Chapter 14 Differentiation of Trigonometric Functions

14.1 An Important Limit of Sine Function

$$\lim_{x\to 0}\frac{\sin x}{x}=1, \text{ where } x \text{ is measured in radians.}$$

Example 14.1

Evaluate

(a) $\lim_{x\to 0}\frac{\sin 3x}{2x}$

(b) $\lim_{x\to 0}\frac{\tan x}{x}$

Solution

(a)
$$\begin{aligned}\lim_{x\to 0}\frac{\sin 3x}{2x} &= \lim_{x\to 0}\left(\frac{\sin 3x}{3x}\cdot\frac{3}{2}\right)\\ &= \frac{3}{2}\lim_{x\to 0}\frac{\sin 3x}{3x}\\ &= \frac{3}{2}(1)\\ &= \frac{3}{2}\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x\to 0}\frac{\tan x}{x} &= \lim_{x\to 0}\frac{\frac{\sin x}{\cos x}}{x}\\ &= \lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right)\\ &= \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x}\\ &= 1\times 1\\ &= 1\end{aligned}$$

Example 14.2

Evaluate $\lim_{x\to 0}\frac{\sin 3x}{\sin 4x}$.

Solution

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin 3x}{\sin 4x} &= \lim_{x\to 0}\left(\frac{\sin 3x}{3x}\cdot\frac{4x}{\sin 4x}\cdot\frac{3}{4}\right) \\
&= \frac{3}{4}\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\frac{1}{\lim_{x\to 0}\frac{\sin 4x}{4x}} \\
&= \frac{3}{4}\times 1\times 1 \\
&= \frac{3}{4}
\end{aligned}$$

Example 14.3

Evaluate $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

Solution

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x}{x^2} &= \lim_{x\to 0}\frac{(1-\cos x)(1+\cos x)}{x^2(1+\cos x)} \\
&= \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)} \\
&= \lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)} \\
&= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2\left(\lim_{x\to 0}\frac{1}{1+\cos x}\right) \\
&= 1^2\times\frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

Checkpoint 14.1

Evaluate the following limits.

(a) $\lim_{x \to 0}\left(\frac{1}{x} \cdot \sin\frac{x}{3}\right)$

(b) $\lim_{x \to 0}\frac{3x}{\sin 4x}$

(c) $\lim_{x \to 0}\frac{\sin^2 x}{x \cos x}$

14.2 Derivatives of Trigonometric Functions

When x is in radian measure, then we have:

(1) $\frac{d}{dx}(\sin x) = \cos x$

(2) $\frac{d}{dx}(\cos x) = -\sin x$

(3) $\frac{d}{dx}(\tan x) = \sec^2 x$

(4) $\frac{d}{dx}(\csc x) = -\csc x \cot x$

(5) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(6) $\frac{d}{dx}(\cot x) = -\csc^2 x$

Example 14.4

In each of the following, find $\frac{dy}{dx}$.

(a) $y = x^3 \sin x$

(b) $y = \frac{\sin x}{1+x}$

Solution

(a)
$$y = x^3 \sin x$$
$$\frac{dy}{dx} = x^3 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^3)$$
Product Rule
$$= x^3 \cos x + \sin x (3x^2)$$
$$= x^3 \cos x + 3x^2 \sin x$$

(b)
$$y = \frac{\sin x}{1+x}$$
$$\frac{dy}{dx} = \frac{(1+x)\frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(1+x)}{(1+x)^2}$$
Quotient Rule
$$= \frac{(1+x)\cos x - \sin x}{(1+x)^2}$$

Checkpoint 14.2

Differentiate $y = \dfrac{x}{x^2 + \sin x}$ with respect to *x*.

Example 14.5

In each of the following, find $\dfrac{dy}{dx}$.

(a) $y = \sin(1 - 3x)$

(b) $y = \sin^3(2x)$

Solution

(a)
$$y = \sin(1 - 3x)$$
$$\frac{dy}{dx} = \frac{d}{d(1-3x)}[\sin(1-3x)] \cdot \frac{d}{dx}(1-3x)$$
Chain Rule
$$= \cos(1-3x) \cdot (-3)$$
$$= -3\cos(1-3x)$$

(b)
$$y = \sin^3(2x)$$
$$\frac{dy}{dx} = \frac{d}{d(\sin 2x)}\sin^3(2x) \cdot \frac{d}{dx}(\sin 2x)$$
Chain Rule
$$= 3\sin^2 2x \cdot \frac{d}{d(2x)}(\sin 2x) \cdot \frac{d}{dx}(2x)$$
Chain Rule
$$= 3\sin^2 2x \cos 2x(2)$$
$$= 6\sin^2 2x \cos 2x$$

Example 14.6

Given that $y = (x + \sin^2 x)^4$, find $\left.\frac{dy}{dx}\right|_{x=\pi}$.

Solution

$$
\begin{aligned}
y &= (x + \sin^2 x)^4 \\
\frac{dy}{dx} &= 4(x + \sin^2 x)^3 \frac{d}{dx}(x + \sin^2 x) \\
&= 4(x + \sin^2 x)^3 \left(1 + 2\sin x \frac{d}{dx}\sin x\right) \\
&= 4(x + \sin^2 x)^3 (1 + 2\sin x \cos x) \\
\left.\frac{dy}{dx}\right|_{x=\pi} &= 4(\pi + \sin^2 \pi)^3 (1 + 2\sin \pi \cos \pi) \\
&= 4(\pi + 0)^3 (1 + 0) \\
&= 4\pi^3
\end{aligned}
$$

Checkpoint 14.3

Differentiate $y = \sin(1 - 2x^2)$ with respect to *x*.

Checkpoint 14.4

Differentiate $y = x^3 \sin^2 3x^2$ with respect to *x*.

Example 14.7

Find $\frac{dy}{dx}$ in each of the following.

(a) $y = (x^3 + 1)\cos x$

(b) $y = \cos\left(\frac{x}{x^2 + 1}\right)$

Solution

(a)
$$\begin{aligned} y &= (x^3 + 1)\cos x \\ \frac{dy}{dx} &= (x^3 + 1)\frac{d}{dx}\cos x + \cos x \frac{d}{dx}(x^3 + 1) \\ &= (x^3 + 1)(-\sin x) + \cos x(3x^2) \\ &= -(x^3 + 1)\sin x + 3x^2 \cos x \end{aligned}$$

(b)
$$\begin{aligned} y &= \cos\left(\frac{x}{x^2 + 1}\right) \\ \frac{dy}{dx} &= \frac{d}{dx}\left[\cos\left(\frac{x}{x^2 + 1}\right)\right] \\ &= -\sin\left(\frac{x}{x^2 + 1}\right)\frac{d}{dx}\left(\frac{x}{x^2 + 1}\right) \\ &= -\sin\left(\frac{x}{x^2 + 1}\right)\frac{(x^2 + 1)\frac{d}{dx}(x) - x\frac{d}{dx}(x^2 + 1)}{(x^2 + 1)^2} \\ &= -\sin\left(\frac{x}{x^2 + 1}\right)\frac{x^2 + 1 - x(2x)}{(x^2 + 1)^2} \\ &= \frac{x^2 - 1}{(x^2 + 1)^2}\sin\left(\frac{x}{x^2 + 1}\right) \end{aligned}$$

Example 14.8

Given that $y = \dfrac{1}{\cos^3\left(ax + \dfrac{\pi}{4}\right)}$, where a is a constant, and $\left.\dfrac{dy}{dx}\right|_{x=0} = \dfrac{3}{2}$, find the value of a.

Solution

$$y = \frac{1}{\cos^3\left(ax + \frac{\pi}{4}\right)}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cos\left(ax + \frac{\pi}{4}\right)\right]^{-3}$$

$$= -3\left[\cos\left(ax + \frac{\pi}{4}\right)\right]^{-4} \frac{d}{dx}\left[\cos\left(ax + \frac{\pi}{4}\right)\right]$$

$$= -3\left[\cos\left(ax + \frac{\pi}{4}\right)\right]^{-4}\left[-\sin\left(ax + \frac{\pi}{4}\right)\right]\frac{d}{dx}\left(ax + \frac{\pi}{4}\right)$$

$$= -3\left[\cos\left(ax + \frac{\pi}{4}\right)\right]^{-4}\left[-\sin\left(ax + \frac{\pi}{4}\right)\right](a)$$

$$= 3a\left[\cos\left(ax + \frac{\pi}{4}\right)\right]^{-4}\left[\sin\left(ax + \frac{\pi}{4}\right)\right]$$

$$\left.\frac{dy}{dx}\right|_{x=0} = 3a\left[\cos\left(\frac{\pi}{4}\right)\right]^{-4}\left[\sin\left(\frac{\pi}{4}\right)\right]$$

$$= 3a\left(\frac{1}{\sqrt{2}}\right)^{-4}\left(\frac{1}{\sqrt{2}}\right)$$

$$= 6a\sqrt{2}$$

$$\therefore \quad 6\sqrt{2}a = \frac{3}{2}$$

$$a = \frac{3}{2} \times \frac{1}{6\sqrt{2}}$$

$$= \frac{\sqrt{2}}{8}$$

Checkpoint 14.5

Differentiate $y = x^3 \cos^2\left(\frac{1}{3x}\right)$ with respect to *x*.

Example 14.9

If $y = x - \tan 2x$, show that $\frac{dy}{dx} = -2\tan^2 2x$.

Solution

$$\begin{aligned} y &= x - \tan 2x \\ \frac{dy}{dx} &= \frac{d}{dx}(2x) - \frac{d}{dx}(\tan 2x) \\ &= 2 - \sec^2 2x \frac{d}{dx}(2x) \\ &= 2 - 2\sec^2 2x \\ &= 2(1 - \sec^2 2x) \\ &= -2\tan^2 2x \end{aligned}$$

Example 14.10

Let $y = (\cot x + \csc x)^2$. Find $\frac{dy}{dx}$.

Solution

$$\begin{aligned} y &= (\cot x + \csc x)^2 \\ \frac{dy}{dx} &= 2(\cot x + \csc x)\frac{d}{dx}(\cot x + \csc x) \\ &= 2(\cot x + \csc x)(-\csc^2 x - \csc x \cot x) \\ &= -2\csc x(\cot x + \csc x)(\csc x + \cot x) \\ &= -2\csc x(\cot x + \csc x)^2 \end{aligned}$$

Checkpoint 14.6

Differentiate $y = \sec^2(x+1)$ with respect to *x*.

Example 14.11

Let $y = \dfrac{1}{\sqrt[3]{(\sec 3x + 1)^2}}$. Find $\dfrac{dy}{dx}$.

Solution

$$y = \frac{1}{\sqrt[3]{(\sec 3x + 1)^2}}$$

$$\frac{dy}{dx} = \frac{d}{dx}(\sec 3x + 1)^{-\frac{2}{3}}$$

$$= -\frac{2}{3}(\sec 3x + 1)^{-\frac{5}{3}}\frac{d}{dx}(\sec 3x + 1)$$

$$= -\frac{2}{3}(\sec 3x + 1)^{-\frac{5}{3}}(\sec 3x \tan 3x)\frac{d}{dx}(3x)$$

$$= -2(\sec 3x + 1)^{-\frac{5}{3}}(\sec 3x \tan 3x)$$

Checkpoint 14.7

Differentiate the following functions with respect to *x*.

(a) $y = \dfrac{\tan 2x}{\cos 3x}$

(b) $y = \dfrac{\cot(3x + 1)}{\csc 3x + 1}$

14.3 Miscellaneous Examples

Example 14.12

Given that $x = y^2 + \sin y$. Find $\frac{dy}{dx}$ in terms of *y*.

Solution

$$x = y^2 + \sin y$$

$$\frac{dx}{dy} = \frac{d}{dy}(y^2 + \sin y)$$

$$= 2y + \cos y$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{2y + \cos y}$$

Example 14.13

If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, where *a* is a constant, find $\frac{dy}{dx}$ in terms of *t*.

Solution

$$x = a(\cos t + t\sin t)$$

$$\frac{dx}{dt} = a\left[\frac{d}{dt}(\cos t) + \frac{d}{dt}(t\sin t)\right]$$

$$= a(-\sin t + t\cos t + \sin t)$$

$$= at\cos t$$

$$y = a(\sin t - t\cos t)$$

$$\frac{dy}{dt} = a\left[\frac{d}{dt}(\sin t) - \frac{d}{dt}(t\cos t)\right]$$

$$= a[\cos t - (-t\sin t + \cos t)]$$

$$= at\sin t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{at\sin t}{at\cos t}$$

$$= \tan t$$

Example 14.14

If $x\cos y = \tan(x+y)$, find $\frac{dy}{dx}$.

Solution

$$
\begin{aligned}
x\cos y &= \tan(x+y) \\
\frac{d}{dx}(x\cos y) &= \frac{d}{dx}[\tan(x+y)] \\
x\frac{d}{dx}(\cos y) + \cos y\frac{d}{dx}(x) &= \sec^2(x+y)\frac{d}{dx}(x+y) \\
x(-\sin y)\frac{dy}{dx} + \cos y &= \sec^2(x+y)\left(1+\frac{dy}{dx}\right) \\
-x\sin y\frac{dy}{dx} + \cos y &= \sec^2(x+y) + \sec^2(x+y)\frac{dy}{dx} \\
\left[x\sin y + \sec^2(x+y)\right]\frac{dy}{dx} &= \cos y - \sec^2(x+y) \\
\frac{dy}{dx} &= \frac{\cos y - \sec^2(x+y)}{x\sin y + \sec^2(x+y)}
\end{aligned}
$$

Checkpoint 14.8

Find $\frac{dy}{dx}$ if $x = \sqrt{y^2 + \tan y}$.

Checkpoint 14.9

Find $\frac{dy}{dx}$ if $\sin^2 x + \sin^2 y = 1$.

Example 14.15

If $f(x) = x\cos \pi x$, find $f''(2)$.

Solution

$$
\begin{aligned}
f(x) &= x\cos \pi x \\
f'(x) &= x\frac{d}{dx}(\cos \pi x) + \cos \pi x \frac{d}{dx}(x) \\
&= x(-\sin \pi x)\frac{d}{dx}(\pi x) + \cos \pi x \\
&= -x\sin \pi x(\pi) + \cos \pi x \\
&= -\pi x\sin \pi x + \cos \pi x \\
f''(x) &= -\pi\left[x\frac{d}{dx}(\sin \pi x) + \sin \pi x\frac{d}{dx}(x)\right] + \frac{d}{dx}(\cos \pi x) \\
&= -\pi\left[x\cos \pi x\frac{d}{dx}(\pi x) + \sin \pi x\right] + (-\sin \pi x)\frac{d}{dx}(\pi x) \\
&= -\pi^2 x\cos \pi x - \pi \sin \pi x - \pi \sin \pi x \\
&= -\pi^2 x\cos \pi x - 2\pi \sin \pi x \\
f''(2) &= -\pi^2(2)\cos 2\pi - 2\pi \sin 2\pi \\
&= -2\pi^2
\end{aligned}
$$

Example 14.16

Let $y = x^2 \cos x$.

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(b) Show that $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (x^2 + 6)y = 0$.

Solution

(a)
$$y = x^2 \cos x$$
$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x^2)$$
$$= -x^2 \sin x + 2x\cos x$$
$$\frac{d^2y}{dx^2} = -\left[x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)\right] + 2\left[x\frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x)\right]$$
$$= -[x^2 \cos x + 2x\sin x] + 2[-x\sin x + \cos x]$$
$$= 2\cos x - x^2 \cos x - 4x\sin x$$

(b)
$$x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (x^2 + 6)y$$
$$= x^2(2\cos x - x^2 \cos x - 4x\sin x) - 4x(2x\cos x - x^2 \sin x) + (x^2 + 6)x^2 \cos x$$
$$= 2x^2 \cos x - x^4 \cos x - 4x^3 \sin x - 8x^2 \cos x + 4x^3 \sin x + x^4 \cos x + 6x^2 \cos x$$
$$= 0$$

Checkpoint 14.10

Find $\left.\frac{d^2y}{dx^2}\right|_{x=\pi}$ if $y = x\sin 2x$.

Checkpoint 14.11

If $x^2 y = a\cos nx$, where a, n are constants, show that $x^2\left(\frac{d^2y}{dx^2}\right) + 4x\left(\frac{dy}{dx}\right) + (n^2x^2 + 2)y = 0$.

Exercise 14 Differentiation of Trigonometric Functions

14.1

1. Evaluate the following limits.

(a) $\lim_{x\to 0}\frac{\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{3}\right)}$ (b) $\lim_{x\to 0}\frac{\tan 4x}{3x}$

(c) $\lim_{x\to 0}\frac{\tan^3 2x}{x\sin^2 5x}$ (d) $\lim_{x\to 0}\frac{1-\cos x}{1-\cos 3x}$

2. Evaluate $\lim_{x\to\infty} x\sin\frac{1}{x}$.

14.2

3. Differentiate the following functions with respect to *x*.

(a) $y=\frac{1}{\sqrt{\sin x}}$ (b) $y=(\sin x^3)^{-4}$

(c) $y=(1-\cos^3 x)^4$ (d) $y=\frac{1}{x}\cos\frac{1}{x}$

(e) $\sin^3 x\cos^2 x$ (f) $y=\frac{(x^2-\sin x)^3}{(x^2+\sin x)^3}$

4. Given that $y=\frac{1+\cos x}{\sin x}$, show that $\frac{dy}{dx}=-\frac{y}{\sin x}$.

5. Differentiate $y=\sin(x^3\cos x)$ with respect to *x*.

6. Differentiate the following functions with respect to *x*.

(a) $y=-\tan(x^2+2)$ (b) $y=\csc\frac{1}{x^2}$

(c) $y=\sqrt[4]{\sec^3(x^2+1)}$ (d) $y=\tan x\sec^2 x$

(e) $y=\frac{\cos^2 x}{1+\tan x}$

7. Given that $y=\frac{1+\tan^2 x}{1-\tan^2 x}$, prove that $\frac{dy}{dx}-\frac{4y\tan x}{1-\tan^2 x}=0$.

14.3

8. In each of the following, find $\frac{dy}{dx}$.

(a) $x = \cos 3y \sin 2y$

(b) $\cos \frac{y}{2} = \tan 4x$

(c) $\begin{cases} x = \cos \theta \\ y = \theta + \sin \theta \end{cases}$

(d) $1 + xy = x \tan y$

9. Find $\frac{d^2 y}{dx^2}$ of the following functions.

(a) $y = \sec x$

(b) $y = 4x \sin x$

(c) $3x = \cos y$

10. Given that $f(x) = x \sin \frac{1}{x}$. Find $f''\left(\frac{\pi}{2}\right)$ and express the answer in terms of π.

11. If $x = 3\cos\theta - \cos 3\theta$, $y = 3\sin\theta - \sin 3\theta$, show that $\frac{dy}{dx} = \tan 2\theta$.

12. If $y = \sec kx$, where k is a constant, show that $\frac{d^2 y}{dx^2} + k^2 y(1 - 2y^2) = 0$.

13. If $xy = \cos x + \sin x$, show that

(a) $x\frac{dy}{dx} + y = \cos x - \sin x$,

(b) $x\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + xy = 0$.