Chapter 12 Limits and Derivatives

12.1 Limit of a Function

Definition of the limit of a function:

If $f(x)$ approaches a fixed number L as x approaches a, then we say that L is the limit of $f(x)$ as x approaches a and is denoted by

$$\lim_{x \to a} f(x) = L$$

Definition of continuity of a function:

A function $f(x)$ is continuous at $x = a$ if and only if $\lim_{x \to a} f(x) = f(a)$.

12.2 Theorems on Limits

When $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} g(x)$ exists, then

(1) $\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$;

(2) $\lim_{x \to a}[k \cdot f(x)] = k \lim_{x \to a} f(x)$, where k is a real constant.;

(3) $\lim_{x \to a}[f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$;

(4) $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided that $\lim_{x \to a} g(x) \neq 0$.

Example 12.1

Given $f(x) = 3x - 4$ and $g(x) = 2x + 3$, evaluate

(a) $\lim_{x \to 1}[f(x) - g(x)]$

(b) $\lim_{x \to 2}[f(x) \cdot g(x)]$

(c) $\lim_{x \to -1}\left[\frac{f(x)}{g(x)}\right]$

Solution

(a)
$$\begin{aligned}\lim_{x \to 1}[f(x) - g(x)] &= \lim_{x \to 1} f(x) - \lim_{x \to 1} g(x) \\ &= \lim_{x \to 1}(3x - 4) - \lim_{x \to 1}(2x + 3) \\ &= (3 \cdot 1 - 4) - (2 \cdot 1 + 3) \\ &= 6\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x \to 2}[f(x) \cdot g(x)] &= \lim_{x \to 2} f(x) \cdot \lim_{x \to 2} g(x) \\ &= \lim_{x \to 2}(3x - 4) \cdot \lim_{x \to 2}(2x + 3) \\ &= (3 \cdot 2 - 4)(2 \cdot 2 + 3) \\ &= 14\end{aligned}$$

(c)
$$\begin{aligned}\lim_{x \to -1}\left[\frac{f(x)}{g(x)}\right] &= \frac{\lim_{x \to -1} f(x)}{\lim_{x \to -1} g(x)} \\ &= \frac{\lim_{x \to -1}(3x - 4)}{\lim_{x \to -1}(2x + 3)} \\ &= \frac{3(-1) - 4}{2(-1) + 3} \\ &= -7\end{aligned}$$

Checkpoint 12.1

Given $F(x) = -2x$ and $G(x) = x - 5$, evaluate

(a) $\lim_{x \to 0}[F(x) - G(x)]$

(b) $\lim_{x \to \frac{1}{2}}[3F(x) \cdot G(x)]$

(c) $\lim_{x \to -2}\left[\frac{G(x)}{F(x)}\right]$

Example 12.2

Evaluate $\lim_{x\to 0}\frac{x^2-x}{x^2+2x}$.

Solution

$$\begin{aligned}\lim_{x\to 0}\frac{x^2-x}{x^2+2x} &= \lim_{x\to 0}\frac{x(x-1)}{x(x+2)}\\ &= \lim_{x\to 0}\frac{x-1}{x+2}\\ &= -\frac{1}{2}\end{aligned}$$

Example 12.3

Evaluate $\lim_{x\to 2}\frac{1}{x-2}$.

Solution

Here, $\lim_{x\to 2}(x-2)=0$ and as the numerator is the constant 1, when x approaches 2, $\frac{1}{x-2}$ tends to $\frac{1}{0}$ which is undefined.

$\therefore$ $\lim_{x\to 2}\frac{1}{x-2}$ does not exist.

Checkpoint 12.2

Evaluate the following limits.

(a) $\lim_{x\to 1}(2-5x)$

(b) $\lim_{x\to 3}\frac{2x-6}{x-3}$

Example 12.4

Evaluate $\lim_{x\to 3}\frac{x-3}{\sqrt{x+1}-2}$.

Solution

$$\begin{aligned}
\lim_{x\to 3}\frac{x-3}{\sqrt{x+1}-2} &= \lim_{x\to 3}\frac{(x-3)(\sqrt{x+1}+2)}{(\sqrt{x+1}-2)(\sqrt{x+1}+2)} \\
&= \lim_{x\to 3}\frac{(x-3)(\sqrt{x+1}+2)}{(x+1)-4} \\
&= \lim_{x\to 3}\sqrt{x+1}+2 \\
&= \sqrt{3+1}+2 \\
&= 4
\end{aligned}$$

Example 12.5

Evaluate $\lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{x}$.

Solution

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{x} &= \lim_{x\to 0}\frac{(\sqrt{1+x^2}-1)(\sqrt{1+x^2}+1)}{x(\sqrt{1+x^2}+1)} \\
&= \lim_{x\to 0}\frac{1+x^2-1}{x(\sqrt{1+x^2}+1)} \\
&= \lim_{x\to 0}\frac{x}{\sqrt{1+x^2}+1} \\
&= \frac{\lim_{x\to 0}x}{\lim_{x\to 0}(\sqrt{1+x^2}+1)} \\
&= 0
\end{aligned}$$

Checkpoint 12.3

Evaluate the following limits.

(a) $\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1}$

(b) $\lim_{x \to 8} \frac{x-8}{\sqrt{x+1}-3}$

(c) $\lim_{x \to -3} \frac{\sqrt{1-x}-2}{x+3}$

12.3 Limits at Infinity

If the value of a function $f(x)$ approaches a number L as the value of x approaches positive infinity or negative infinity, then correspondingly we have

$$\lim_{x \to +\infty} f(x) = L \qquad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$

In particular, $\lim_{x \to \pm\infty} \frac{1}{x} = 0$.

Example 12.6

Evaluate

(a) $\lim_{x \to \infty} \frac{x+3}{x-2}$

(b) $\lim_{x \to -\infty} \frac{x^2 + x - 1}{2x^3 + 4}$

Solution

(a)
$$\begin{aligned}\lim_{x \to \infty} \frac{x+3}{x-2} &= \lim_{x \to \infty} \frac{1 + \frac{3}{x}}{1 - \frac{2}{x}} \\ &= \frac{1+0}{1-0} \\ &= 1\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x \to -\infty} \frac{x^2 + x - 1}{2x^3 + 4} &= \lim_{x \to -\infty} \frac{\frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3}}{2 + \frac{4}{x^3}} \\ &= \frac{0+0-0}{2+0} \\ &= 0\end{aligned}$$

Checkpoint 12.4

Evaluate the following limits. If the limit does not exist, say so.

(a) $\lim_{x\to\infty}\frac{4x+1}{x-3}$

(b) $\lim_{x\to\infty}\frac{x^3}{x^2+1}$

Example 12.7

Evaluate $\lim_{x\to\infty}(\sqrt{x}-\sqrt{x+1})$.

Solution

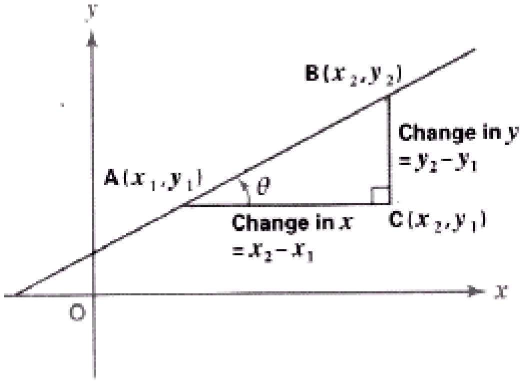
$$\begin{aligned}\lim_{x\to\infty}(\sqrt{x}-\sqrt{x+1}) &= \lim_{x\to\infty}\frac{(\sqrt{x}-\sqrt{x-1})(\sqrt{x}+\sqrt{x+1})}{\sqrt{x}+\sqrt{x+1}}\\ &= \lim_{x\to\infty}\frac{x-(x+1)}{\sqrt{x}+\sqrt{x+1}}\\ &= \lim_{x\to\infty}\frac{-1}{\sqrt{x}+\sqrt{x+1}}\\ &= 0\end{aligned}$$

Checkpoint 12.5

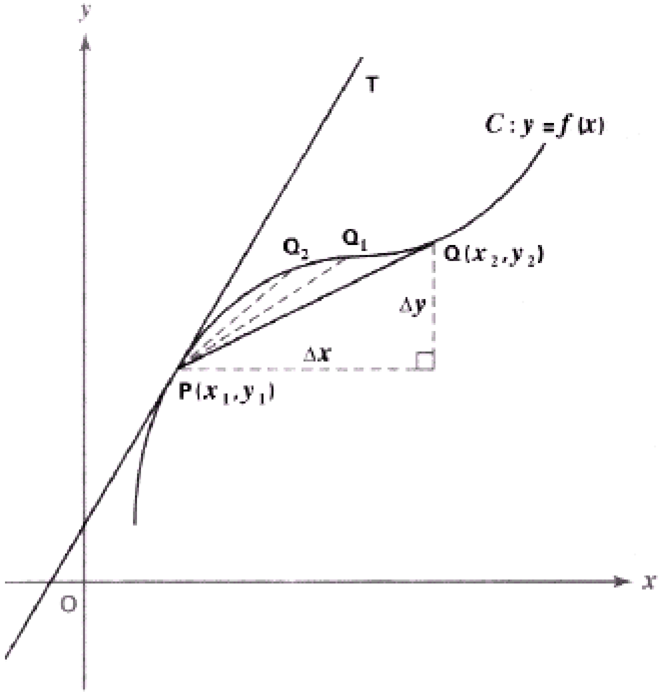
Evaluate $\lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x+2})$.

12.4 Slope of a Curve

Recall that the slope of a straight line is defined as $\dfrac{y_2 - y_1}{x_2 - x_1}$



In the figure, PT is the tangent of the curve C at the point P. The slope of PT is called the slope of the curve at P.



Denote $x_2 - x_1$ by Δx and $y_2 - y_1$ by Δy. The slope of the curve at the point $\mathrm{P}(x_1, y_1)$ is given by

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

Example 12.8

(a) Find the slope of the tangent to the curve $y = x^2$ at any point (x, y).

(b) Hence find the slope of the curve $y = x^2$ at (1, 1).

Solution

(a) Let $f(x) = x^2$.

The slope of the tangent at (x, y)

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)(2x + \Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x)$$

$$= 2x$$

(b) The slope of the curve at (1, 1) = 2(1) = 2.

Checkpoint 12.6

(a) Find the slope of the tangent to the curve $y = 2x^2 - x$ at any point (x, y).

(b) Hence find the slope of the curve $y = 2x^2 - x$ at (2, 6).

12.5 Derivative of a Function

The derivative of the function $y = f(x)$ with respect to x is defined as

$$\frac{dy}{dx} \text{ (or } f'(x)) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}, \text{ if the limit exists.}$$

Example 12.9

Find the derivative of $y = x^3$ with respect to x from first principles.

Solution

[From first principles means using the definition of derivative to find $\frac{dy}{dx}$.]

Let $f(x) = x^3$.

$$\begin{aligned} f(x+\Delta x) &= (x+\Delta x)^3 \\ &= x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 \\ &= f(x+\Delta x) - f(x) = 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 \\ \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \frac{\Delta x[3x^2 + 3x(\Delta x) + (\Delta x)^2]}{\Delta x}, \ \Delta x \neq 0 \\ &= 3x^2 + 3x(\Delta x) + (\Delta x)^2 \end{aligned}$$

$$\therefore \quad \begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} [3x^2 + 3x(\Delta x) + (\Delta x)^2] \\ &= 3x^2 \end{aligned}$$

Checkpoint 12.7

Find the derivative of $y = f(x) = \frac{1}{x+4}$ with respect to x for $x \neq -4$ from first principles.

Example 12.10

Let $f(x)=\sqrt{3x}$. Find the derivative of $f(x)$ with respect to x from first principles and hence find the slope of $f(x)$ at $x = 3$.

Solution

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\sqrt{3(x+\Delta x)}-\sqrt{3x}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{[\sqrt{3(x+\Delta x)}-\sqrt{3x}][\sqrt{3(x+\Delta x)}+\sqrt{3x}]}{\Delta x[\sqrt{3(x+\Delta x)}+\sqrt{3}x]}$$

$$=\lim_{\Delta x\to 0}\frac{3(x+\Delta x)-3x}{\Delta x[\sqrt{3(x+\Delta x)}+\sqrt{3}x]}$$

$$=\lim_{\Delta x\to 0}\frac{3\Delta x}{\Delta x[\sqrt{3(x+\Delta x)}+\sqrt{3}x]}$$

$$=\lim_{\Delta x\to 0}\frac{3}{[\sqrt{3(x+\Delta x)}+\sqrt{3}x]}$$

$$=\frac{3}{\sqrt{3x}+\sqrt{3x}}$$

$$=\frac{3}{2\sqrt{3}x}$$

$$f'(3)=\frac{3}{2\sqrt{3(3)}}$$

$$=\frac{1}{2}$$

Checkpoint 12.8

Find the derivative of $y = \sqrt{1 - x}$ with respect to *x* for $x \geq 1$ from first principles.

Exercise 12 Limits and Derivatives

12.2

1. Evaluate the following limits.

(a) $\lim_{x\to 5} 8$ (b) $\lim_{x\to 1}(x^3 + 2x + 1)$

(c) $\lim_{x\to 0}(x-4)^2$ (d) $\lim_{x\to 0}\sqrt[3]{x}$

(e) $\lim_{x\to 2}\frac{x+1}{x+3}$

2. Evaluate the following limits. If the limit does not exist, say so.

(a) $\lim_{x\to 0}\frac{x^3 + 2x^2 + 3x}{x}$ (b) $\lim_{x\to -2}\frac{x^2 + 5x + 6}{x^2 + 6x + 8}$

(c) $\lim_{x\to 7}\frac{x-7}{\sqrt{x+2}-3}$ (d) $\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$

(e) $\lim_{x\to -3}\frac{1}{x+3}$ (f) $\lim_{x\to 2}\frac{x^2 - 4}{\sqrt[3]{x-2}}$

(g) $\lim_{x\to -3}\frac{x^2 + 2x}{x^3}$

3. Evaluate $\lim_{x\to 5}\frac{x^3 - 125}{x^2 - 25}$.

12.3

4. Evaluate the following limits. If the limit does not exist, say so.

(a) $\lim_{x\to\infty}\frac{\sqrt{2}}{x}$ (b) $\lim_{x\to\infty}\frac{2x+3}{3x+1}$

(c) $\lim_{x\to\infty}\frac{x^2 + 3x + 2}{2x^2 - 2x + 1}$ (d) $\lim_{x\to\infty}\frac{3 - 2x - x^3}{1 + 4x^2}$

(e) $\lim_{x\to\infty}\frac{2x(3x+1)}{(2x+1)(x-3)}$

5. Evaluate the following limits. If the limit does not exist, say so.

(a) $\lim_{x\to\infty}\frac{\sqrt{x^2+4}}{x+4}$ (b) $\lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x-1})$

(c) $\lim_{x\to\infty}(\sqrt{x^2+x}-x)$

12.4, 12.5

6. Given the function $f(x)=x^2-x+2$, find

(a) $f(1)$,

(b) $f(1+\Delta x)$,

(c) the slope of the curve $y=x^2-x+2$ at the point (1, 2).

7. Find the gradient of the tangent to the curve $y=x^3-2x$ at (0, 0).

8. Find the derivative $\frac{dy}{dx}$ of $y=4x+3$ from first principles.

9. Find the derivative $\frac{dy}{dx}$ of $y=\frac{1}{5+x}$ from first principles.

10. Find the derivative $f'(x)$ of $f(x)=x^3+x^2$ from first principles.

11. Find the derivative $f'(x)$ of $f(x)=\sqrt{4-x^2}$ from first principles.

12. Find the derivative of the function $y=\sqrt[3]{x}$ with respect to x from first principles.

13. Find the derivative y' of $y=\frac{x}{x+2}$ from first principles.

14. Given $y=\frac{2}{x^3}$. Find $\frac{dy}{dx}$ and hence find the value of $\left.\frac{dy}{dx}\right|_{x=2}$.

15. Given $f(x)=\frac{1}{\sqrt{x+1}}$. Find $f'(x)$ and hence the slope of $f(x)$ at x = 3.