\, dx$.

Solution

$$\begin{aligned}\int x\sqrt{x-1}\, dx &= \int [(x-1)+1]\sqrt{x-1}\, dx \\ &= \int (x-1)\sqrt{x-1}\, dx + \int \sqrt{x-1}\, dx \\ &= \int (x-1)^{\frac{3}{2}}\, dx + \int (x-1)^{\frac{1}{2}}\, dx \\ &= \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{2}{3}(x-1)^{\frac{3}{2}} + C\end{aligned}$$

Checkpoint 19.2

Find $\int x(2x+1)^{2005}\, dx$.

Checkpoint 19.3

Find $\int 2(x-5)(2x+1)^{17}\ dx$.

Checkpoint 19.4

Find $\int \frac{1}{\sqrt{2x+1}-\sqrt{2x}}\,dx$.

Example 19.5

Find $\int \frac{x^2 + 2x}{(x+1)^2}\,dx$.

Solution

$$\begin{aligned}
\int \frac{x^2 + 2x}{(x+1)^2}\,dx &= \int \frac{x^2 + 2x + 1 - 1}{(x+1)^2}\,dx \\
&= \int \left[\frac{(x+1)^2}{(x+1)^2} - \frac{1}{(x+1)^2} \right] dx \\
&= \int dx - \int \frac{1}{(x+1)^2}\,dx \\
&= x - \left[-\frac{1}{(x+1)} \right] + C \\
&= x + \frac{x}{x+1} + C
\end{aligned}$$

Checkpoint 19.5

Find $\int \frac{x^2 - 4x}{x^2 - 4x + 4}\,dx$.

Example 19.6

Find $\int \frac{2x^2+10x+9}{(x^2+4x+4)^2}\,dx$.

Solution

$$\begin{aligned}
\int \frac{2x^2+10x+9}{(x^2+4x+4)^2}\,dx &= \int \frac{2x^2+10x+9}{(x+2)^4}\,dx \\
&= \int \frac{2(x^2+4x+4)+(2x+1)}{(x+2)^4}\,dx \\
&= \int \frac{2(x+2)^2+2(x+2)-3}{(x+2)^4}\,dx \\
&= \int \frac{2(x+2)^2}{(x+2)^4}\,dx + \int \frac{2(x+2)}{(x+2)^4}\,dx - \int \frac{3}{(x+2)^4}\,dx \\
&= 2\int (x+2)^{-2}\,dx + 2\int (x+2)^{-3}\,dx - 3\int (x+2)^{-4}\,dx \\
&= \frac{2(x+2)^{-1}}{-1} + \frac{2(x+2)^{-2}}{-2} - \frac{3(x+2)^{-3}}{-3} + C \\
&= -\frac{2}{x+2} - \frac{1}{(x+2)^2} + \frac{1}{(x+2)^3} + C
\end{aligned}$$

Checkpoint 19.6

Find $\int \frac{2x^2-7x-2}{(x-2)^5}\,dx$.

19.3 More About Integration of Trigonometric Functions

For some complicated trigonometric functions, we may use product-to-sum formulae to reduce the integrand into a sum of simple trigonometric functions, which is easier to be integrated.

Recall the three useful product-to-sum formulae:

(1) $\sin x\cos y = \frac{1}{2}[\sin(x+y)+\sin(x-y)]$

(2) $\sin x\sin y = -\frac{1}{2}[\cos(x+y)+\cos(x-y)]$

(3) $\cos x\cos y = \frac{1}{2}[\cos(x+y)+\cos(x-y)]$

Example 19.7

Find $\int \sin 2x\cos 4x\,dx$.

Solution

$$\begin{aligned}\int \sin 2x\cos 4x\,dx &= \frac{1}{2}\int [\sin(2x+4x)+\sin(2x-4x)\,dx \\ &= \frac{1}{2}\int [\sin 6x+\sin(-2x)]\,dx \\ &= \frac{1}{2}\int (\sin 6x-\sin 2x)\,dx \\ &= \frac{1}{2}\left(\frac{-\cos 6x}{6}-\frac{-\cos 2x}{2}\right)+C \\ &= \frac{1}{4}\cos 2x-\frac{1}{12}\cos 6x+C\end{aligned}$$

Example 19.8

Find $\int \sin 4x\sin 3x\,dx$.

Solution

$$\begin{aligned}\int \sin 4x\sin 3x\,dx &= -\frac{1}{2}\int [\cos(4x+3x)-\cos(4x-3x)]\,dx \\ &= -\frac{1}{2}\int (\cos 7x-\cos x)\,dx \\ &= -\frac{1}{2}\left(\frac{\sin 7x}{7}-\sin x\right)+C \\ &= \frac{1}{2}\sin x-\frac{1}{14}\sin 7x+C\end{aligned}$$

Checkpoint 19.7

Find the following integrals:

(a) $\int \sin 5x \cos 3x \, dx$

(b) $\int \sin 2x \sin 3x \, dx$

Example 19.9

Find $\int \sin x \cos^3 x \, dx$.

Solution

$$
\begin{aligned}
\int \sin x \cos^3 x \, dx &= \frac{1}{2}\int (2\sin x \cos x)\cos^2 x \, dx \\
&= \frac{1}{2}\int \sin 2x \left[\frac{1}{2}(1+\cos 2x)\right] dx \\
&= \int \left(\frac{1}{4}\sin 2x + \frac{1}{4}\sin 2x \cos 2x\right) dx \\
&= \int \left(\frac{1}{4}\sin 2x + \frac{1}{8}\sin 4x\right) dx \\
&= \frac{-\cos 2x}{8} - \frac{1}{32}\cos 4x + C
\end{aligned}
$$

Checkpoint 19.8

Find $\int \sin^3 x \cos^2 x \, dx$.

19.4 Reduction Formulae

We may denote I_n be $\int \sin^n x\,dx$, where n is a positive integers. Then

$$I_1 = \int \sin x\,dx,\ I_2 = \int \sin^2 x\,dx,\ I_3 = \int \sin^3 x\,dx, \text{ etc.}$$

For such integrals, there are reduction formulae which express the integrals in terms of simpler ones. For $\int \sin^n x\,dx$,

$$I_n = \frac{n-1}{n}\int \sin^{n-2} x\,dx - \frac{1}{n}\sin^{n-1} x\cos x, \text{ where } n \geq 2$$

Example 19.10

(a) Show that $\frac{d}{dx}(\sin^{n-1} x\cos x) = (n-1)\sin^{n-2} x - n\sin^n x$.

(b) Let $I_n = \int \sin^n x\,dx$. Using the result of (a), prove that, for any positive integer $n \geq 2$,

$$I_n = \frac{n-1}{n}I_{n-2} - \frac{1}{n}\sin^{n-1} x\cos x.$$

(c) Hence find $\int \sin^2 x\,dx$, $\int \sin^4 x\,dx$ and $\int \sin^5 x\,dx$.

Solution

(a)

$$\begin{aligned}
\text{L.H.S.} &= \frac{d}{dx}(\sin^{n-1} x\cos x) \\
&= \sin^{n-1} x\frac{d}{dx}\cos x + \cos x\frac{d}{dx}\sin^{n-1} x \\
&= \sin^{n-1} x(-\sin x) + \cos x(n-1)\sin^{n-2} x\cos x \\
&= -\sin^n x + (n-1)\sin^{n-2} x\cos^2 x \\
&= -\sin^n x + (n-1)\sin^{n-2} x(1-\sin^2 x) \\
&= -\sin^n x + (n-1)\sin^{n-2} x - (n-1)\sin^n x \\
&= (n-1)\sin^{n-2} x - n\sin^n x \\
&= \text{R.H.S.}
\end{aligned}$$

$\therefore$ L.H.S. = R.H.S.

(b) $$\frac{d}{dx}(\sin^{n-1} x\cos x) = (n-1)\sin^{n-2} x - n\sin^n x$$

$$\int \frac{d}{dx}(\sin^{n-1} x\cos x)\, dx = \int [(n-1)\sin^{n-2} x - n\sin^n x]\, dx$$

$$\sin^{n-1} x\cos x = (n-1)\int \sin^{n-2} x\, dx - n\int \sin^n x\, dx$$

$$= (n-1)I_{n-2} - nI_n$$

$$nI_n = (n-1)I_{n-2} - \sin^{n-1} x\cos x$$

$$I_n = \frac{n-1}{n} I_{n-2} - \frac{1}{n}\sin^{n-1} x\cos x$$

(c) $$\int \sin^2 x\, dx = I_2$$

$$= \frac{2-1}{2} I_0 - \frac{1}{2}\sin^{2-1} x\cos x$$

$$= \frac{1}{2}\int \sin^0 x\, dx - \frac{1}{2}\sin x\cos x$$

$$= \frac{1}{2}x - \frac{1}{2}\sin x\cos x + C$$

$$\int \sin^4 x\, dx = I_4$$

$$= \frac{4-1}{4} I_{4-2} - \frac{1}{4}\sin^{4-1} x\cos x$$

$$= \frac{3}{4} I_2 - \frac{1}{4}\sin^3 x\cos x$$

$$= \frac{3}{8}x - \frac{3}{8}\sin x\cos x - \frac{1}{4}\sin^3 x\cos x + C$$

$$\int \sin^5 x\, dx = I_5$$

$$= \frac{5-1}{5} I_{5-2} - \frac{1}{5}\sin^{5-1} x\cos x$$

$$= \frac{4}{5} I_3 - \frac{1}{5}\sin^4 x\cos x$$

$$= \frac{4}{5}\left(\frac{3-1}{3} I_{3-2} - \frac{1}{3}\sin^{3-1} x\cos x\right) - \frac{1}{5}\sin^4 x\cos x$$

$$= \frac{4}{5}\left[\frac{2}{3}(-\cos x) - \frac{1}{3}\sin^2 x\cos x\right] - \frac{1}{5}\sin^4 x\cos x + C$$

$$= -\frac{8}{15}\cos x - \frac{4}{15}\sin^2 x\cos x - \frac{1}{5}\sin^4 x\cos x + C$$

Checkpoint 19.9

(a) Show that $\frac{d}{d\theta}\cos^{n-1}\theta\sin\theta = -(n-1)\cos^{n-2}\theta + n\cos^n\theta$ for any integer $n \geq 2$.

(b) Let $I_n = \int \cos^n\theta\, d\theta$. Using the result of (a), prove that, for any positive integer $n \geq 2$,

$$I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n}\cos^{n-1}\theta\sin\theta .$$

(c) (i) Find I_0 and I_1.

(ii) Hence evaluate the integrals $\int \cos^3\theta\, d\theta$ and $\int \cos^4\theta\, d\theta$.

Checkpoint 19.10

(a) Show that $\frac{d}{dx}x^n(1+x)^{\frac{1}{2}} = \frac{nx^{n-1}}{\sqrt{1+x}} + \frac{(2n+1)x^n}{2\sqrt{1+x}}$ for any positive integer n.

(b) Let $I_n = \int \frac{x^n}{\sqrt{1+x}}dx$. By the result of (a), show that $I_n = \frac{2}{1+2n}x^n\sqrt{1+x} - \frac{2n}{1+2n}I_{n-1}$ for any positive integer n.

Exercise 19 Techniques of Integration

19.1

1. Find the following indefinite integrals.

(a) $\int (8x-1)^3\,dx$

(b) $\int \frac{4}{(1-x)^3}\,dx$

(c) $\int \sqrt{3-\frac{x}{2}}\,dx$

(d) $\int \frac{dx}{x^3-3x^2+3x-1}$

2. Find the following indefinite integrals.

(a) $\int \cos(4x-7)\,dx$

(b) $\int \cos\left(\frac{5-4x}{3}\right)dx$

(c) $\int (2\sin\frac{x}{2}-3\cos 2x)\,dx$

19.2

3. Find the following indefinite integrals.

(a) $\int x(x-3)^{15}\,dx$

(b) $\int (4x+1)(2x+1)^{20}\,dx$

(c) $\int 2x\sqrt{x-1}\,dx$

(d) $\int \frac{x\,dx}{\sqrt{3x-2}}$

4. Find the following indefinite integrals.

(a) $\int \frac{3x(3x-2)}{(3x-1)^2}\,dx$

(b) $\int \frac{3x^3+27x^2+81x+21}{(x+3)^3}\,dx$

5. (a) Let $\frac{-x^2+14x-1}{(x^2-1)^2}=\frac{A}{(x-1)^2}+\frac{B}{(x+1)^2}$, where A and B are constant. Find A and B.

(b) Hence find $\int \frac{-x^2+14x-1}{(x^2-1)^2}\,dx$.

19.3

6. Find the following indefinite integrals.

(a) $\int \cos 4x\cos 5x\,dx$

(b) $\int \sin 2x\cos^2 2x\,dx$

(c) $\int \sin^4 x\cos^4 x\,dx$

(d) $\int \left[1-\sin\left(x+\frac{\pi}{5}\right)\right]^2 dx$

(e) $\int \sin 4x\cos x\cos 3x\,dx$

7. (a) Find $\int \sin 3x \sin x \, dx$.

(b) Show that $\dfrac{\cos 5x - \cos x}{\cos x} = -4\sin 3x \sin x$. Hence find $\int \dfrac{\cos 5x}{\cos x} dx$.

19.4

8. (a) Show that $\dfrac{d}{dx} x^n (1+x)^{\frac{3}{2}} = \dfrac{3+2n}{2} x^n (1+x)^{\frac{1}{2}} + nx^{n-1}(1+x)^{\frac{1}{2}}$ for any positive integer n.

(b) Let $I_n = \int x^n \sqrt{1+x} \, dx$. By the result of (a), show that, for any positive integer n,

$$I_n = \frac{2}{3+2n} x^n (1+x)^{\frac{3}{2}} - \frac{2n}{3+2n} I_{n-1}$$

(c) Hence evaluate $\int x^2 \sqrt{1+x} \, dx$

9. (a) Let n be an integer greater than 1. Show that

$$\frac{d}{d\theta} \sec^{n-2}\theta \tan\theta = (n-1)\sec^n\theta - (n-2)\sec^{n-2}\theta .$$

(b) Let $J_n = \int \sec^n \theta \, d\theta$. Show that $J_n = \dfrac{1}{n-1} \sec^{n-2}\theta \tan\theta + \dfrac{n-2}{n-1} J_{n-2}$ for any integer $n > 1$.

(c) Hence evaluate $\int \sec^6 \theta \, d\theta$.

10. (a) Prove that for all integers n, $\sin x \sin nx = \cos(n-1)x - \cos nx \cos x$.

(b) Hence prove that $\dfrac{d}{dx} \cos^m x \sin nx = (m+n)\cos^m x \cos nx - m\cos^{m-1} x \cos(n-1)x$ for any positive integer m.

(c) By the result of (b), show that

$$\int \cos^n x \cos nx \, dx = \frac{1}{2} \int \cos^{n-1} x \cos(n-1)x \, dx + \frac{1}{2n} \cos^n x \sin nx .$$

(d) Hence find $\int \cos^3 x \cos 3x \, dx$.

Exercise 18 Answers

(*Indefinite Integrals*)

1. (a) $2x^4 + C$ (b) $\frac{12}{7}x^{\frac{7}{3}} + C$ (c) $\frac{15}{7}x^{\frac{7}{5}} + C$

   (d) $\frac{3}{10}x^{\frac{10}{3}} + C$

2. (a) $\frac{3}{4}x^4 + \frac{2}{3}x^3 - 10x + C$ (b) $-2t^{-3} + 2t^{-1} + 5t + C$ (c) $6t^{\frac{4}{3}} + 6t^{\frac{1}{2}} + C$

3. (a) $\frac{x^4}{4} - \frac{x^2}{2} + C$ (b) $\frac{x^3}{3} + 4x + \frac{1}{x} + C$ (c) $-\frac{1}{x} + \frac{2}{x^2} + C$

   (d) $\frac{5}{3}x^3 + \frac{7}{2}x^2 + 2x + C$ (e) $\frac{2}{5}s^{\frac{5}{2}} + 2s^{\frac{1}{2}} + C$

   (f) $\frac{6}{11}t^{\frac{11}{6}} - \frac{3}{4}t^{\frac{4}{3}} + \frac{2}{3}t^{\frac{3}{2}} - t + C$ (g) $\frac{9}{5}x^5 - 8kx^3 + 16k^2x + C$

4. $y = \frac{x^4}{4} + \frac{(a-b)}{3}x^3 - \frac{ab}{2}x^2 + C$

5. (a) $-7\csc x$ (b) $5\tan x + 2\cot x + C$ (c) $2\tan x + C$

   (d) $2x - 3\cot x + C$

6. (a) $-\cos x + \sin x + C$ (b) $3x + 2\cot x + C$ (c) $2x + 2\sin x + C$

   (d) $x - \cos x + C$

7. (a) $3\sec^3 x \tan x$ (b) $\frac{1}{3}\sec^3 x + C$

8. $2\sin x + C$

9. $y = x^5 - 2x^3 - 4$

10. $y = \frac{3}{2}x^2 - 2x + \frac{1}{2}$

11. $y = -\sin x + x + \frac{\pi}{2}$

12. $y = \frac{x^3}{6} + 2x - \frac{5}{2}$

13. (a) – (b) $y = \frac{3}{4}x^4 + 2x + \frac{5}{4}$

14. (a) -4 (b) $y = \frac{3}{2}x^2 - 4x + 6$

15. (a) $10\sqrt{6}$ ms^{-1} (b) 20 ms^{-1}

16. 1 s, 4 s

17. (a) 75 m (b) 85 m

18. (a) $(20t^2 - 120t + 180)$ cm^3 (b) 3 min