Chapter 18 Indefinite Integrals

18.1 Concept of Indefinite Integrals

Definition:

If $F(x)$ is a primitive function of $f(x)$, i.e. $\frac{d}{dx}[F(x)] = f(x)$, the ***indefinite integral*** of $f(x)$ with respect to *x*, denoted by $\int f(x)dx$, is defined as $F(x) + C$,

i.e. $\int f(x)dx = F(x) + C$, where *C* is an arbitrary constant.

Note: In the notation $\int f(x)dx$, $\int$ is called the *integral sign*; $f(x)$ is called the *integrand*.

Example 18.1

(a) Find $\frac{d}{dx}(3x^5 - 2x^2)$.

(b) Using the result of (a), find $\int (15x^4 - 4x)dx$

Solution

(a)
$$\frac{d}{dx}(3x^5 - 2x^2) = 3(5x^4) - 2(2x)$$
$$= 15x^4 - 4x$$

(b) By the definition of indefinite integral,

$$\int (15x^4 - 4x)dx = 3x^5 - 2x^2 + C$$

18.2 The Integration Formula for $\int x^n dx$

For $n \neq 1$, $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where *C* is a constant.

Example 18.2

Find $\int x^4 dx$.

Solution

$$\int x^4 dx = \frac{x^{4+1}}{4+1} + C$$
$$= \frac{x^5}{5} + C$$

Example 18.3

Find $\int \frac{dx}{x^7}$.

Solution

$$\begin{aligned}\int \frac{dx}{x^7} &= \int x^{-7}\,dx \\ &= \frac{x^{-7+1}}{-7+1} + C \\ &= -\frac{1}{6x^6} + C\end{aligned}$$

Example 18.4

Find $\int x\sqrt{x}dx$.

Solution

$$\begin{aligned}\int x\sqrt{x}dx &= \int x^{\frac{3}{2}}dx \\ &= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + C \\ &= \frac{2}{5}x^{\frac{5}{2}} + C\end{aligned}$$

Checkpoint 18.1

Find the following indefinite integrals.

(a) $\int 5dx$ (b) $\int x^6 dx$ (c) $\int x^{-3} dx$

Checkpoint 18.2

Find the following indefinite integrals.

(a) $\int \sqrt[3]{x}dx$ (b) $\int \sqrt{x\sqrt{x}}dx$

18.3 Theorems of Integration

For any value of *k* and functions $f(x)$ and $g(x)$,

(1) $\int k \cdot f(x)dx = k\int f(x)dx$

(2) $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$

(3) $\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$

Example 18.5

Find $\int 6x^{\frac{2}{3}}dx$.

Solution

$$\begin{aligned}\int 6x^{\frac{2}{3}}dx &= 6\int x^{\frac{2}{3}}dx \\ &= 6\left(\frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + C_1\right) \\ &= \frac{18}{5}x^{\frac{5}{3}} + 6C_1 \\ &= \frac{18}{5}x^{\frac{5}{3}} + 6C\end{aligned}$$

Example 18.6

Find $\int \frac{-4x}{\sqrt{x}} dx$.

Solution

$$\begin{aligned}\int \frac{-4x}{\sqrt{x}} dx &= -4\int x^{\frac{1}{2}} dx \\ &= -4\left(\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right) + C \\ &= -\frac{8}{3}x^{\frac{3}{2}} + C\end{aligned}$$

Checkpoint 18.3

Find the following indefinite integrals.

(a) $\int 6x^2 dx$

(b) $\int 3x^2\sqrt{x} dx$

Example 18.7

Find $\int (x^2 + x^3 - 2x) dx$.

Solution

$$\begin{aligned}\int (x^2 + x^3 - 2x) dx &= \int x^2 dx + \int x^3 dx - \int 2x dx \\ &= \left(\frac{1}{3}x^3 + C_1\right) + \left(\frac{1}{4}x^4 + C_2\right) - \left[2\left(\frac{1}{2}\right)x^2 + C_3\right] \\ &= \frac{1}{3}x^3 + \frac{1}{4}x^4 - x^2 + C\end{aligned}$$

Example 18.8

Find $\int \frac{6x-2x^2}{\sqrt{x}}dx$.

Solution

$$\int \frac{6x-2x^2}{\sqrt{x}}dx = \int\left(\frac{6x}{\sqrt{x}}-\frac{2x^2}{\sqrt{x}}\right)dx$$

$$= \int 6x^{\frac{1}{2}}dx - \int 2x^{\frac{3}{2}}dx$$

$$= 6\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right) - 2\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right) + C$$

$$= 4x^{\frac{3}{2}} - \frac{4}{5}x^{\frac{5}{2}} + C$$

Checkpoint 18.4

Find the following indefinite integrals.

(a) $\int (3x^{-2} + 7x^{\frac{1}{2}})dx$

(b) $\int x^2(3-x)dx$

Example 18.9

Given that $\frac{dy}{dx} = \frac{3x^5 + 6x^4 - 2x + 7}{x^3}$, find y in terms of x.

Solution

$$\begin{aligned}
\frac{dy}{dx} &= \frac{3x^5 + 6x^4 - 2x + 7}{x^3} \\
y &= \int \frac{3x^5 + 6x^4 - 2x + 7}{x^3} dx \\
&= \int 3x^2 dx + 6\int x dx - 2\int x^{-2} dx + 7\int x^{-3} dx \\
&= 3\left(\frac{x^3}{3}\right) + 6\left(\frac{1}{2}x^2\right) - 2\left(\frac{x^{-1}}{-1}\right) + 7\left(\frac{x^{-2}}{-2}\right) + C \\
&= x^3 + 3x^2 + \frac{2}{x} - \frac{7}{2x^2} + C
\end{aligned}$$

Example 18.10

Given that $\frac{dy}{dt} = (2t - 3)(t - 8)$, find y in terms of t.

Solution

$$\begin{aligned}
\frac{dy}{dt} &= (2t - 3)(t - 8) \\
y &= \int (2t^2 - 19t + 24) dt \\
&= \frac{2}{3}t^3 - \frac{19}{2}t^2 + 24t + C
\end{aligned}$$

Checkpoint 18.5

Given that $\frac{dy}{dx} = \frac{2x + 3x^2}{x^4}$, find y in terms of x.

Checkpoint 18.6

Given that $\frac{dy}{dt} = t(t-1)(t+2)$, find y in terms of t.

18.4 Integration of Trigonometric Functions

(1) $\int \cos x dx = \sin x + C$

(2) $\int \sin x dx = -\cos x + C$

(3) $\int \sec^2 x dx = \tan x + C$

(4) $\int \csc^2 x dx = -\cot x + C$

(5) $\int \sec x \tan x dx = \sec x + C$

(6) $\int \csc x \cot x dx = -\csc x + C$

Example 18.11

Find $\int (\sin x + \cos x) dx$.

Solution

$$\begin{aligned}\int (\sin x + \cos x) dx &= \int \sin x dx + \int \cos x dx \\ &= -\cos x + \sin x + C\end{aligned}$$

Example 18.12

Find $\int \sec x(\sec x + \tan x)dx$.

Solution

$$\begin{aligned}\int \sec x(\sec x + \tan x)dx &= \int (\sec^2 x + \sec x \tan x)dx \\ &= \tan x + \sec x + C\end{aligned}$$

Example 18.13

Find $\int \frac{1}{\sin x}\left(\frac{1}{\sin x} - \frac{1}{\tan x}\right)dx$.

Solution

$$\begin{aligned}\int \frac{1}{\sin x}\left(\frac{1}{\sin x} - \frac{1}{\tan x}\right)dx &= \int \left(\frac{1}{\sin^2 x} - \frac{1}{\sin x \tan x}\right)dx \\ &= \int (\csc^2 x - \csc x \cot x)dx \\ &= -\cot x - (-\csc x) + C \\ &= \csc x - \cot x + C\end{aligned}$$

Checkpoint 18.7

Find the following indefinite integrals.

(a) $\int 4\sec^2 x dx$ (b) $\int -2\csc x \cot x dx$ (c) $\int \frac{\sin x}{\cos^2 x} dx$

Recall some useful formulae which are useful in finding some indefinite integrals.

(1) $\sin^2\theta + \cos^2\theta = 1$ (2) $1 + \tan^2\theta = \sec^2\theta$

(3) $1 + \cot^2\theta = \csc^2\theta$ (4) $\sin 2\theta = 2\sin\theta\cos\theta$

(5) $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$ (6) $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

Example 18.14

Find $\int (6 + \tan^2 x)dx$.

Solution

$$\begin{aligned}\int (6 + \tan^2 x)dx &= \int (5 + 1 + \tan^2 x)dx \\ &= \int (5 + \sec^2 x)dx \\ &= 5x + \tan x + C\end{aligned}$$

Example 18.15

Find $\int 4\sin\frac{x}{2}\cos\frac{x}{2}dx$.

Solution

$$\begin{aligned}\int 4\sin\frac{x}{2}\cos\frac{x}{2}dx &= 2\int 2\sin\frac{x}{2}\cos\frac{x}{2}dx \\ &= 2\int \sin 2\left(\frac{x}{2}\right)dx \\ &= 2\int \sin x dx \\ &= -2\cos x + C\end{aligned}$$

Checkpoint 18.8

Find the following indefinite integrals.

(a) $\int \tan x(\sec x + \tan x)dx$ (b) $\int \left(2\cos^2 \frac{x}{2} - 2\sin^2 \frac{x}{2}\right)dx$ (c) $\int \sin^2 \frac{x}{2} dx$.

18.5 Geometric Meaning of Indefinite Integrals

$\int f(x)dx = F(x) + C$ represents a family of curves $y = F(x) + C$ whose slope is the function $\frac{dy}{dx} = f(x)$.

Example 18.16

The slope of a curve at a given point (x, y) is $\frac{dy}{dx} = 2x$. If the curve passes through (0, –1), find its equation.

Solution

The family of curves is $y = \int 2x dx$

$$y = x^2 + C$$

Since the curve passes through (0, –1),

$\therefore \quad -1 = 0^2 + C$

$$C = -1$$

The equation of the required curve is $y = x^2 - 1$

Example 18.17

The slope of a curve at a given point (x, y) is $3x^2 - 8x + 3$. If the curve passes through the point (1, 3), find its equation.

Solution

$$\frac{dy}{dx} = 3x^2 - 8x + 3$$

$$y = x^3 - 4x^2 + 3x + C$$

Since the curve passes through (1, 3),

$\therefore \quad 3 = (1)^2 - 4(1) + 3(1) + C$

$$C = 3$$

The equation of the required curve is $y = x^3 - 4x^2 + 3x + 3$.

Checkpoint 18.9

In each of the following, the slope function $\frac{dy}{dx}$ of a curve is given together with the coordinates of a point P lying on the curve. Find the equation of the curve.

(a) $\frac{dy}{dx} = 3x^2 - 2x$, P(−1, 1)

(b) $\frac{dy}{dx} = \cos x$, P(π, 4)

Checkpoint 18.10

Find a relation between x and y if $\frac{dy}{dx} = 3x^2 - x$ and when $x = 2$, $y = 7$.

Example 18.18

At every point of a certain curve, $\frac{d^2y}{dx^2} = 2x+1$. If the slope of the curve at (1, 2) is 3, find the equation of the curve.

Solution

$$\frac{d^2y}{dx^2} = 2x+1$$

$$\frac{dy}{dx} = \int (2x+1)dx$$

$$= x^2 + x + C$$

Since the slope at (1, 2) is 3, we have

$$3 = 1^2 + 1 + C$$

$$C = 1$$

$\therefore \quad \frac{dy}{dx} = x^2 + x + 1$

Thus $y = \int (x^2 + x + 1)dx$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C_1$$

Since the curve passes through (1, 2), we have

$$2 = \frac{1}{3}(1) + \frac{1}{2}(1)^2 + 1 + C_1$$

$$C_1 = \frac{1}{6}$$

$\therefore$ The equation of the curve is $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \frac{1}{6}$.

Checkpoint 18.11

At each point of a certain curve, $\frac{d^2y}{dx^2} = \sin x$. If the slope of the curve at (π, π) is 2, find the equation of the curve.

18.6 Physical Applications of Indefinite Integrals

18.6.1 Application to Problems on Motions

If the displacement, velocity and acceleration at time t are s, v and a respectively, we have

$$s = \int v\,dt \quad \text{and} \quad v = \int a\,dt\,.$$

Example 18.19

A particle moves along a straight line in such a way that its velocity $v\ \text{ms}^{-1}$ after t seconds from the initial position O is given by $v = 4t + 2$. Find the distance that the particle moves in the first 3 seconds.

Solution

Let s metres be the displacement of the particle from O.

$$\begin{aligned} s &= \int v\,dt \\ &= \int (4t+2)\,dt \\ &= 2t^2 + 2t + C \end{aligned}$$

When $t = 0$, $s = 0$.

$\therefore \quad 0 = 2(0)^2 + 2(0) + C$

$\quad\quad C = 0$

So $\quad s = 2t^2 + 2t$

When $t = 3$, $s = 2(3)^2 + 2(3) = 24$

Therefore, the particle has moved 24 metres in the first 3 seconds.

Checkpoint 18.12

If $a = 48t - 24$ and $v = 6$ when $t = 0$, find the time when the particle is at rest, where a and v are the acceleration and the velocity of the moving particle at time t.

Checkpoint 18.13

A particle moves along a straight line in such a way that its acceleration $a\,\text{ms}^{-2}$ at time t seconds is given by $a = 54 - 162t$. Suppose that the particle starts from rest at a point O, find

(a) v and s in terms of t if , at time t seconds, the velocity of the particle is $v\,\text{ms}^{-1}$ and its displacement from O is s metres.

(b) the time when the particle changes its direction of motion after starting.

(c) the distance of the particle from O at the time obtained in (b).

18.6.2 Application to Problems on Rate of Change

Example 18.20

The volume of a balloon after t seconds (where $0 \le t \le 10$) is decreasing at a rate of $(70 - 7t)\ \text{cm}^3\text{s}^{-1}$. If the volume of the balloon is $900\ \text{cm}^3$ at $t = 0$, find its volume at $t = 8$.

Solution

Let the volume of the balloon after t seconds be $V\ \text{cm}^3$. Since the volume of the balloon is decreasing, $\frac{dV}{dt}$ is negative.

$$\therefore \quad \frac{dV}{dt} = -(70 - 7t)$$

$$V = \int -(70 - 7t)\, dt$$

$$= -70t + \frac{7}{2}t^2 + C$$

When $t = 0$, $V = 900$

$$\therefore \quad 900 = -70(0) + \frac{7}{2}(0)^2 + C$$

$$C = 900$$

$$\therefore \quad V = -70t + \frac{7}{2}t^2 + 900$$

When $t = 8$, $V = -70(8) + \frac{7}{2}(8)^2 + 900 = 564$

$\therefore$ The volume of the balloon when $t = 8$ is $564\ \text{cm}^3$.

Checkpoint 18.14

An empty tank is filled with water at a rate of $(36 - t^2)\ \text{m}^3\text{h}^{-1}$ where t is the time in hours. If it takes 6 hours to fill the tank, find the capacity of the tank.

Exercise 18 Indefinite Integrals

18.2, 18.3

1. Find the following indefinite integrals.

(a) $\int 8x^3 dx$

(b) $\int 4x^{\frac{4}{3}} dx$

(c) $\int 3x^{0.4} dx$

(d) $\int x^2 (\sqrt[3]{x}) dx$

2. Find the following indefinite integrals.

(a) $\int (3x^3 + 2x^2 - 10)\, dx$

(b) $\int \left(\frac{6}{t^4} - \frac{2}{t^2} + 5 \right) dt$

(c) $\int \left(8\sqrt[3]{t} + \frac{3}{\sqrt{t}} \right) dt$

3. Find the following indefinite integrals.

(a) $\int x(x^2 - 1)\, dx$

(b) $\int \frac{1}{x^2} (x^4 + 4x^2 - 1)\, dx$

(c) $\int \frac{x-4}{x^3}\, dx$

(d) $\int (x+1)(5x+2)\, dx$

(e) $\int \frac{s^2 + 1}{\sqrt{s}}\, ds$

(f) $\int (t^{\frac{1}{3}} + 1)(t^{\frac{1}{2}} - 1)\, dt$

(g) $\int (3x^2 - 4k)^2\, dx$, where k is a constant.

4. Given that $\frac{dy}{dx} = x(x+a)(x-b)$, where a and b are constants, find y as a function of x.

18.4

5. Find the following indefinite integrals.

(a) $\int 7 \csc x \cot x\, dx$

(b) $\int (5 \sec^2 x - 2 \csc^2 x)\, dx$

(c) $\int \frac{2}{\cos^2 x}\, dx$

(d) $\int \left(2 + \frac{3}{\sin^2 x} \right) dx$

6. Find the following indefinite integrals.

(a) $\int \cos x(\tan x + 1)\, dx$ (b) $\int (1 - 2\cot^2 x)\, dx$

(c) $\int 4\cos^2 \frac{x}{2}\, dx$ (d) $\int \left(\cos\frac{x}{2} + \sin\frac{x}{2} \right)^2 dx$

7. (a) Find $\frac{d}{dx}\sec^3 x$.

(b) Hence, or otherwise, find $\int \frac{\sin x}{\cos^4 x}\, dx$.

8. Find $\int \frac{1 + \cos 2x + \sin 2x}{\cos x + \sin x}\, dx$.

18.5

9. Given the slope at any point (x, y) of a curve is $\frac{dy}{dx} = 5x^4 - 6x^2$. Find the equation of the curve if P(1, –5) lies on the curve.

10. The slope at any point (x, y) of a curve is given by $\frac{dy}{dx} = 3x - 2$.

If the curve cuts the x-axis at x = 1, find the equation of the curve.

11. Given $\frac{d^2 y}{dx^2} = \sin x$ and $\left.\frac{dy}{dx}\right|_{x=0} = 0$. Find the function y satisfying $y = \pi - 1$ and $x = \frac{\pi}{2}$.

12. Find the equation of the curve $y = f(x)$ if $f(3) = 8$, $f'(0) = 2$ and $f''(x) = x$.

13. At each point (x, y) of the curve $y = f(x)$, $\frac{d^2 y}{dx^2} = 9x^2$.

(a) Show that $y = \frac{3}{4}x^4 + k_1 x + k_2$, where k_1 and k_2 are constants.

(b) If the curve passes through (1, 4) and (–1, 0), find the equation of the curve.

14. A curve $y = f(x)$ has slope $3x + k$ at any point (x, y), where k is a constant. It is known that the tangent to the curve at the point (2, 4) passes through the origin.

(a) Find the value of k.

(b) Using the result of (a), find the equation of the curve.

18.6

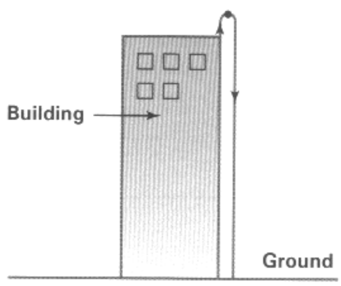
15. A stone falling freely from a cliff 30 m above the ground has a constant acceleration $10\ \text{ms}^{-2}$. Find its velocity when

(a) it reaches the ground,

(b) it is 10 m above the ground.

16. A particle starts moving at a velocity of $24\ \text{ms}^{-1}$. Its acceleration after t seconds is $a\ \text{ms}^{-2}$ where $a = 12t - 30$. Find the time when the particle is temporarily at rest.

17.



A person stands at the edge of the top of a building and throws a ball vertically upwards at $10\ \text{ms}^{-1}$. The ball reaches the ground after 5 s. If the downward acceleration is $10\ \text{ms}^{-2}$, find

(a) the height of the building,

(b) the total distance the ball has travelled.

18. An ice block with volume $180\ \text{cm}^3$ melts at a rate of $(120 - 40t)\ \text{cm}^3\text{min}^{-1}$ where t min is the time after the ice block begins to melt.

(a) After t min, what is the volume of the ice block in terms of t?

(b) Find the time required to melt the whole ice block.

Exercise 18 Answers

(*Indefinite Integrals*)

1. (a) $2x^4 + C$ (b) $\frac{12}{7}x^{\frac{7}{3}} + C$ (c) $\frac{15}{7}x^{\frac{7}{5}} + C$

   (d) $\frac{3}{10}x^{\frac{10}{3}} + C$

2. (a) $\frac{3}{4}x^4 + \frac{2}{3}x^3 - 10x + C$ (b) $-2t^{-3} + 2t^{-1} + 5t + C$ (c) $6t^{\frac{4}{3}} + 6t^{\frac{1}{2}} + C$

3. (a) $\frac{x^4}{4} - \frac{x^2}{2} + C$ (b) $\frac{x^3}{3} + 4x + \frac{1}{x} + C$ (c) $-\frac{1}{x} + \frac{2}{x^2} + C$

   (d) $\frac{5}{3}x^3 + \frac{7}{2}x^2 + 2x + C$ (e) $\frac{2}{5}s^{\frac{5}{2}} + 2s^{\frac{1}{2}} + C$

   (f) $\frac{6}{11}t^{\frac{11}{6}} - \frac{3}{4}t^{\frac{4}{3}} + \frac{2}{3}t^{\frac{3}{2}} - t + C$ (g) $\frac{9}{5}x^5 - 8kx^3 + 16k^2x + C$

4. $y = \frac{x^4}{4} + \frac{(a-b)}{3}x^3 - \frac{ab}{2}x^2 + C$

5. (a) $-7\csc x$ (b) $5\tan x + 2\cot x + C$ (c) $2\tan x + C$

   (d) $2x - 3\cot x + C$

6. (a) $-\cos x + \sin x + C$ (b) $3x + 2\cot x + C$ (c) $2x + 2\sin x + C$

   (d) $x - \cos x + C$

7. (a) $3\sec^3 x \tan x$ (b) $\frac{1}{3}\sec^3 x + C$

8. $2\sin x + C$

9. $y = x^5 - 2x^3 - 4$

10. $y = \frac{3}{2}x^2 - 2x + \frac{1}{2}$

11. $y = -\sin x + x + \frac{\pi}{2}$

12. $y = \frac{x^3}{6} + 2x - \frac{5}{2}$

13. (a) – (b) $y = \frac{3}{4}x^4 + 2x + \frac{5}{4}$

14. (a) -4 (b) $y = \frac{3}{2}x^2 - 4x + 6$

15. (a) $10\sqrt{6}\ \text{ms}^{-1}$ (b) $20\ \text{ms}^{-1}$

16. 1 s, 4 s

17. (a) 75 m (b) 85 m

18. (a) $(20t^2 - 120t + 180)\ \text{cm}^3$ (b) 3 min