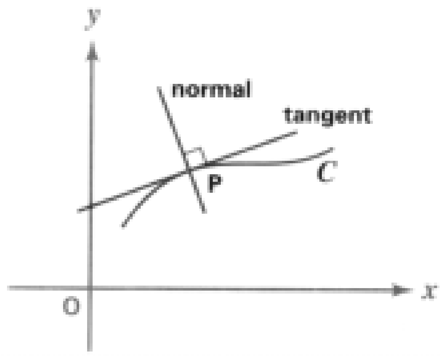
Chapter 15 Applications of Differentiation in Coordinate Geometry

15.1 Tangents and Normals to a Curve

Definition:

The normal to a curve C at a point P on C is the line which passes through P and is perpendicular to the tangent of the curve at P.



At the point (x_1, y_1) on the curve $y = f(x)$,

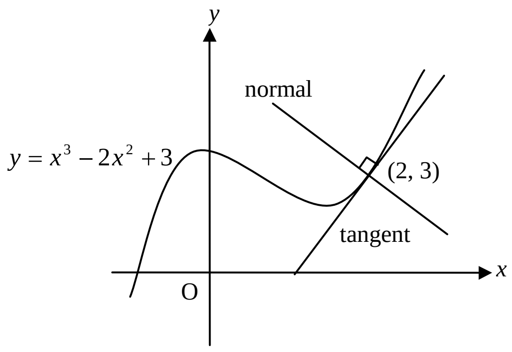
$$\text{slope of tangent} = f'(x_1)$$

$$\text{slope of normal} = \frac{-1}{f'(x_1)}$$

By using the point-slope form, we can find the equation of the tangent or the normal.

Example 15.1

Find the equations of the tangent and the normal to the curve $y = x^3 - 2x^2 + 3$ at the point (2, 3).



Solution

$$y = x^3 - 2x^2 + 3$$

$$\frac{dy}{dx} = 3x^2 - 4x$$

$$\left.\frac{dy}{dx}\right|_{x=2} = 3(2)^2 - 4(2) = 4$$

$\therefore$ The equation of the tangent at (2, 3) is

$$y - 3 = 4(x - 2)$$

$$4x - y - 5 = 0$$

$$\text{slope of normal} = \frac{-1}{\text{slope of tangent}} = -\frac{1}{4}$$

$\therefore$ The equation of the normal at (2, 3) is

$$y - 3 = -\frac{1}{4}(x - 2)$$

$$x + 4y - 14 = 0$$

Example 15.2

Find the equations of the tangent and the normal to the curve $y = \frac{1}{8}x^2 + 24\sqrt[3]{x}$ at the origin.

Solution

$$y = \frac{1}{8}x^2 + 24\sqrt[3]{x}$$

$$\frac{dy}{dx} = \frac{1}{8}(2x) + 24\left(\frac{1}{3}x^{-\frac{2}{3}}\right)$$

$$= \frac{1}{4}x + 8x^{-\frac{2}{3}}$$

$$\left.\frac{dy}{dx}\right|_{x=0} = \frac{1}{4}(0) + \frac{8}{0^{\frac{2}{3}}} \text{ which is undefined.}$$

$\therefore$ The tangent is a vertical line passing through the origin.

$\therefore$ The equation of the tangent at the origin is $x = 0$.

The normal is a horizontal line passing through the origin.

$\therefore$ The equation of the normal at the origin is $y = 0$.

Checkpoint 15.1

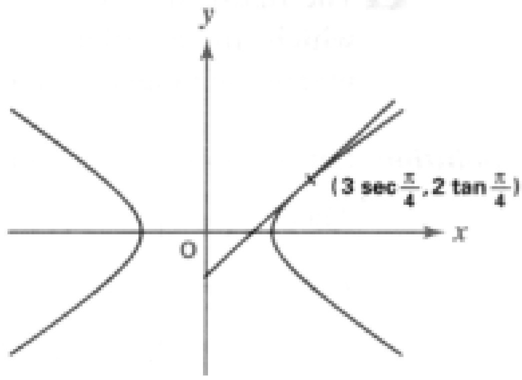
Find the equations of the tangent and the normal to the curve $x^2 y = 12$ at the point (2, 3).

Example 15.3

A curve is given by the parametric equations $x = 3\sec\theta$ and $y = 2\tan\theta$. Find the equations of the tangent and the normal to the curve at the point where $\theta = \frac{\pi}{4}$.

Solution

When $\theta = \frac{\pi}{4}$,



$$x = 3\sec\frac{\pi}{4} = 3\sqrt{2} \quad \text{and} \quad y = 2\tan\frac{\pi}{4} = 2$$

$\therefore$ The point is $(3\sqrt{2}, 2)$.

$\because$ $\frac{dx}{d\theta} = 3\sec\theta\tan\theta$ and $\frac{dy}{d\theta} = 2\sec^2\theta$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2\sec^2\theta}{3\sec\theta\tan\theta}$$

$$= \frac{2}{3}\csc\theta$$

When $\theta = \frac{\pi}{4}$,

$$\frac{dy}{dx} = \frac{2}{3}\csc\frac{\pi}{4}$$

$$= \frac{2\sqrt{2}}{3}$$

$\therefore$ The equation of the tangent at $\theta = \frac{\pi}{4}$ is

$$y - 2 = \frac{2\sqrt{2}}{3}(x - 3\sqrt{2})$$

$$2\sqrt{2}x - 3y - 6 = 0$$

$$\text{slope of normal} = \frac{-1}{\text{slope of tangent}} = -\frac{3}{2\sqrt{2}}$$

$\therefore$ The equation of the normal at $\theta = \frac{\pi}{4}$ is

$$y - 2 = -\frac{3}{2\sqrt{2}}(x - 3\sqrt{2})$$

$$3x + 2\sqrt{2}y - 13\sqrt{2} = 0$$

Checkpoint 15.2

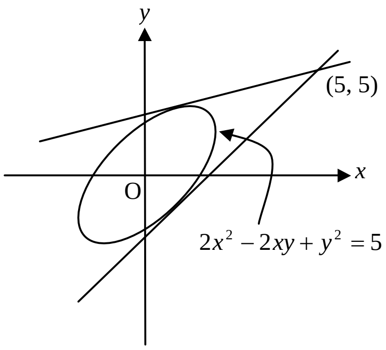
Find the equations of the tangent and the normal to the curve $\begin{cases} x = 2t \\ y = t^2 - 1 \end{cases}$ at the point where $t = 2$.

Example 15.4

The figure shows the figure $2x^2 - 2xy + y^2 = 5$. Two tangents are drawn from an external point (5, 5) to the curve. Find

(a) the equations of the two tangents;

(b) the acute angle between the two tangents correct to the nearest degree.



Solution

(a) Let the point of contact be (x_1, y_1).

Then $2x_1^{\ 2} - 2x_1y_1 + y_1^{\ 2} = 5$ ……… (1)

Differentiating both sides of $2x^2 - 2xy + y^2 = 5$ with respect to x, we have

$$4x - 2x\frac{dy}{dx} - 2y + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{y - 2x}{y - x}$$

$$\text{slope of the tangent at } (x_1, y_1) = \left.\frac{dy}{dx}\right|_{(x_1, y_1)}$$

$$= \frac{y_1 - 2x_1}{y_1 - x_1}$$

Since the tangent passes through (5, 5), the slope of the tangent is also equal to $\frac{y_1 - 5}{x_1 - 5}$.

$\therefore$ $$\frac{y_1 - 5}{x_1 - 5} = \frac{y_1 - 2x_1}{y_1 - x_1}$$

$$y_1^{\ 2} - 5y_1 - x_1y_1 + 5x_1 = x_1y_1 - 2x_1^{\ 2} - 5y_1 + 10x_1$$

$$2x_1^{\ 2} - 2x_1y_1 + y_1^{\ 2} = 5x_1$$

$$5 = 5x_1 \qquad \text{From (1)}$$

$$x_1 = 1$$

Substituting $x_1 = 1$ into (1), we have

$$2 - 2y_1 + y_1^{\ 2} = 5$$

$$y_1^{\ 2} - 2y_1 - 3 = 0$$

$$y_1 = -1 \quad \text{or} \quad 3$$

$\therefore$ The two points of contact are (1, –1) and (1, 3).

Slope of tangent at (1, –1) $= \dfrac{-1-2(1)}{-1-1} = \dfrac{3}{2}$

Slope of tangent at (1, 3) $= \dfrac{3-2(1)}{3-1} = \dfrac{1}{2}$

∴ The equations of the two tangents are

$$y-(-1) = \frac{3}{2}(x-1) \quad \text{and} \quad y-3 = \frac{1}{2}(x-1)$$

i.e. $3x-2y-5=0$ and $x-2y+5=0$

(b) Let the angle between the two tangents be θ, then

$$\tan\theta = \left|\frac{\frac{3}{2}-\frac{1}{2}}{1+\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}\right| = \frac{4}{7}$$

$$\theta = 30° \qquad \text{(corr. to the nearest degree)}$$

∴ The required angle is $30°$.

Checkpoint 15.3

Find the equations of the tangent to the curve $y = x^4 - 2x^2 + 3x + 1$ at the point where the curve cuts the *y*-axis.

15.2 Maxima and Minima

15.2.1 Increasing and Decreasing Functions

For any function $f(x)$ defined on an interval,

(1) if $f'(x) > 0$, then $f(x)$ is increasing;

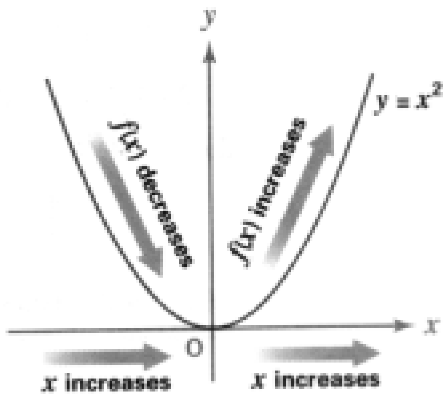
(2) if $f'(x) < 0$, then $f(x)$ is decreasing.

e.g. Consider the function $f(x) = x^2$.

$$f(x) = x^2$$

$$f'(x) = 2x \begin{cases} \geq 0 \text{ for } x \geq 0 \\ \leq 0 \text{ for } x \leq 0 \end{cases}$$

$$\therefore \quad f(x) \begin{cases} \text{is increasing for } x \geq 0. \\ \text{is decreasing for } x \leq 0. \end{cases}$$



Example 15.5

Let $f(x) = 2x^3 + x^2 - 8x + 1$. Find the range of values of x for which $f(x)$ is decreasing.

Solution

$$f(x) = 2x^3 + x^2 - 8x + 1$$

$$f'(x) = 6x^2 + 2x - 8$$

For $f(x)$ to be decreasing, $f'(x) < 0$.

$$\therefore \quad 6x^2 + 2x - 8 < 0$$

$$2(3x + 4)(x - 1) < 0$$

$$-\frac{4}{3} < x < 1$$

$\therefore$ $f(x)$ is decreasing when $-\frac{4}{3} < x < 1$.

Checkpoint 15.4

Find the range of values of x for which $f(x) = x^2 + 2x - 4$ is increasing.

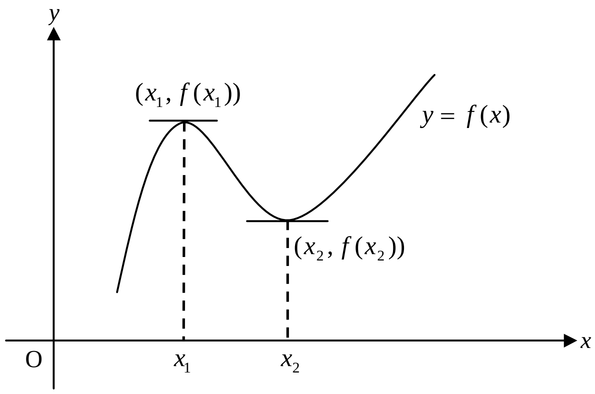
15.2.2 Relative Extrema and Stationary Points

Definitions:

(1) A function $y = f(x)$ is said to have a relative maximum (value) at x_1 if $f(x_1) \geq f(x)$ for all x in an open interval containing x_1.
The point $(x_1, f(x_1))$ on the curve $y = f(x)$ is called a maximum point.

(2) A function $y = f(x)$ is said to have a relative minimum (value) at x_2 if $f(x_2) \leq f(x)$ for all x in an open interval containing x_2.
The point $(x_2, f(x_2))$ on the curve $y = f(x)$ is called a minimum point.

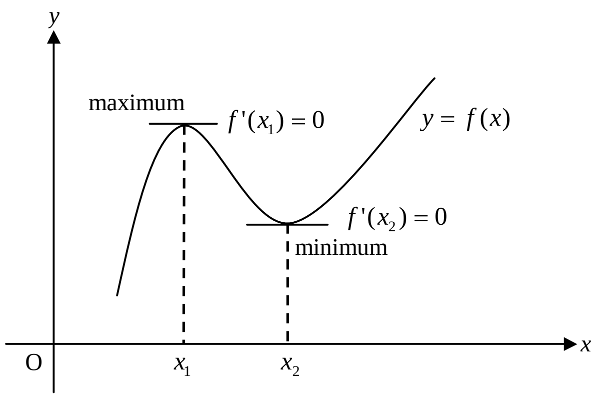
(3) A stationary point is a point of the curve where $f'(x) = 0$.



15.2.3 Tests for Maximum and Minimum Points – First Derivative Test

(1) If $f'(x) = 0$ and $f'(x)$ changes from positive to negative as x increases through x_1, then $f(x_1)$ is a relative maximum.

(2) If $f'(x) = 0$ and $f'(x)$ changes from negative to positive as x increases through x_2, then $f(x_2)$ is a relative minimum.



Example 15.6

Find the relative maximum and relative minimum of the function $f(x) = (x+1)^2(x-2)$

Solution

$$f(x) = (x+1)^2(x-2)$$
$$f'(x) = 2(x+1)(x-2) + (x+1)^2$$
$$= 3(x+1)(x-1)$$

When $f'(x) = 0$,

$$3(x+1)(x-1) = 0$$
$$x = -1 \quad \text{or} \quad 1$$

x	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$f(x)$	/	0	\	– 4	/
$f'(x)$	+ ve	0	– ve	0	+ ve

$\therefore$ Relative maximum = 0 and relative minimum = –4.

Example 15.7

Find the stationary points of the function $y = 3x^4 - 4x^3$ and determine whether they are maximum points or minimum points or neither.

Solution

$$y = 3x^4 - 4x^3$$
$$\frac{dy}{dx} = 12x^3 - 12x^2$$

For stationary points, $\frac{dy}{dx} = 0$,

i.e. $12x^3 - 12x^2 = 0$

$$12x^2(x-1) = 0$$
$$x = 0 \quad \text{or} \quad 1$$

x	$x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$x > 1$
$f(x)$	\	0	\	– 1	/
$f'(x)$	– ve	0	– ve	0	+ ve

From the table, (0, 0) and (1, –1) are stationary points.
(1, –1) is a minimum point and (0, 0) is neither a maximum point nor minimum point.

Checkpoint 15.5

Find the stationary points of the curve $y = x^3 - 3x^2$. For each stationary point, test whether it is a maximum point or a minimum point, or neither.

15.2.4 Tests for Maximum and Minimum Points − Second Derivative Test

(1) If $f'(x) = 0$ and $f''(x) < 0$, then $f(x_0)$ is a relative maximum.

(2) If $f'(x) = 0$ and $f''(x) > 0$, then $f(x_0)$ is a relative minimum.

Example 15.8

Find the relative maximum and relative minimum of the function $f(x) = x^3 - 5x^2 + 3x - 2$.

Solution

$$f(x) = x^3 - 5x^2 + 3x - 2$$
$$f'(x) = 3x^2 - 10x + 3$$
$$f''(x) = 6x - 10$$

For turning points (maximum points or minimum points), $f'(x) = 0$,

$\therefore \quad 3x^2 - 10x + 3 = 0$

$$(3x-1)(x-3) = 0$$
$$x = \frac{1}{3} \quad \text{or} \quad 3$$

$\because \quad f''\left(\frac{1}{3}\right) = 6\left(\frac{1}{3}\right) - 10 = -8 < 0$

$\therefore \quad f(x)$ attains a maximum at $x = \frac{1}{3}$.

The maximum value of $f(x) = f\left(\frac{1}{3}\right)$

$$= \left(\frac{1}{3}\right)^3 - 5\left(\frac{1}{3}\right)^2 + 3\left(\frac{1}{3}\right) - 2$$
$$= -\frac{41}{27}$$

$\because \quad f''(3) = 6(3) - 10 = 8 > 0$

$\therefore \quad f(x)$ attains a minimum at $x = 3$.

The minimum value of $f(x) = f(3)$

$$= (3)^3 - 5(3)^2 + 3(3) - 2$$
$$= -11$$

Checkpoint 15.6

Let $f(x) = 8 - 4x - x^2$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Find the turning point(s) and determine whether it is (they are) maximum or minimum or neither.

Example 15.9

Find the relative maximum and relative minimum of the function $f(x) = x^8 - 2x^4$.

Solution

$$f(x) = x^8 - 2x^4$$
$$f'(x) = 8x^7 - 8x^3$$
$$f''(x) = 56x^6 - 24x^2$$

For turning points, $f'(x) = 0$.

$\therefore$ $8x^7 - 8x^3 = 0$

$8x^3(x^4 - 1) = 0$

$x = 0$ or ± 1

$f''(0) = 56(0)^6 - 24(0)^2 = 0$.

The second derivative test fails to work since $f''(0) = 0$. The first derivative test is used instead.

x	$-1 < x < 0$	$x = 0$	$0 < x < 1$
$f(x)$	↗	0	↘
$f'(x)$	+ ve	0	– ve

$\therefore$ The relative maximum of $f(x) = 0$.

$f''(-1) = 56(-1)^6 - 24(-1)^2$

$= 32 > 0$

Also $f''(1) = 32 > 0$

$\therefore$ $f(x)$ attains two minima at x = 1 and x = –1, where $f(-1) = (-1)^8 - 2(-1)^4 = -1$

$f(1) = (1)^8 - 2(1)^4 = -1$

Checkpoint 15.7

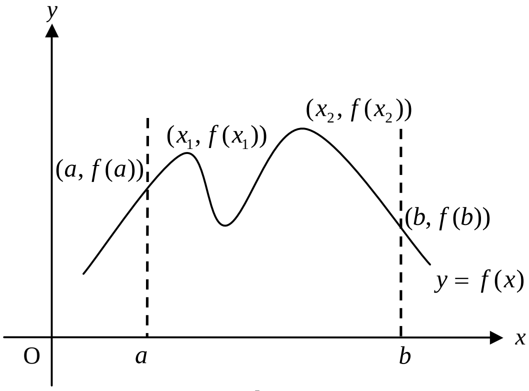
Find the relative maximum and relative minimum of the function $f(x) = x^3 - 2x^{\frac{3}{2}} + 5$.

15.2.5 Greatest Value and Least Value of a Function in a Given Range

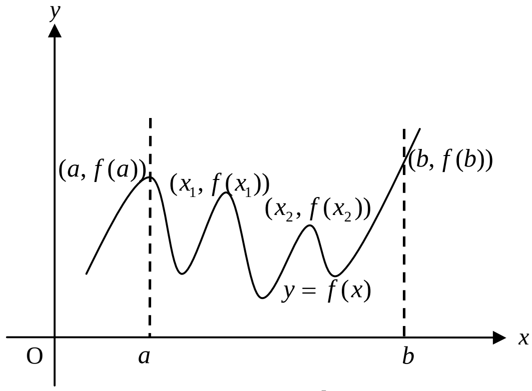
We can find the greatest (least) value of a function $f(x)$ in a given range, say $a \le x \le b$, by comparing

(1) all relative maxima (minima) for $a \le x \le b$, and

(2) the values at the end points *a* and *b*, that is $f(a)$ and $f(b)$.



Greatest value: $f(x_2)$



Greatest value: $f(b)$

Example 15.10

Let $f(x) = x^3 - 6x^2 + 9x + 2$ for $-1 \le x \le 5$. Find the least value of $f(x)$ for $-1 \le x \le 5$.

Solution

$$f(x) = x^3 - 6x^2 + 9x + 2$$
$$f'(x) = 3x^2 - 12x + 9$$
$$f''(x) = 6x - 12$$

For turning points, $f'(x) = 0$,

$\therefore$ $3x^2 - 12x + 9 = 0$

$3(x-1)(x-3) = 0$

$x = 1 \quad \text{or} \quad 3$

$f''(1) = 6(1) - 12 = -6 < 0$

$f''(3) = 6(3) - 12 = 6 > 0$

$\therefore$ $f(x)$ attains a minimum at *x* = 3, where

$f(3) = 3^3 - 6(3)^2 + 9(3) + 2 = 2$

Consider the end points –1 and 5,

$$f(-1) = (-1)^3 - 6(-1)^2 + 9(-1) + 2 = -14$$
$$f(5) = (5)^3 - 6(5)^2 + 9(5) + 2 = 22$$

$\therefore$ The least value of $f(x)$ for $-1 \le x \le 5$ is –14.

Checkpoint 15.8

For a given function $f(x) = 3x^{\frac{1}{3}} - 4x$, find

(a) its greatest value and least value for $-1 \le x \le 1$,

(b) its least value for $1 \le x \le 8$.

15.3 Curve Sketching

General procedures to sketch a curve $y = f(x)$:

(1) Identify any restriction on *x*.

(2) Find the *x*-intercepts and *y*-intercepts if any.

(3) Find the turning points by the first or second derivative test.

(4) Sketch the curve and label it properly.

Example 15.11

Sketch the graph of $y = x - 3\sqrt{x}$.

Solution

For $y = x - 3\sqrt{x}$, $x \geq 0$.

restrictions

When $y = 0$,

$$x - 3\sqrt{x} = 0$$
$$\sqrt{x}(\sqrt{x} - 3) = 0$$
$$x = 0 \quad \text{or} \quad 9$$

$\therefore$ The *x*-intercepts are 0 and 9.

When $x = 0$,

$$y = 0 - 3\sqrt{0}$$
$$= 0$$

$\therefore$ The *y*-intercepts is 0.

x- and *y*- intercepts

$$y = x - 3\sqrt{x}$$
$$\frac{dy}{dx} = 1 - \frac{3}{2}x^{-\frac{1}{2}}$$
$$\frac{d^2y}{dx^2} = \frac{3}{4}x^{-\frac{3}{2}}$$

For turning points, $\frac{dy}{dx} = 0$,

$$\therefore \quad 1 - \frac{3}{2\sqrt{x}} = 0$$
$$x = \frac{9}{4}$$

turning points

When $x = \frac{9}{4}$,

$$\frac{d^2y}{dx^2} = \frac{3}{4}\left(\frac{9}{4}\right)^{-\frac{3}{2}} > 0$$

$\therefore$ y attains a minimum at $x = \frac{9}{4}$.

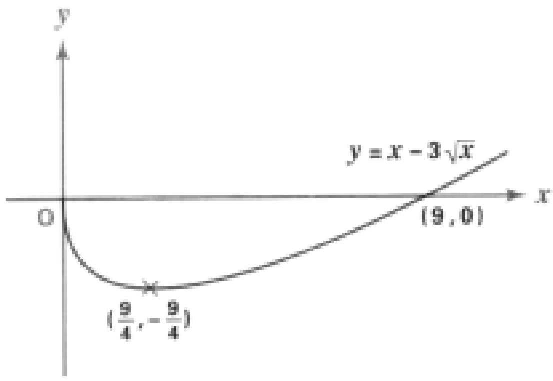
When $x = \frac{9}{4}$,

$$y = \frac{9}{4} - 3\sqrt{\frac{9}{4}} = -\frac{9}{4}$$

$\therefore$ The minimum point is $\left(\frac{9}{4}, -\frac{9}{4}\right)$.

turning points

$\therefore$ The sketch of the curve:



Sketch the curve

Example 15.12

Sketch the curve $y = (x-1)^2(x+2)$.

Solution

When $y = 0$,

$$(x-1)^2(x+2) = 0$$

$$x = 1 \quad \text{or} \quad -2$$

$\therefore$ The x-intercepts are 1 and –2.

When $x = 0$,

$$y = (0-1)^2(0+2)$$
$$= 2$$

$\therefore$ The y-intercept is 2.

$$y = (x-1)^2(x+2)$$
$$= x^3 - 3x + 2$$
$$\frac{dy}{dx} = 3x^2 - 3$$
$$\frac{d^2y}{dx^2} = 6x$$

For turning points, $\frac{dy}{dx} = 0$,

$\therefore$ $3x^2 - 3 = 0$

$x = \pm 1$

When $x = -1$,

$$\frac{d^2y}{dx^2} = 6(-1) = -6 < 0 \text{ and } y = (-1-1)^2(-1+2) = 4.$$

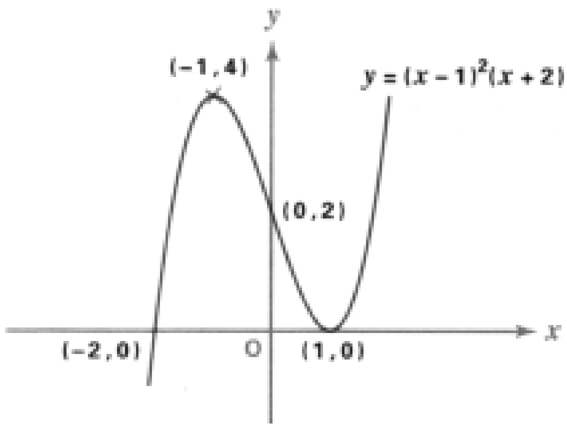
$\therefore$ (–1, 4) is a maximum point.

When $x = 1$,

$$\frac{d^2y}{dx^2} = 6(1) = 6 > 0 \text{ and } y = (1-1)^2(1+2) = 0.$$

$\therefore$ (1, 0) is a minimum point.

$\therefore$ The sketch of the curve:



Checkpoint 15.9

Given the curve $y=(x-1)^2(x-4)$.

(a) Find the x- and y- intercepts of the curve.

(b) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(c) Find the turning points of the curve.

(d) Sketch the curve.

Example 15.13

Sketch the curve $y=\frac{3-2x}{x^2+4}$.

Solution

When $y=0$,

$$3-2x=0$$

$$x=\frac{3}{2}$$

$\therefore$ The x-intercept is $\frac{3}{2}$.

When $x=0$,

$$y=\frac{3-2(0)}{0^2+4}=\frac{3}{4}$$

$\therefore$ The y-intercept is $\frac{3}{4}$.

$$y=\frac{3-2x}{x^2+4}$$

$$\frac{dy}{dx}=\frac{(x^2+4)(-2)-(3-2x)(2x)}{(x^2+4)^2}$$

$$=\frac{2(x+1)(x-4)}{(x^2+4)^2}$$

For turning points, $\frac{dy}{dx}=0$,

$\therefore$ $\frac{2(x+1)(x-4)}{(x^2+4)^2}=0$

$$x=-1 \quad \text{or} \quad 4$$

x	$x<-1$	$x=-1$	$-1<x<4$	$x=4$	$x>4$
$\frac{dy}{dx}$	>0	0	<0	0	>0
y	↑	maximum	↓	minimum	↑

When $x=-1$,

$$y=\frac{3-2(-1)}{(-1)^2+4}=1$$

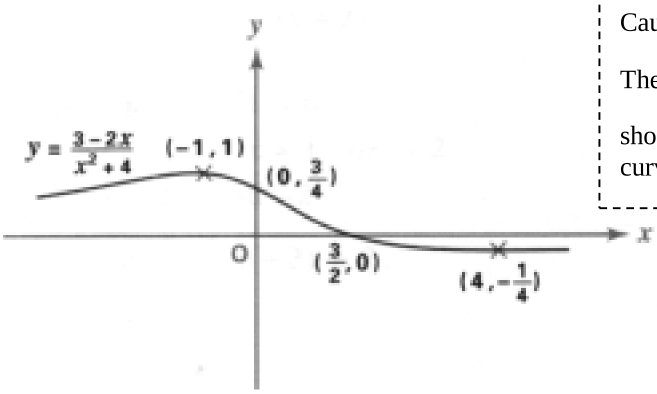
$\therefore$ (–1, 1) is a maximum point.

When $x = 4$,

$$y = \frac{3 - 2(4)}{(4)^2 + 4} = -\frac{1}{4}$$

$\therefore$ $\left(4, -\frac{1}{4}\right)$ is a minimum point.

$\therefore$ The sketch of the curve:



Caution:

There is only one x-intercept, $\frac{3}{2}$. We should be careful not to draw the curve to cut the x-axis at other points.

Example 15.14

Sketch the curve $y = \frac{x^2 - 3}{2x - 4}$ for $-2 \le x \le 5$.

Solution

For $y = \frac{x^2 - 3}{2x - 4}$, $x \ne 2$.

When $x = 2$, $\frac{x^2 - 3}{2x - 4}$ is undefined.

When $y = 0$,

$$x^2 - 3 = 0$$
$$x = \pm\sqrt{3}$$

$\therefore$ The x-intercept are $\pm\sqrt{3}$.

When $x = 0$,

$$y = \frac{(0)^2 - 3}{2(0) - 4} = \frac{3}{4}$$

$\therefore$ The y-intercept is $\frac{3}{4}$.

$$\frac{dy}{dx} = \frac{(2x-4)(2x)-(x^2-3)(2)}{(2x-4)^2}$$
$$= \frac{2x^2-8x+6}{(2x-4)^2}$$
$$= \frac{(x-3)(x-1)}{2(x-2)^2}$$

For turning points, $\frac{dy}{dx} = 0$.

$\therefore \quad \frac{(x-3)(x-1)}{2(x-2)^2} = 0$

$x = 3 \quad \text{or} \quad 1$

x	$x<1$	$x=1$	$1<x<2$	$2<x<3$	$x=3$	$x>3$
$\frac{dy}{dx}$	>0	0	<0	<0	0	>0
y	↑	maximum	↓	↓	minimum	↑

When $x = 1$, $y = \frac{(1)^2-3}{2(1)-4} = 1$.

$\therefore$ (1, 1) is a maximum point.

When $x = 3$, $y = \frac{(3)^2-3}{2(3)-4} = 3$.

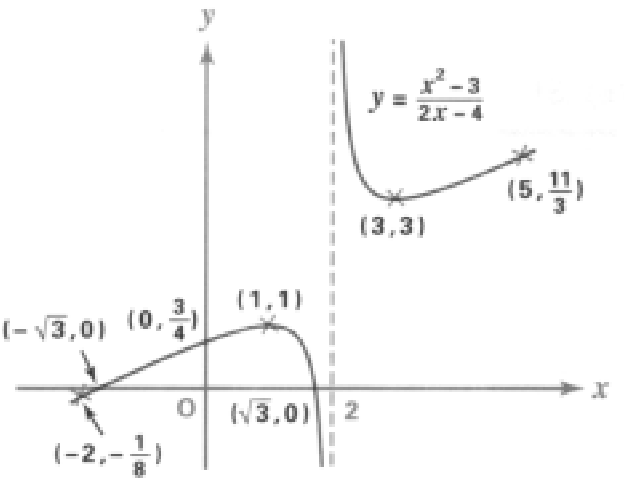
$\therefore$ (3, 3) is a minimum point.

Also, when $x = -2$, $y = \frac{(-2)^2-3}{2(-2)-4} = -\frac{1}{8}$;

when $x = 5$, $y = \frac{(5)^2-3}{2(5)-4} = \frac{11}{3}$

It is required to sketch the curve for $-2 \le x \le 5$, so we need to find y at $x = -2$ and $x = 5$.

$\therefore$ The sketch of the curve:



Checkpoint 15.10

Given the curve $y = \frac{x+1}{x^2+3}$.

(a) Find the x- and y- intercepts of the curve.

(b) Find $\frac{dy}{dx}$.

(c) Hence find the turning points of the curve.

(d) Sketch the curve.

Exercise 15 Applications of Differentiation in Coordinate Geometry

15.1

1. Find the equations of the tangent and the normal to the curve $y = \dfrac{2}{3-x}$ at the point where $x = 1$.

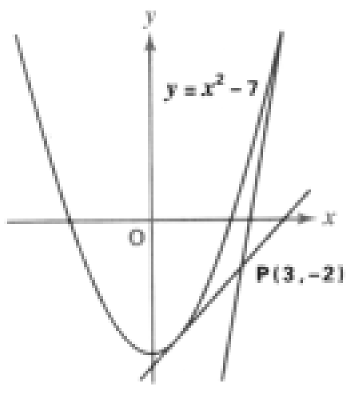
2. Find the equations of the tangent and the normal to the curve $(y-1)^2 = x+2$ at the point (7, 4).

3. Find the equations of the tangent and the normal to the curve $x^2 - x\sin y = 5$ at the point $\left(-2, \dfrac{\pi}{6}\right)$.

4. Find the equations of the tangent and the normal to the curves $x = 2\sin^2\theta$, $y = \cos\theta$ at the point where $\theta = \dfrac{\pi}{3}$.

5. Find the coordinates of the points on the curve $y = x^3 - 2$ at which the tangents to the curve have a slope of 3.
   Hence find the equations of these tangents.

6. In the figure, two tangents are drawn from the point P(3, –2) to the curve $y = x^2 - 7$.



   (a) Find the two points at which the tangents touch the curve.
   (b) Using the result of (a), find the equations of the two tangents.

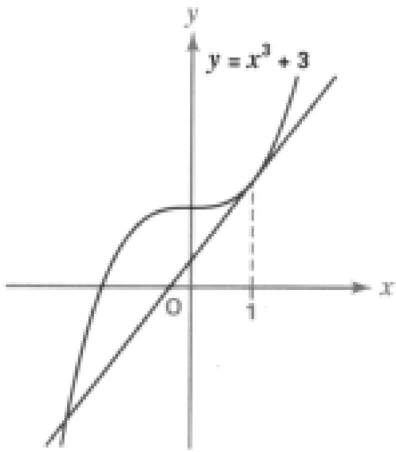
7. Given the curve $x^2 + axy + by^2 = 28$, where *a* and *b* are constants.

(a) Find $\frac{dy}{dx}$ in terms of *a*, *b*, *x* and *y*.

(b) If the slope of the tangent at the point (–8, 2) on the curve is –1, find the values of *a* and *b*.

8. (a) Find the equation of the tangent to the curve $y = x^3 + 3$ at $x = 1$.

(b) Find the point(s) where this tangent will meet the curve again.



15.2

9. Find the relative maximum and relative minimum of $y = x^3 - 8x^2 + 5x + 2$.

10. Find the relative maximum and relative minimum of $y = \frac{2x^2 + 3x + 3}{x^2 + x + 1}$.

11. Find the relative maximum and relative minimum of $y = \frac{x^3 - 2x}{\sqrt{x^2 + 1}}$.

12. The relative maximum value of $y = 4k + 2x - kx^2$ is 5. Find the value(s) of the constant *k*.

13. Given the curve $y = (x-1)^3$.

(a) Find the value of *x* for $\frac{dy}{dx} = 0$.

(b) Show that the curve does not have a maximum or a minimum point.

14. Let $y = x^4 - 2x^2$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Hence find the relative maximum and relative minimum of the function.

15. Find the relative maximum and relative minimum of the function $y = \cos x(1 + \sin x)$ where $-\pi \le x \le \pi$.

16. Given the function $f(x) = x - \frac{4}{x^2}$.

(a) Find the stationary point(s) of $f(x)$.

(b) Find the least value of $f(x)$ for $1 \le x \le 5$.

17. Let $y = \sin x + \cos x$.

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(b) Find the greatest and least values of *y*.

15.3

18. Given the curve $y = x^2 + \frac{1}{x^2}$.

(a) Find the *x*- and *y*- intercepts of the curve.

(b) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(c) Find the maximum and minimum points if any.

(d) Sketch the curve.

19. Given the curve $y = \sin 6x - \sqrt{3}\cos 6x$, where $-\frac{\pi}{6} \le x \le \frac{\pi}{6}$.

(a) Find the *x*- and *y*- intercepts of the curve.

(b) Find the maximum and minimum points if any.

(c) Sketch the curve.

20. The curve $y = mx^2(x + n)$, where *m* and *n* are constants, has a turning point at (2, –1).

(a) Find the values of *m* and *n*. What kind of turning point is (2, –1)? Show that the curve also has a maximum point at the origin.

(b) Find the *x*- and *y*- intercepts of the curve.

(c) Sketch the curve.

21. Given the curve $y = \dfrac{x-1}{x+1}$.

(a) Find the x- and y- intercepts of the curve.

(b) Show that

(i) $\dfrac{dy}{dx} > 0$ when $x \neq -1$;

(ii) $y < -1$ when $x > -1$.

(c) Sketch the curve for $-1 < x \leq 5$.

22. Given the curve $y = x\sqrt{x+3}$.

(a) Find the x- and y- intercepts of the curve.

(b) Find the range of values of x for which the slope of the curve is

(i) positive;

(ii) negative.

(c) Find the turning points of the curve. Are they maximum or minimum points?

(d) Sketch the curve for $x \geq -3$.

(e) Hence sketch the curve $y^2 = x^2(x+3)$ in the same figure.