Chapter 11 Loci and Parametric Equations

11.1 Loci

11.1.1 Meaning of Loci

The path traced by a movable point under certain conditions is called the locus of the movable point.

11.1.2 Equation of Locus

If the conditions on a movable point P(x, y) are given, then the equation expressing the relation between x and y is called the equation of the locus of P.

e.g. The equation $y = 2$ represents the locus of a movable point which is always 2 units from the x-axis.

Example 11.1

Find the equation of the locus of a variable point P such that its distance from the x-axis is equal to a distance d.

Solution

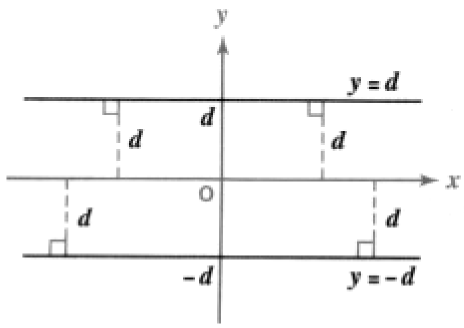
Let the coordinates of P be (x, y). Then the distance from P to the x-axis is $|y|$.

$$|y| = d$$

$$y = d \quad \text{or} \quad -d$$

$\therefore$ The equation of the locus is $y = d$ or $y = -d$.

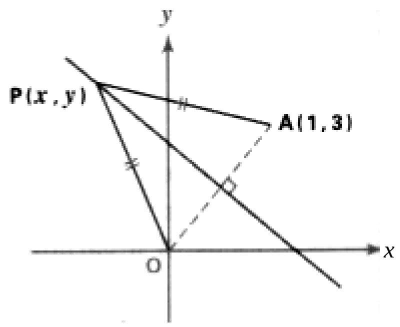
From the above equations, we know that the locus consists of two horizontal lines, one on each side of the x-axis. The distance of each horizontal line from the x-axis is d units.



Very often, we do not know the shape of the locus beforehand. However, once we know the condition for the locus and find the equation and find the equation of the locus, then we can draw the locus and identify its shape easily.

Example 11.2

Find the equation of the locus of a point P whose distances from the point O and A(1, 3) are equal.



Solution

Let the coordinates of P be (x, y).

Distance between P and O = $\sqrt{x^2 + y^2}$

Distance between P and A = $\sqrt{(x-1)^2 + (y-3)^2}$

$$\therefore \quad \sqrt{x^2 + y^2} = \sqrt{(x-1)^2 + (y-3)^2}$$

$$x^2 + y^2 = (x-1)^2 + (y-3)^2$$

$$x^2 + y^2 = x^2 - 2x + 1 + y^2 - 6y + 9$$

$$2x + 6y - 10 = 0$$

$$x + 3y - 5 = 0$$

$\therefore$ The equation of the locus of P is $x + 3y - 5 = 0$.

Checkpoint 11.1

A is the point (1, 2) and P(x, y) is a variable point such that the distance PA is always equal to 2. Find the equation of the locus of P.

Checkpoint 11.2

Given two points O(0, 0) and A(2, 2). P is a variable point such that PA = PO. Find the equation of the locus of P.

Example 11.3

S and T are points (–1, –2) and (–1, 0) respectively. P is a variable point such that SP + TP = 4. Find the equation of the locus of P.

Solution

Let the coordinates of P be (x, y).

$$
\begin{aligned}
\text{SP} + \text{TP} &= 4 \\
\sqrt{(x+1)^2 + (y+2)^2} + \sqrt{(x+1)^2 + y^2} &= 4 \\
\sqrt{(x+1)^2 + (y+2)^2} &= 4 - \sqrt{(x+1)^2 + y^2} \\
(x+1)^2 + (y+2)^2 &= 16 - 8\sqrt{(x+1)^2 + y^2} + (x+1)^2 + y^2 \\
4y - 12 &= -8\sqrt{(x+1)^2 + y^2} \\
y - 3 &= -2\sqrt{(x+1)^2 + y^2} \\
(y-3)^2 &= 4[(x+1)^2 + y^2] \\
y^2 - 6y + 9 &= 4x^2 + 8x + 4 + 4y^2 \\
4x^2 + 3y^2 + 8x + 6y - 5 &= 0
\end{aligned}
$$

∴ The equation of the locus of P is $4x^2 + 3y^2 + 8x + 6y - 5 = 0$.

Checkpoint 11.3

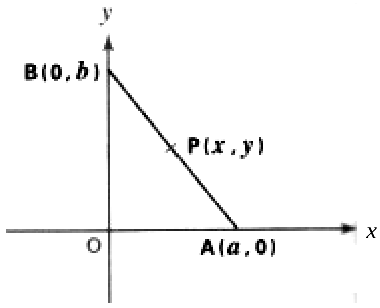
A and B are the points (–4, 0) and (4, 0) respectively. P(x, y) is a variable point such that PA + PB = 12. Find the equation of the locus of P in the form $ax^2 + by^2 + cx + dy + e = 0$.

Example 11.4

A is a variable point on the *x*-axis and B is a variable point on the *y*-axis. P is the mid-point of AB. If the length of AB is equal to 4, find the equation of the locus of P.

Solution

Let the coordinates of A, B and P be $(a, 0)$, $(0, b)$ and (x, y) respectively.



$\therefore$ Distance between A and B = $\sqrt{a^2+b^2}$

$$\sqrt{a^2+b^2}=4$$

$$a^2+b^2=16 \qquad \ldots\ldots(1)$$

P is the mid-point of AB. We have

$$\begin{cases} a=2x & \ldots\ldots(2) \\ b=2y & \ldots\ldots(3) \end{cases}$$

Substituting (2), (3) into (1), we have

$$(2x)^2+(2y)^2=16$$

i.e. $x^2+y^2=4$

$\therefore$ The equation of the locus of P is $x^2+y^2=4$.

Example 11.5

A and B are variable points on the lines $y=-1$ and $x=2y-2$ respectively. If the length of AB is always equal to 3, find the equation of the locus of the mid-point of AB.

Solution

Let the coordinates of A, B and the mid-point of AB, P, be $(s, -1)$, (p, q) and (x, y) respectively.

Then $\sqrt{(p-s)^2+(q+1)^2}=3$

i.e. $(p-s)^2+(q+1)^2=9 \qquad \ldots\ldots(1)$

Also
$$p=2q-2 \qquad \ldots\ldots(2)$$

$$x=\frac{p+s}{2} \qquad \ldots\ldots(3)$$

$$y=\frac{q-1}{2} \qquad \ldots\ldots(4)$$

$$q=2y+1 \qquad \ldots\ldots(5)$$

Substituting (5) into (2), we have

$$p=2(2y+1)-2$$

$$p=4y \qquad \ldots\ldots(6)$$

Substituting (6) into (3), we have

$$x = \frac{4y + s}{2}$$

$$s = 2x - 4y \qquad \ldots\ldots(7)$$

Substituting (5), (6), (7) into (1), we have

$$[4y - (2x - 4y)]^2 + (2y + 1 + 1)^2 = 9$$

$$(8y - 2x)^2 + (2y + 2)^2 = 9$$

$$4x^2 - 32xy + 68y^2 + 8y - 5 = 0$$

$\therefore$ The equation of the locus of the mid-point of AB is $4x^2 - 32xy + 68y^2 + 8y - 5 = 0$.

Checkpoint 11.4

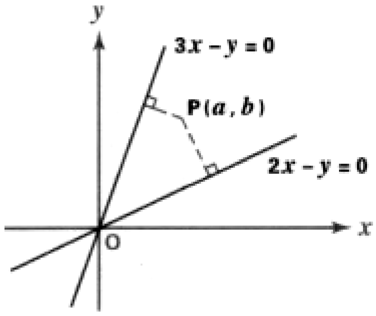
A is a variable point on the line $2x + 3y = 1$ and B is the point (3, 0). P is the mid-point of AB. Find the equation of the locus of P.

Example 11.6

Find the equation of the locus of a point P such that its distances from the two lines $2x - y = 0$ and $3x - y = 0$ are in the ratio $\sqrt{2}:1$.

Solution

Let the coordinates of P be (a, b).



Distance from P to the line $2x - y = 0$ is $\left|\dfrac{2a-b}{\sqrt{2^2+(-1)^2}}\right| = \left|\dfrac{2a-b}{\sqrt{5}}\right|$.

Distance from P to the line $3x - y = 0$ is $\left|\dfrac{3a-b}{\sqrt{3^2+(-1)^2}}\right| = \left|\dfrac{3a-b}{\sqrt{10}}\right|$.

$$\therefore \quad \left|\frac{2a-b}{\sqrt{5}}\right| \left|\frac{3a-b}{\sqrt{10}}\right| = \sqrt{2}:1$$

$$\frac{\left|\frac{2a-b}{\sqrt{5}}\right|}{\left|\frac{3a-b}{\sqrt{10}}\right|} = \sqrt{2}$$

$$\left|\frac{2a-b}{3a-b}\right|\sqrt{2} = \sqrt{2}$$

$$\left|\frac{2a-b}{3a-b}\right| = 1$$

$$\frac{2a-b}{3a-b} = \pm 1$$

When $\dfrac{2a-b}{3a-b} = 1$,

$$2a - b = 3a - b$$

$$a = 0$$

When $\dfrac{2a-b}{3a-b} = -1$,

$$2a - b = -(3a - b)$$

$$5a - 2b = 0$$

$\therefore$ The equation of the locus of P is $x = 0$ or $5x - 2y = 0$ and the point P does not include the origin (0, 0).

Note that if $b = 2a$ or $b = 3a$, the ratio will become zero or undefined. Thus $y \neq 2x$ and $y \neq 3x$; (0, 0) is then excluded from the locus of P.

Checkpoint 11.5

Given two lines

$$L_1 : 3x - y + 1 = 0 \text{ and } L_2 : 2x - y + 1 = 0.$$

Let P(*x*, *y*) be a variable point and d_1, d_2 be the perpendicular distances from P to L_1, L_2 respectively. Find the equation of the locus of P such that

(a) $d_1 = d_2$;

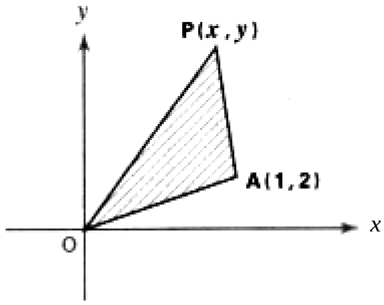
(b) $d_1 : d_2 = 1 : 3\sqrt{2}$.

Example 11.7

Given that A is the point (1, 2). P is a variable point such that the area of ΔPOA $=\frac{5}{2}$, where O is the origin. Find the equation of the locus of P.

Solution

Let the coordinates of P be (x, y).



$$\frac{1}{2}\begin{vmatrix} 1 & 2 \\ x & y \\ 0 & 0 \\ 1 & 2 \end{vmatrix} = \frac{1}{2}(y-2x)$$

∴ Area of $\Delta POA = \frac{1}{2}|y-2x|$

i.e. $\frac{1}{2}(y-2x)=5$ or $\frac{1}{2}(y-2x)=-5$

$y-2x=5$ or $y-2x=-5$

∴ The equation of the locus of P is $2x-y-5=0$ or $2x-y+5=0$.

Checkpoint 11.6

A(–2, 1) and B(3, 4) are two vertices of ΔPAB. If P is a moving point and the area of ΔPAB is 6 square units, find the equation of the locus of P.

11.2 Concept of Parametric Equations

The x- and y-coordinates of a point on a locus may be expressed in terms of another variable,

e.g. $\begin{cases} x = 2\cos\theta \\ y = 2\sin\theta \end{cases}$, $\begin{cases} x = t+1 \\ y = t+2 \end{cases}$.

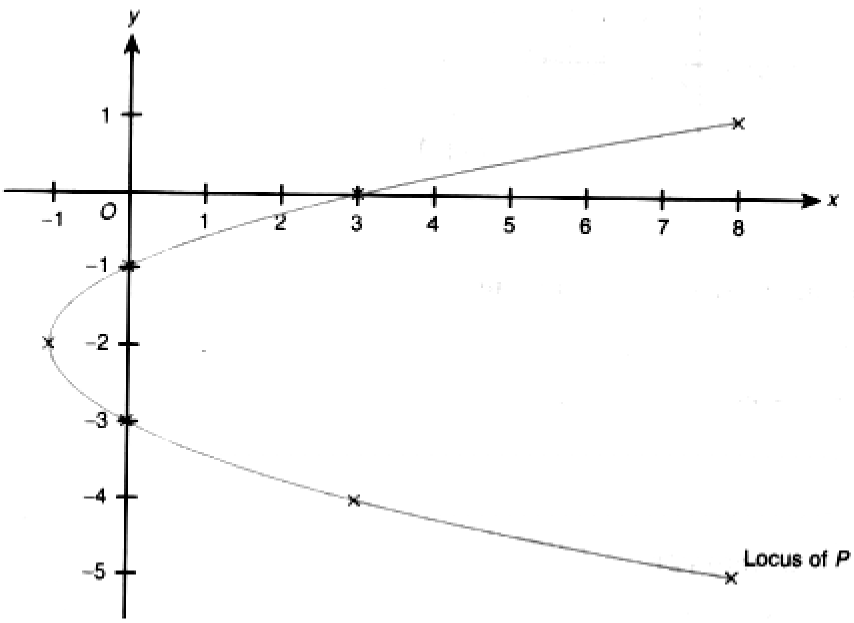
Such equations are called parametric equations of the locus and the variable used is called the parameter. Hence θ and t each represents the parameter of a parametric equation. That is to say, both the x-coordinate and y-coordinate of the point are expressed as functions of t.

Example 11.8

Plot the locus of the parametric equations $\begin{cases} x = t^2 + 2t \\ y = t - 1 \end{cases}$.

Solution

t	–4	–3	–2	–1	0	1	2
x	8	3	0	–1	0	3	8
y	–5	–4	–3	–2	–1	0	1



Example 11.9

Convert the parametric equations $\begin{cases} x = \sqrt{3 - t^2} \\ y = t - 3 \end{cases}$ into an equation in *x* and *y*, where $0 \le t \le \sqrt{3}$.

Solution

$$\begin{cases} x = \sqrt{3 - t^2} \text{ ......(1)} \\ y = t - 3 \quad \text{......(2)} \end{cases}$$

$\because \quad 0 \le t \le \sqrt{3}$

$\therefore \quad \sqrt{3 - (\sqrt{3})^2} \le \sqrt{3 - t^2} \le \sqrt{3 - 0^2}$

i.e. $\quad 0 \le x \le \sqrt{3}$

From (2), $t = y + 3$......(3)

Substituting (3) into (1), we have

$$\begin{aligned} x &= \sqrt{3 - (y + 3)^2} \\ &= \sqrt{3 - y^2 - 6y - 9} \\ x^2 &= 3 - y^2 - 6y - 9 \end{aligned}$$

$$x^2 + y^2 + 6y + 6 = 0, \text{ where } 0 \le x \le \sqrt{3}$$

Checkpoint 11.7

Convert the parametric equations $\begin{cases} x = 2t + 1 \\ y = t^2 + t \end{cases}$ into an equation in *x* and *y*.

Example 11.10

Convert the parametric equations $\begin{cases} x = 2\cos t \\ y = 3\sin t \end{cases}$ into an equation in *x* and *y*.

Solution

$$\begin{cases} x = 2\cos t \text{ ......(1)} \\ y = 3\sin t \text{ ......(2)} \end{cases}$$

From (1), $\cos t = \dfrac{x}{2}$

From (2), $\sin t = \dfrac{y}{3}$

$\because \quad \sin^2 t + \cos^2 t = 1$

$\therefore \quad \left(\dfrac{x}{2}\right) + \left(\dfrac{y}{3}\right)^2 = 1$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

Checkpoint 11.8

Convert the parametric equations $\begin{cases} x = \tan\theta - 1 \\ y = \sec\theta \end{cases}$ into an equation in *x* and *y*.

Checkpoint 11.9

Convert the parametric equations $\begin{cases} x = t^2 + 2t \\ y = 3t^2 + t \end{cases}$ into an equation in x and y.

Example 11.11

Obtain a set of parametric equations for the curve $y^2 = 3x$ by putting

(a) $y = t$, (b) $y = 3t$.

Solution

(a) Putting $y = t$, we have

$$t^2 = 3x$$

$$x = \frac{t^2}{3}$$

∴ The parametric equations are $\begin{cases} x = \dfrac{t^2}{3} \\ y = t \end{cases}$.

(b) Putting $y = 3t$, we have

$$(3t)^2 = 3x$$

$$x = 3t^2$$

∴ The parametric equations are $\begin{cases} x = 3t^2 \\ y = t \end{cases}$.

Checkpoint 11.10

Obtain a set of parametric equations for the curve $y=(x-1)^3$ by putting $x=t+1$.

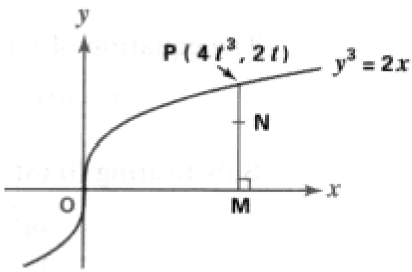
11.3 Using Parameters in Locus Problems

For some locus problems, it may be easier to find the equation of the locus by obtaining parametric equations of the locus first.

Example 11.12

P($4t^3$, $2t$) is a variable point on the curve $y^3=2x$. The perpendicular from P to the *x*-axis cuts the *x*-axis at M. N is a point on PM such that PN : NM = 1 : 2. Find the equation of the locus of N.

Solution



Let the coordinates of N be (x, y).

$\because$ The coordinates of P are ($4t^3$, $2t$).

$\therefore$ The coordinates of M are ($4t^3$, 0).

Then $x=4t^3$ ……(1)

$$y=\frac{0+2(2t)}{1+2}$$

$$=\frac{4}{3}t$$

$$t=\frac{3}{4}y \quad \text{……(2)}$$

Substituting (2) into (1), we have

$$x=4\left(\frac{3}{4}y\right)^3$$

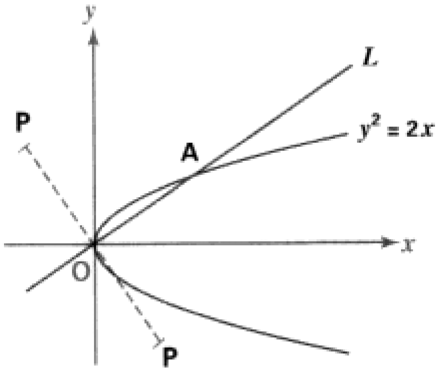
$$16x=27y^3$$

At (0, 0), the distance between the points P and M is zero.

$\therefore$ The equation of the locus of N is $16x=27y^3$, where $x\neq 0$, $y\neq 0$.

Example 11.13

In the figure, L is a variable line passing through the origin O. It cuts the curve $y^2 = 2x$ at O and A. P is another point such that OP = OA and OP $\perp$ OA. Find the equation of the locus of P.



Solution

Let the coordinates of P be (a, b) and the slope of L be m, where $m \neq 0$.
The equation of L is $y = mx$ ……(1)
Substituting (1) into the equation $y^2 = 2x$, we have

$$m^2x^2 = 2x$$
$$m^2x^2 - 2x = 0$$
$$x(m^2x - 2) = 0$$
$$x = 0 \text{ (rejected)} \quad \text{or} \quad \frac{2}{m^2}$$

From (1), when $x = \dfrac{2}{m^2}$, $y = m\left(\dfrac{2}{m^2}\right) = \dfrac{2}{m}$,

the coordinates of A are $\left(\dfrac{2}{m^2}, \dfrac{2}{m}\right)$.

$\because$ OP $\perp$ OA

$\therefore$ The equation of OP is $y = -\dfrac{1}{m}x$.

$\therefore$ $b = -\dfrac{1}{m}a$

$$m = -\frac{a}{b} \text{ ......(2)}$$

$$\text{OA} = \text{OP}$$

$$\sqrt{\left(\frac{2}{m^2}\right)^2 + \left(\frac{2}{m}\right)^2} = \sqrt{a^2 + b^2}$$

$$\frac{4}{m^4} + \frac{4}{m^2} = a^2 + b^2 \text{ ......(3)}$$

Substituting (2) into (3), we have

$$\frac{4}{\left(-\frac{a}{b}\right)^4}+\frac{4}{\left(-\frac{a}{b}\right)^2}=a^2+b^2$$

$$4b^4+4a^2b^2=a^6+a^4b^2$$

$$4b^2(b^2+a^2)=a^4(a^2+b^2)$$

$$(4b^2-a^4)(a^2+b^2)=0$$

$$4b^2-a^4=0 \quad \text{or} \quad a^2+b^2=0\,\text{(rejected)}$$

$a^2+b^2=0$ implies a = 0 and b = 0. Then P is simply the origin (0, 0).

$\therefore$ The equation of the locus of P is $4y^2-x^4=0$, where $x\neq 0$, $y\neq 0$.

Checkpoint 11.11

P(t^2, 2t) is a variable point on the curve $y^2=4x$. M is the point (0, 1). Find the equation of the locus of the mid-point of MP.

Checkpoint 11.12

$P(3t, 3t^2)$ is a variable point on the curve $3y = x^2$. PX is a vertical line with P and X on opposite sides of the *x*-axis. PX cuts the *x*-axis at M such that 2PM = MX. Find the equation of the locus of X as P varies.

Example 11.14

L_1 is a line passing through the origin with variable slope m ($m \neq 0$). L_2 is a line passing through (3, 0) and perpendicular to L_1. As m varies, find the equation of the locus of the point of intersection of L_1 and L_2.

Solution

Let the coordinates of the point of intersection be (a, b).

The equation of L_1 is $y = mx$ ……(1)

The equation of L_2 is $y = -\frac{1}{m}(x-3)$ ……(2)

Substituting (1) into (2), we have

$$mx = -\frac{1}{m}(x-3)$$

$$m^2 x = -x + 3$$

$$x = \frac{3}{m^2+1}$$

$$a = \frac{3}{m^2+1} \text{ ......(3)}$$

$\because$ (a, b) lies on L_1,

$$b = ma$$

$$m = \frac{b}{a} \text{ ......(4)}$$

Substituting (4) into (3), we have

$$a = \frac{3}{\left(\frac{b}{a}\right)^2 + 1}$$

$$b^2 + a^2 = 3a$$

$\therefore$ The equation of the locus of the point of intersection of L_1 and L_2 is $x^2 + y^2 - 3x = 0$.

Checkpoint 11.13

L_1 is a variable line passing through (0, 1) with slope *t*, where $t \neq -1$. L_2 is another variable line passing through (1, 0) with slope $\frac{1}{t}$. Find the equation of the locus of the point of intersection of L_1 and L_2.

Exercise 11 Loci and Parametric Equations

11.1

1. P(x, y) is a variable point on the line L: $x - 1 = 0$. Let M be the mid-point between P and the origin (0, 0). Find the equation of the locus of M.

2. S and T are two variable points on the lines $x = 0$ and $x - 2y = 0$ respectively such that the length of ST is always equal to 3. Find the equation of the locus of the mid-point of ST.

3. L is a variable straight line parallel to $x - y = 0$. If L cuts the x-axis at a point M and the y-axis at a point N, find the equation of the locus of the mid-point of MN.

4. Given two points O(0, 0) and A(4, 0). P(x, y) is a variable point such that PO + PA = 6. Find the equation of the locus of P, and give the answer in the form $ax^2 + by^2 + cx + d = 0$.

5. Given two points A(3, 0) and A'(–3, 0). P(x, y) is a variable point such that $|\text{PA} - \text{PA'}| = 10$. Find the equation of the locus of P.

6. Q is the fixed point (2, 3). P is a variable point moving on the curve $x^2 + y^2 = 25$. If M is the mid-point of PQ, find the equation of the locus of M.

7. Given the line $L : x + 1 = 0$ and the point A(1, 0). Let P(x, y) be a variable point and d the distance from P to L. Find the equation of the locus of P such that
   (a) d = PA;
   (b) d : PA = 2 : 1

8. Given the point A(2, 0). P(x, y) is a variable point such that the area of ΔOPA = 10, where O is the origin. Find the equation of the locus of the mid-point of OP.

9. Two lines
$$L_1 : 4x - 3y + 1 = 0 \text{ and } L_2 : 3x - 4y + 3 = 0$$
intersect at N. P(x, y) is a variable point such that PN bisects the acute angle between L_1 and L_2. Find the equation of the locus of P.

11.2

10. Convert each of the following parametric equations into an equation in *x* and *y*.

(a) $\begin{cases} x = 3t^2 + t \\ y = t + 1 \end{cases}$

(b) $\begin{cases} x = 2t^2 + 3 \\ y = t^2 + 2t - 2 \end{cases}$

11. Convert the parametric equations $\begin{cases} x = 3\tan\theta \\ y = 4\sec\theta \end{cases}$ into an equation in *x* and *y*.

12. Find the equation, in *x* and *y*, for the curves $x = \sec\theta$, $y = \sin^2\theta + 1$.

13. Find the equation, in *x* and *y*, of the locus of the point $\left(\frac{a}{2}\left(t + \frac{1}{t}\right), \frac{b}{2}\left(t - \frac{1}{t}\right)\right)$, where *a* and *b* are non-zero constants.

14. For each of the following equations in *x* and *y*, find the parametric equations by substituting *y* = *tx*.
(a) $y^4 = x^5$
(b) $y^2 = x^2 + 2x$

11.3

15. P(t, t^2) is a variable point on the curve $y = x^2$. Let M be the mid-point of OP, where O is the origin.

(a) Show that the coordinates of M are $\left(\frac{t}{2}, \frac{t^2}{2}\right)$.

(b) Find the equation of the locus of M.

16. Let $L: y = tx + 1$ be a variable line. *L* cuts the *x*-axis at a point M and the *y*-axis at a point N.
(a) Find the mid-point of MN in terms of *t*.
(b) Find the equation of the locus of the mid-point of MN.

17. Given that $L: y = 2x + t$ is a variable line. L cuts the x-axis at a point M and the y-axis at a point N.

(a) If K is a point on L that lies between M and N and MK : KN = 1 : 2, find the equation of the locus of K.

(b) If K is a point on L and lies on MN produced such that MN = NK, find the equation of the locus of K.

18. A variable line L with constant slope 2 cuts the line $2x + y - 1 = 0$ at a point M and the x-axis at a point N. Find the equation of the locus of the mid-point of MN.

19. M(0, 1) is a point on the curve $x^2 = y - 1$, L is a variable line passing through M, and L cuts the curve $x^2 = y - 1$ again at N. P is a point on L such that MN = NP. Find the equation of the locus of the point P.

20. Given a line $L: y = -3x + c$ and a curve $P: y = 6x^2$.

(a) Find the range of values of c such that L cuts P at two distinct points.

(b) If L cuts P at two distinct points A and B, find the equation of the locus of the mid-point of AB.

21. A is a point (8, 0). P is a variable point on the curve $C: 9x^2 + 25y^2 = 1$.

(a) Show that $\left(\frac{\cos\theta}{3}, \frac{\sin\theta}{5}\right)$ is a point on C.

(b) If M is the mid-point of AP, find the equation of the locus of M.