Chapter 13 Techniques of Differentiation

13.1 Basic Theorems of Differentiation

Let $f(x)$ and $g(x)$ be differentiable functions.

(1) $\frac{d}{dx}(c) = 0$, where c is a constant.

(2) The Power Rule:

$\frac{dx^n}{dx} = nx^{n-1}$, where n is a rational number.

(3) $\frac{d}{dx}[c \cdot f(x)] = c\frac{d}{dx}[f(x)]$

(4) $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

(5) $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$

(6) The Product Rule:

If $u = f(x)$ and $v = g(x)$ are differentiable functions, then $u \cdot v$ is also differentiable, and $\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$.

(7) The Quotient Rule:

If $u = f(x)$ and $v = g(x)$ are differentiable functions and $g(x) \neq 0$, then $\frac{u}{v}$ is also differentiable and $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, where $v \neq 0$.

Example 13.1

Differentiate $y = x^3$ with respect to x.

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^3)\\ &= 3x^{3-1}\\ &= 3x^2\end{aligned}$$

Example 13.2

Find $\frac{dy}{dx}$ if $y = \frac{1}{\sqrt{x^5}}$.

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{1}{\sqrt{x^5}}\right)\\ &= \frac{d}{dx}x^{-\frac{5}{2}}\\ &= -\frac{5}{2}x^{-\frac{5}{2}-1}\\ &= -\frac{5}{2}x^{-\frac{7}{2}}\end{aligned}$$

Checkpoint 13.1

Find the derivatives of the following functions with respect to x.

(a) $y = \pi$ (b) $f(x) = x^{1234}$ (c) $F(x) = \frac{\sqrt[3]{x}}{\sqrt{x}}$

Example 13.3

Differentiate $y = 4\sqrt{x}$ with respect to x.

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(4\sqrt{x}) \\ &= 4\frac{d}{dx}(\sqrt{x}) \\ &= 4\frac{d}{dx}\left(x^{\frac{1}{2}}\right) \\ &= 4\left(\frac{1}{2}x^{\frac{1}{2}-1}\right) \\ &= 2x^{-\frac{1}{2}} \\ &= \frac{2}{\sqrt{x}}\end{aligned}$$

Example 13.4

Find the derivative of $y = \dfrac{7\cdot\sqrt[7]{x^3}}{3x^6}$ with respect to x.

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{7\cdot\sqrt[7]{x^3}}{3x^6}\right) \\ &= \frac{d}{dx}\left(\frac{7}{3}x^{\frac{3}{7}-6}\right) \\ &= \frac{7}{3}\frac{d}{dx}\left(x^{-\frac{39}{7}}\right) \\ &= \frac{7}{3}\left(-\frac{39}{7}\right)x^{-\frac{39}{7}-1} \\ &= -13x^{-\frac{46}{7}}\end{aligned}$$

Checkpoint 13.2

Find the derivatives of the following functions with respect to x.

(a) $y = 7x$

(b) $y = 2x^4$

(c) $y = \frac{4}{3}\pi x^3$

(d) $y = 6\sqrt{x}$

Example 13.5

Differentiate $y = 2x^3 + 7x$ with respect to x.

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(2x^3 + 7x) \\ &= \frac{d}{dx}(2x^3) + \frac{d}{dx}(7x) \\ &= 2\frac{d}{dx}(x^3) + 7\frac{d}{dx}(x) \\ &= 2(3x^2) + 7(x^0) \\ &= 6x^2 + 7\end{aligned}$$

Example 13.6

Find the derivative of $y = 5x^4 - 12x^2 + 11x$ with respect to x.

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(5x^4 - 12x^2 + 11x) \\ &= \frac{d}{dx}(5x^4) - \frac{d}{dx}(12x^2) + \frac{d}{dx}(11x) \\ &= 5\frac{d}{dx}(x^4) - 12\frac{d}{dx}(x^2) + 11\frac{d}{dx}(x) \\ &= 5(4x^3) - 12(2x) + 11(x^0) \\ &= 20x^3 - 24x + 11\end{aligned}$$

Checkpoint 13.3

Find the derivatives of $y = x^3 + 4x^2 - 5x$ with respect to x.

Checkpoint 13.4

Find the derivatives of the following functions with respect to x.

(a) $y = 7x + 2\sqrt{x}$ (b) $y = 2 \cdot \sqrt[3]{x} - \sqrt{x} + 1$

Example 13.7

Differentiate $y = x^2(x+2)$ with respect to x.

Solution

$$\begin{aligned}\frac{dy}{dx} &= x^2 \frac{d}{dx}(x+2) + (x+2)\frac{d}{dx}(x^2) \\ &= x^2(1+0) + (x+2)(2x^1) \\ &= x^2 + (x+2)(2x) \\ &= 3x^2 + 4x\end{aligned}$$

Alternative Solution

$$\begin{aligned}y &= x^2(x+2) = x^3 + 2x^2 \\ \frac{dy}{dx} &= \frac{d}{dx}(x^3) + \frac{d}{dx}(2x^2) \\ &= 3x^2 + 2(2x) \\ &= 3x^2 + 4x\end{aligned}$$

Example 13.8

Let $y = x^2\sqrt{x}$. Find $\left.\frac{dy}{dx}\right|_{x=1}$.

Solution

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(x^2\sqrt{x}) \\
&= x^2\frac{d}{dx}(\sqrt{x}) + \sqrt{x}\frac{d}{dx}(x^2) \\
&= x^2\frac{d}{dx}(x^{\frac{1}{2}}) + x^{\frac{1}{2}}\frac{d}{dx}(x^2) \\
&= x^2\left(\frac{1}{2}x^{\frac{1}{2}-1}\right) + x^{\frac{1}{2}}(2x) \\
&= \frac{1}{2}x^{\frac{3}{2}} + 2x^{\frac{3}{2}} \\
&= \frac{5}{2}x^{\frac{3}{2}} \\
\left.\frac{dy}{dx}\right|_{x=1} &= \frac{5}{2}(1)^{\frac{3}{2}} \\
&= \frac{5}{2}
\end{aligned}$$

Checkpoint 13.5

Differentiate $y = x^2(x^2+1)$ with respect to x.

Solution

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}_____ \\
&= _____\frac{d}{dx}(x^2+1) + _____\frac{d}{dx}(x^2)
\end{aligned}$$

Checkpoint 13.6

Differentiate $F(x) = (x+3)(x^2 - x)$ with respect to x.

Solution

$$\frac{dF(x)}{dx} = \frac{d}{dx}\_\_\_\_\_$$

$$= \_\_\_\_\_\frac{d}{dx}(x^2 - x) + \_\_\_\_\_\frac{d}{dx}(x+3)$$

Example 13.9

Differentiate $y = (3x-1)(x+4) - (x^3+1)^2$ with respect to x.

Solution

$$\frac{dy}{dx} = \frac{d}{dx}[(3x-1)(x+4) - (x^3+1)^2]$$

$$= \frac{d}{dx}(3x-1)(x+4) - \frac{d}{dx}(x^3+1)(x^3+1)$$

$$= \left[(3x-1)\frac{d}{dx}(x+4) + (x+4)\frac{d}{dx}(3x-1)\right] - \left[(x^3+1)\frac{d}{dx}(x^3+1) + (x^3+1)\frac{d}{dx}(x^3+1)\right]$$

$$= [(3x-1)(1) + (x+4)(3)] - [(x^3+1)(3x^2) + (x^3+1)(3x^2)]$$

$$= (3x-1+3x+12) - (6x^5 + 3x^2 + 3x^5 + 3x^2)$$

$$= (6x+11) - (6x^5 + 6x^2)$$

$$= -6x^5 - 6x^2 + 6x + 11$$

Checkpoint 13.7

Differentiate $y = (x^2 + 3)^2 - (2x - 1)(x^2 + x)$ with respect to x.

Example 13.10

Differentiate $y = \dfrac{2x + 3}{x - 1}$ with respect to x.

Solution

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{2x+3}{x-1}\right) \\
&= \frac{(x-1)\dfrac{d}{dx}(2x+3) - (2x+3)\dfrac{d}{dx}(x-1)}{(x-1)^2} \\
&= \frac{(x-1)(2) - (2x+3)(1)}{(x-1)^2} \\
&= -\frac{5}{(x-1)^2}
\end{aligned}$$

Example 13.11

Differentiate $y = \dfrac{2x^3 + 3x^2 - 1}{x^2}$ with respect to x.

Solution

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{2x^3 + 3x^2 - 1}{x^2}\right) \\
&= \frac{x^2 \dfrac{d}{dx}(2x^3 + 3x^2 - 1) - (2x^3 + 3x^2 - 1)\dfrac{d}{dx}(x^2)}{(x^2)^2} \\
&= \frac{x^2[2(3x^2) + 3(2x)] - (2x^3 + 3x^2 - 1)(2x)}{x^4} \\
&= \frac{x^2(6x^2 + 6x) - (4x^4 + 6x^3 - 2x)}{x^4} \\
&= \frac{2x^4 + 2x}{x^4} \\
&= 2 + 2x^{-3}
\end{aligned}$$

Alternative Solution

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{2x^3 + 3x^2 - 1}{x^2}\right) \\
&= \frac{d}{dx}(2x + 3 - x^{-2}) \\
&= 2 + 0 - (-2x^{-2-1}) \\
&= 2 + 2x^{-3}
\end{aligned}$$

Checkpoint 13.8

Differentiate $y = \dfrac{x-3}{x+2}$ with respect to x.

Solution

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{x-3}{x+2}\right) \\
&= \frac{_____\dfrac{d}{dx}_____ - _____\dfrac{d}{dx}_____}{(x+2)^2}
\end{aligned}$$

Checkpoint 13.9

Differentiate $y = \dfrac{x^2 - 1}{2x + 3}$ with respect to x.

Solution

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\quad}{\quad}\right)$$

$$= \frac{\left(\frac{\quad}{\quad}\right)\frac{d}{dx}\left(\frac{\quad}{\quad}\right) - \left(\frac{\quad}{\quad}\right)\frac{d}{dx}\left(\frac{\quad}{\quad}\right)}{\left(\frac{\quad}{\quad}\right)}$$

Checkpoint 13.10

Find the derivatives of $y = \dfrac{2x - 1}{3x^2 + 2x}$ with respect to x.

13.2 Differentiation of Composite Functions

Suppose $C = f(n) = n^2 - 1$ and $n = g(t) = 3t + 2$. We can express C as a function of t by substituting $n = g(t)$ into $C = f(n)$.

$$\begin{aligned} C &= f[g(t)] \\ &= [g(t)]^2 - 1 \\ &= (3t+2)^2 - 1 \\ &= 9t^2 + 12t + 3 \end{aligned}$$

We say that $f[g(t)]$ is a composite function of $C = f(n)$ and $n = g(t)$.

To differentiate a composite function, we have the following theorem to help.

The Chain Rule:

Suppose *y* is a differentiable function of *u*, and *u* is a differentiable function of *x*. Then *y* is also a differentiable function of *x* and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 13.12

Given that $y = f(u) = u^4$ and $u = g(x) = x^2 + 1$, find

(a) $f[g(x)]$, (b) $\frac{dy}{dx}$.

Solution

(a) $$\begin{aligned} f[g(x)] &= f(x^2+1) \\ &= (x^2+1)^4 \end{aligned}$$

(b) $\frac{dy}{du} = 4u^3$ and $\frac{du}{dx} = 2x$

$$\begin{aligned} \therefore \quad \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 4u^3(2x) \\ &= 4(x^2+1)^3(2x) \\ &= 8x(x^2+1)^3 \end{aligned}$$

Example 13.13

Differentiate $y = (2x+5)^3$ with respect to x.

Solution

Let $y = u^3$ and $u = 2x+5$.

$$\frac{dy}{du} = \frac{d}{du}u^3 = 3u^2$$
$$\frac{du}{dx} = \frac{d}{dx}(2x+5) = 2$$
$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$
$$= 3u^2 \cdot 2$$
$$= 6u^2$$
$$= 6(2x+5)^2$$

Example 13.14

Differentiate $y = \dfrac{1}{\sqrt{5x^2-4}}$ with respect to x.

Solution

Let $y = \dfrac{1}{\sqrt{u}} = u^{-\frac{1}{2}}$ and $u = 5x^2 - 4$.

$$\frac{dy}{du} = \frac{d}{du}u^{-\frac{1}{2}} = -\frac{1}{2}u^{-\frac{3}{2}}$$
$$\frac{du}{dx} = \frac{d}{dx}(5x^2-4) = 10x$$
$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = -\frac{1}{2}u^{-\frac{3}{2}}(10x)$$
$$= -5xu^{-\frac{3}{2}}$$
$$= -5x(5x^2-4)^{-\frac{3}{2}}$$
$$= -\frac{5x}{(5x^2-4)^{\frac{3}{2}}}$$

Checkpoint 13.11

Differentiate $y = (2x^2 + 1)^3$ with respect to x.

Solution

Let $y = u^3$ and $u = 2x^2 + 1$

$$\frac{dy}{du} = \_\_\_\_\_$$

$$\frac{du}{dx} = \_\_\_\_\_$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Checkpoint 13.12

Differentiate $y = \dfrac{1}{\sqrt{2x-5}}$ with respect to x.

Solution

Let _____ and _____.

$$\frac{dy}{du} = \_\_\_\_\_$$

$$\frac{du}{dx} = \_\_\_\_\_$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 13.15

Differentiate $y = \left(\frac{1+x}{1+x^2}\right)^3$ with respect to x.

Solution

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1+x}{1+x^2}\right)^3$$

$$= 3\left(\frac{1+x}{1+x^2}\right)^2 \frac{d}{dx}\left(\frac{1+x}{1+x^2}\right)$$

$$= 3\left(\frac{1+x}{1+x^2}\right)^2 \frac{(1+x^2)\frac{d}{dx}(1+x) - (1+x)\frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$$= 3\left(\frac{1+x}{1+x^2}\right)^2 \frac{(1+x^2)(1) - (1+x)(2x)}{(1+x^2)^2}$$

$$= 3\left(\frac{1+x}{1+x^2}\right)^2 \frac{1+x^2-2x-2x^2}{(1+x^2)^2}$$

$$= \frac{3(1+x)^2(1-2x-x^2)}{(1+x^2)^4}$$

Example 13.16

Differentiate $y = \sqrt[3]{x^2-2x+3}$ with respect to x.

Solution

$$\frac{dy}{dx} = \frac{d}{dx}\sqrt[3]{x^2-2x+3}$$

$$= \frac{d}{dx}(x^2-2x+3)^{\frac{1}{3}}$$

$$= \frac{1}{3}(x^2-2x+3)^{-\frac{2}{3}} \frac{d}{dx}(x^2-2x+3)$$

$$= \frac{1}{3}(x^2-2x+3)^{-\frac{2}{3}}(2x-2)$$

$$= \frac{2(x-1)}{3(x^2-2x+3)^{\frac{2}{3}}}$$

Checkpoint 13.13

Differentiate $y = (3x + 2)^4$ with respect to x.

Checkpoint 13.14

Find the derivatives of $y = \dfrac{1}{\sqrt{x^3 + 1}}$ with respect to x.

13.3 Differentiation of Parametric Equations

For a pair of parametric equations $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ with parameter t, we can apply to chain rule to find $\frac{dy}{dx}$:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Example 13.17

Given that $x = 2t - 1$ and $y = 1 - 4t^2$, find $\frac{dy}{dx}$ in terms of x.

Solution

$$\frac{dx}{dt} = 2 \text{ and } \frac{dy}{dt} = -8t$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{-8t}{2}$$

$$= -4t$$

$$= -4\left(\frac{x+1}{2}\right)$$

$$= -2(x+1)$$

Example 13.18

Given that $\begin{cases} x = 2t^2 \\ y = 3 + 5t \end{cases}$, find $\left.\frac{dy}{dx}\right|_{t=3}$.

Solution

$$\frac{dx}{dt} = 4t \text{ and } \frac{dy}{dt} = 5$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{5}{4t}$$

$$\left.\frac{dy}{dx}\right|_{t=3} = \frac{5}{12}$$

Checkpoint 13.15

Given that $\begin{cases} x = 2t - 1 \\ y = t^2 - 1 \end{cases}$, find $\left.\frac{dy}{dx}\right|_{(3,3)}$.

13.4 Differentiation of Inverse Functions

Let $y = f(x)$. If $x = g(y)$, we say that $g(y)$ is the inverse function of $f(x)$.

For example, if $y = f(x) = 2x - 1$, then we have $x = g(y) = \frac{y+1}{2}$. $g(y)$ is the inverse function of $f(x)$.

Let $y = f(x)$ and its inverse function be $x = g(y)$. Then.

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \text{ for } \frac{dy}{dx} \neq 0.$$

Example 13.19

If $x = 3y^2 - 2y + 1$, find $\frac{dy}{dx}$ in terms of y.

Solution

$$\begin{aligned}\frac{dx}{dy} &= \frac{d}{dy}(3y^2 - 2y + 1) \\ &= 6y - 2 \\ \frac{dy}{dx} &= \frac{1}{\frac{dx}{dy}} \\ &= \frac{1}{6y - 2}\end{aligned}$$

Example 13.20

If $x = \dfrac{y}{1+y}$, find $\dfrac{dy}{dx}$.

Solution

$$\frac{dx}{dy} = \frac{d}{dy}\left(\frac{y}{1+y}\right)$$

$$= \frac{(1+y)\dfrac{d}{dy}(y) - y\dfrac{d}{dy}(1+y)}{(1+y)^2}$$

$$= \frac{(1+y) - y(1)}{(1+y)^2}$$

$$= \frac{1}{(1+y)^2}$$

$$\frac{dy}{dx} = \frac{1}{\dfrac{dx}{dy}}$$

$$= (1+y)^2$$

Alternative Solution

$$x = \frac{y}{1+y}$$

$$y = \frac{x}{1-x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{1-x}\right)$$

$$= \frac{(1-x)\dfrac{d}{dx}(x) - x\dfrac{d}{dx}(1-x)}{(1-x)^2}$$

$$= \frac{(1-x) - x(-1)}{(1-x)^2}$$

$$= \frac{1}{(1-x)^2}$$

Checkpoint 13.16

In each of the following, find $\frac{dx}{dy}$ in terms of x.

(a) $y = x^3 - x^2 + 3x$

(b) $y = \sqrt{1 - 3x + x^2}$

13.5 Differentiation of Implicit Functions

13.5.1 Explicit Functions and Implicit Functions

(1) y is an ***explicit function*** of x if dependent variable y of the function is expressed in terms of the independent variable x.

e.g. $y = \sqrt{x^3 + 1}$

(2) y is an ***implicit function*** of x if the equations relating to x and y does not have an explicit subject.

e.g. $x^2 y + 2x = x^3 y$, $xy^2 + 2x = x^2 y^3$

13.5.2 Differentiation of Implicit Functions

To differentiate an implicit function with respect to x, we can either

(1) change the subject to y,

e.g. $x^2 y + 2x = (x+1)^3 y$

$$y = \frac{2x}{(x+1)^3 - x^2},$$

and then use any theorems (Quotient rule, here) to find $\frac{dy}{dx}$, or

(2) differentiate both sides of the equations simultaneously with respect to x, and then make $\frac{dy}{dx}$ as the subject.

Example 13.21

If $x^3 + y^3 - y = 10$, find $\frac{dy}{dx}$.

Solution

$$x^3 + y^3 - y = 10$$

$$\frac{d}{dx}(x^3 + y^3 - y) = \frac{d}{dx}(10)$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(y) = 0$$

$$3x^2 + \frac{d}{dy}(y^3)\frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$3x^2 + 3y^2\frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$3x^2 = (1 - 3y^2)\frac{dy}{dx}$$

$\therefore$ $$\frac{dy}{dx} = \frac{3x^2}{1 - 3y^2}$$

Example 13.22

If $x^2y - 8 = 3y^3 + 2x$, find $\frac{dy}{dx}$.

Solution

$$x^2y - 8 = 3y^3 + 2x$$

$$\frac{d}{dx}(x^2y - 8) = \frac{d}{dx}(3y^3 + 2x)$$

$$x^2\frac{d}{dx}(y) + y\frac{d}{dx}(x^2) - 0 = 3\frac{d}{dx}(y^3) + 2$$

$$x^2\frac{dy}{dx} + y(2x) = 3(3y^2)\frac{dy}{dx} + 2$$

$$x^2\frac{dy}{dx} - 9y^2\frac{dy}{dx} = 2 - 2xy$$

$$(x^2 - 9y^2)\frac{dy}{dx} = 2 - 2xy$$

$$\frac{dy}{dx} = \frac{2 - 2xy}{x^2 - 9y^2}$$

Checkpoint 13.17

In each of the following, find $\frac{dy}{dx}$.

(a) $xy = 4$

(b) $x^3 + y^3 = 6xy^4$

Example 13.23

If $\dfrac{x^2y+1}{3xy^2-1}=1$, find $\left.\dfrac{dy}{dx}\right|_{x=2}$.

Solution

$$\frac{x^2y+1}{3xy^2-1}=1$$

$$x^2y+1=3xy^2-1$$

$$\frac{d}{dx}(x^2y+1)=\frac{d}{dx}(3xy^2-1)$$

$$2x\frac{dy}{dx}+y(2x)+0=3x(2y)\frac{dy}{dx}+3y^2$$

$$(2x-6xy)\frac{dy}{dx}=3y^2-2xy$$

$$\frac{dy}{dx}=\frac{3y^2-2xy}{2x-6xy}$$

When $x = 2$, $\dfrac{(2)^2y+1}{3(2)y^2-1}=1$

$$4y+1=6y^2-1$$

$$6y^2-4y-2=0$$

$$2(y-1)(3y+1)=0$$

$$y=1 \quad \text{or} \quad y=-\frac{1}{3}$$

When $x = 2$ and $y = 1$, $\dfrac{dy}{dx}=\dfrac{3(1)^2-2(2)(1)}{2(2)-6(2)(1)}=\dfrac{1}{8}$

When $x = 2$ and $y = -\dfrac{1}{3}$, $\dfrac{dy}{dx}=\dfrac{3\left(-\frac{1}{3}\right)^2-2(2)\left(-\frac{1}{3}\right)}{2(2)-6(2)\left(-\frac{1}{3}\right)}=\dfrac{5}{24}$

$\therefore \quad \left.\dfrac{dy}{dx}\right|_{x=2}=\dfrac{1}{8} \quad \text{or} \quad \dfrac{5}{24}$

Checkpoint 13.18

Let $xy + y^2 = 6$. Find $\left.\frac{dy}{dx}\right|_{x=1}$

13.6 Second Derivatives

Sometimes in physical application of differentiation, we need to consider the derivative of yet another derivative.

For example, let $y = 3x^2$. Then $\frac{dy}{dx} = 6x$. The derivative of $\frac{dy}{dx}$, i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, is $\frac{d}{dx}(6x) = 6$.

We call $\frac{dy}{dx}$ the ***first derivative*** of *y* with respect to *x*, and

$\frac{d}{dx}\left(\frac{dy}{dx}\right)$ the ***second derivative*** of *y* with respect to *x*.

The second derivative is usually denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$ or y''.

Note that $\frac{d^2y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2$. In the above example, $\frac{d^2y}{dx^2} = 6$ and $\left(\frac{dy}{dx}\right)^2 = (6x)^2 = 36x^2$.

Example 13.24

Let $y = 2x^3 - x^2 + 4x - 7$. Find $\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}}$.

Solution

$$\begin{aligned} y &= 2x^3 - x^2 + 4x - 7 \\ \frac{dy}{dx} &= 2(3x^2) - 2x + 4 \\ &= 6x^2 - 2x + 4 \\ \frac{d^2y}{dx^2} &= 6(2x) - 2 \\ &= 12x - 2 \\ \left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{2}} &= 12\left(\frac{1}{2}\right) - 2 \\ &= 4 \end{aligned}$$

Checkpoint 13.19

Let $y = 3x^2 - \frac{1}{x}$. Find $\frac{d^2y}{dx^2}$.

Example 13.25

Let $y = \frac{1}{1+x}$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and hence show that $(1+x)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$.

Solution

$$y = \frac{1}{1+x} = (1+x)^{-1}$$

$$\frac{dy}{dx} = -(1+x)^{-2}$$

$$= -\frac{1}{(1+x)^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[-(1+x)^{-2}\right]$$

$$= -(-2)(1+x)^{-3}$$

$$= 2(1+x)^{-3}$$

$$= \frac{2}{(1+x)^3}$$

$\therefore$ $(1+x)\frac{d^2y}{dx^2} + 2\frac{dy}{dx}$

$$= (1+x)\left[\frac{2}{(1+x)^3}\right] + 2\left[-\frac{1}{(1+x)^2}\right]$$

$$= \frac{2}{(1+x)^2} - \frac{2}{(1+x)^2}$$

$$= 0$$

Checkpoint 13.20

Let $y = x + \sqrt{x^2 - 1}$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and hence show that $(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$.

Example 13.26

Find $\frac{d^2y}{dx^2}$ if $x^4 + y^4 = 1$.

Solution

$$x^4 + y^4 = 1$$

$$\frac{d}{dx}(x^4) + \frac{d}{dx}(y^4) = 0$$

$$4x^3 + 4y^3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^3}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right)$$

$$= -\frac{y^3\frac{d}{dx}(x^3) - x^3\frac{d}{dx}(y^3)}{(y^3)^3}$$

$$= -\frac{y^3(3x^2) - x^3(3y^2)\frac{dy}{dx}}{y^6}$$

$$= -\frac{3x^2y^3 - 3x^2y^2\left(-\frac{x^3}{y^3}\right)}{y^6}$$

$$= -\frac{3x^2y^4 + 3x^6}{y^7} = -\frac{3x^2(1-x^4) + 3x^6}{y^7}$$

$$= -\frac{3x^2}{y^7}$$

Alternative Solution

$$x^4 + y^4 = 1$$

$$4x^3 + 4y^3\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^3}{y^3}$$

$$\frac{d}{dx}\left[4x^3 + 4y^3\frac{dy}{dx}\right] = 0$$

$$4(3x^2) + 4y^3\frac{d}{dx}\left(\frac{dy}{dx}\right) + 4(3y^2)\frac{dy}{dx}\cdot\frac{dy}{dx} = 0$$

$$12x^2 + 4y^3\frac{d^2y}{dx^2} + 12y^2\left(\frac{dy}{dx}\right)^2 = 0$$

$$3x^2 + y^3\frac{d^2y}{dx^2} + 3y^2\left(-\frac{x^3}{y^3}\right)^2 = 0$$

$$3x^2 + y^3\frac{d^2y}{dx^2} + \frac{3x^6}{y^4} = 0$$

$$3x^2y^4 + y^7\frac{d^2y}{dx^2} + 3x^6 = 0$$

$$\frac{d^2y}{dx^2} = -\frac{3x^2y^4 + 3x^6}{y^7}$$

$$\frac{d^2y}{dx^2} = -\frac{3x^2}{y^7}$$

Checkpoint 13.21

Let $\sqrt{x} - y = 2xy^2$. Find $\frac{d^2y}{dx^2}$.

Exercise 13 Techniques of Differentiation

13.1

1. Write down the derivatives of the following functions with respect to *x*.

   (a) $y = 5x^4$ (b) $y = -4x^{-3}$

   (c) $y = x^{1.2}$ (d) $y = 7x^{-\frac{4}{3}}$

   (e) $y = \dfrac{\pi}{\sqrt{x}}$

2. Find the derivatives $\dfrac{dy}{dx}$ of the following functions.

   (a) $y = 3x^2 - 5x$ (b) $y = 6x^3 - 2x^2 + x - 16$

   (c) $y = \dfrac{x\sqrt{x}}{3x^4}$ (d) $y = x^{-4} + 4x^{-2} - 3x^{-1}$

   (e) $y = 2x^{-\frac{1}{5}} - x^{-\frac{1}{4}} + x^{-\frac{1}{3}}$ (f) $y = (7x+1)(x+3)$

   (g) $y = \dfrac{x^8 + 6x^7 - 18x^2 + 4x}{x^3}$

3. If $f(x) = 3x^2 - x^{-1} + 2$, find $f'(3)$.

4. Let $f(x) = ax + bx^{\frac{1}{2}} + 4x^{\frac{3}{2}}$. If $f'(1) = 11$ and $f'(4) = 28$, find the values of *a* and *b*.

5. Find the derivatives of the following functions with respect to *x*.

   (a) $y = (2-x)(3x^2+1)$ (b) $y = (2x+1)(x^2 - 5x + 7)$

   (c) $y = (2-\sqrt{x})(3x^2+2)$ (d) $y = (x^3 - x^{-2})^2$

   (e) $y = (x^2+1)(2x-3)(x+1)$

6. Find the derivatives of the following functions with respect to *x*.

   (a) $y = \dfrac{2}{x+4}$ (b) $y = \dfrac{3x-5}{x^2+4}$

   (c) $y = \dfrac{2x-3}{x^2+3x+7}$ (d) $f(x) = \dfrac{1}{x+5} + \dfrac{1}{x-5}$

   (e) $f(x) = \dfrac{x^2+\sqrt{x}}{x^2-\sqrt{x}}$ (f) $f(x) = \dfrac{3x^2-2}{x^2-1}(x+3)$

7. If $f(x) = (3x^{-2} - 2x^{-3} + x^{-1})^2$, find $f'(2)$.

8. Given that $f(x)$ and $g(x)$ are differentiable functions such that $f(x) \neq g(x)$. If $f(2) = 3$, $g(2) = 4$, $\frac{d}{dx}[f(x)] = -1$ at $x = 2$ and $\frac{d}{dx}[g(x)] = 2$ at $x = 2$, find the values of $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$ at $x = 2$.

13.2

9. Use the chain rule to find $\frac{dy}{dx}$ in each of the following.

   (a) $y = u(u^2 - 1)$, $u = 2x^3 + 3x^2$

   (b) $y = \sqrt[3]{u}$, $u = \frac{1+x}{x}$

10. Find $\frac{dy}{dx}$ of the following functions.

   (a) $y = (x^2 + 2x - 4)^3$ (b) $y = (x^2 - 4)^{-2}$

   (c) $y = \frac{1}{(x^2 + 3x + 5)^3}$ (d) $y = \sqrt[3]{(4x^2 - 1)^2}$

11. Find the derivatives of the following functions with respect to x.

   (a) $y = (5x + 2)^2\sqrt{5x + 2}$ (b) $y = (3x + 1)^{\frac{2}{3}}(x - 2)^3$

   (c) $y = \frac{3x+1}{\sqrt{2x+3}}$ (d) $y = \left(\frac{x^2 - 2}{3x + 1}\right)$

   (e) $y = \sqrt{\left(\frac{2x-3}{x^2+4}\right)^3}$ (f) $y = \left(\frac{x+5}{x+1}\right)(x^2 + 1)^{\frac{1}{3}}$

12. If $f(x) = \frac{\left(\frac{5x+3}{5x-3}\right)^4}{16}$, find $f'(1)$.

13. Find the derivative of $y = [(x^2 + 3)^3 - 5]^4$ with respect to x.

14. Find the derivative of $y = \sqrt{x + 3\sqrt{x+1}}$ with respect to x.

13.3

15. Find, in terms of x, $\frac{dy}{dx}$ in each of the following.

(a) $x = 3t$, $y = \frac{3}{t}$ (b) $x = 2t^2$, $y = 2t$, where $t > 0$

16. Find, in terms of t, $\frac{dy}{dx}$ in each of the following.

(a) $x = \frac{t^2+3}{9}$, $y = \frac{t^2-4t}{2}$ (b) $x = \sqrt{t^2-1}$, $y = (t^2+1)^2$

(c) $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$

17. If $x = t^3 + 3t - 2$ and $y = t^3 + 2$, find $\frac{dy}{dx}$ at $t = -2$.

13.4

18. Find $\frac{dy}{dx}$ in each of the following.

(a) $x = y + \frac{1}{y}$ (b) $x = \sqrt{y^2 + 2y}$

(c) $x = \frac{y^2-1}{y^2+1}$

19. Find $\frac{dy}{dx}$ in each of the following.

(a) $x = (y-1)^3(2y+3)^2$ (b) $x = \sqrt{\frac{2-y}{y+1}}$

20. Let $3xy^2 - 2xy = 1$. Find $\frac{dy}{dx}$ in terms of y.

13.5

21. Find $\frac{dy}{dx}$ in each of the following.

(a) $4x^2 + y^2 = 16$ (b) $3xy + y^2 = 1$

(c) $x^2 - 2xy + 3y^2 = 0$ (d) $y = \frac{x}{x+y}$

(e) $x(x-2y)^2 = 2$ (f) $\sqrt{x+y} + \sqrt{xy} = 6$

22. Given that $x^3 + y^3 - 6xy + 1 = 0$, find $\frac{dy}{dx}$ at $x = 2$.

23. Given the equation $x^2 + y^2 = 25$ representing a circle with radius 5 and centre at the origin. The point (a, b) lies on the circle and $\frac{dy}{dx} = -\frac{4}{3}$.

(a) Show that $a = \frac{4}{3}b$.

(b) Find the possible values of a and b.

13.6

24. Find $\frac{d^2y}{dx^2}$ of the following functions.

(a) $y = 2x^4 - x^3$ (b) $y = 5 - 3x^{-1} + 2x^{-2}$

(c) $y = (3x-4)^{-5}$ (d) $y = \frac{x^2+1}{x+1}$

(e) $y = (x+1)^3(x-3)^2$ (f) $\sqrt{x} + y^2 = 2$

25. If $f(x) = (2-3x)^{-\frac{1}{2}}$, find $\frac{d}{dx}[f(x)]$ and $\frac{d^2}{dx^2}[f(x)]$ at $x = 0$.

26. If $y = \sqrt{3x+2}$, show that $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$.

27. Given that $x = 2t - 1,\ y = t^2 - 1$, find $\frac{d^2y}{dx^2}$.