

Chapter 10 Vectors

10.1 Definition of Vectors

Definitions:

- (1) Quantities that are determined by a magnitude alone are called *scalars*.
e.g. The length of the Yellow River is 4855 km.
- (2) Quantities that have both magnitude and direction are called *vectors*.
e.g. The velocity of the wind is 20 km/h from the northeast.

For a vector from an initial point A to a terminal point B,

- (1) it is denoted by \overrightarrow{AB} , \mathbf{a} , \vec{a} or \underline{a} ;
- (2) its magnitude is denoted by $|\overrightarrow{AB}|$, $|\mathbf{a}|$, $|\vec{a}|$ or $|\underline{a}|$.



Definitions:

- (1) Two vectors are said to be *equal* if and only if they have the same magnitude and direction.

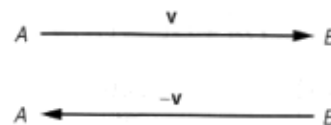
e.g. In the figure, ABCD is a parallelogram, then

$$\overrightarrow{AB} = \overrightarrow{DC} \text{ and } \overrightarrow{AD} = \overrightarrow{BC}.$$



- (2) A vector with zero magnitude is called a *zero vector*. Therefore, \overrightarrow{AA} is always a zero vector. A zero vector is denoted by $\vec{0}$ (or $\mathbf{0}$, etc.). It is the only vector that has no specific direction.
- (3) A vector with magnitude 1 unit is called a *unit vector*. That is, \mathbf{u} is a unit vector if and only if $|\mathbf{u}| = 1$.

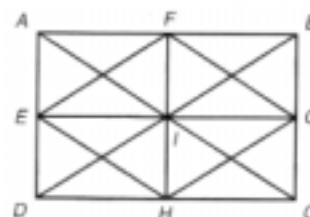
- (4) The *negative vector* of \mathbf{v} , denoted by $-\mathbf{v}$, is a vector having equal magnitude but opposite direction to \mathbf{v} . Therefore, $\overrightarrow{AB} = -\overrightarrow{BA}$ and $-\overrightarrow{AB} = \overrightarrow{BA}$.



Checkpoint 10.1

In the figure, ABCD is a rectangle and E, F, G, H are the mid-points of the sides.

- (a) Write down all vectors that are equal to \overrightarrow{EF} .
- (b) Write down all vectors that are the negative of \overrightarrow{IE} .



10.2 Operations of Vectors

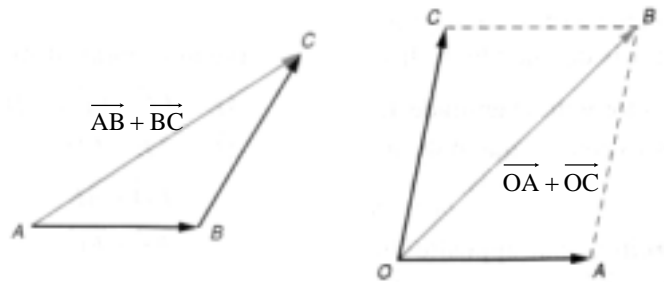
10.2.1 Addition

(1) The triangle law of addition:

$$\vec{AB} + \vec{BC} = \vec{AC}$$

(2) The parallelogram law of addition:

$$\vec{OA} + \vec{OC} = \vec{OB}$$



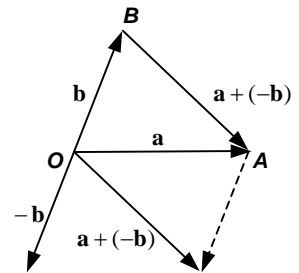
10.2.2 Subtraction

The difference of two vectors are given by

$$\mathbf{a} + (-\mathbf{b}) = \mathbf{a} - \mathbf{b}$$

For two vectors \vec{OA} and \vec{OB} ,

$$\vec{OA} - \vec{OB} = \vec{BA}$$



Example 10.1

Consider a quadrilateral ABCD. The diagonal AC and BD bisect each other at E. Prove that ABCD is a parallelogram.

Solution

\therefore E is the mid-point of AC and BD,

$\therefore \vec{AE} = \vec{EC}$ and $\vec{BE} = \vec{ED}$

$$\vec{AD} = \vec{AE} + \vec{ED}$$

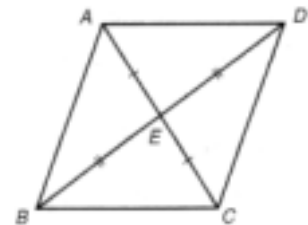
$$= \vec{ED} + \vec{AE}$$

$$= \vec{BE} + \vec{EC}$$

$$= \vec{BC}$$

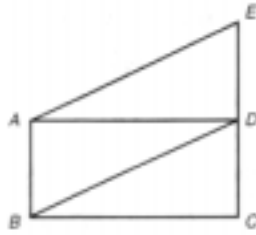
$\therefore AD \parallel BC$ and $AD = BC$.

\therefore ABCD is a parallelogram.



Checkpoint 10.2

In the figure, ABCD is a rectangle. E is a point on CD produced such that $CD = DE$.

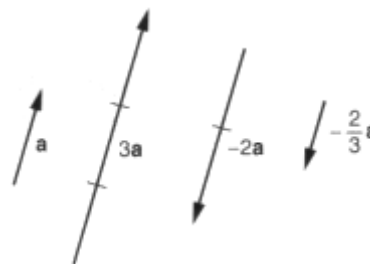


- (a) Show that $\overrightarrow{AE} = \overrightarrow{BD}$.
- (b) Express the following as a single vector in terms of the vertices A, B, C, D and E.
- (i) $\overrightarrow{AE} + \overrightarrow{DC}$ (ii) $\overrightarrow{DE} - \overrightarrow{AB}$ (iii) $\overrightarrow{DC} + \overrightarrow{DB}$

10.2.3 Scalar Multiplication

For any vector \mathbf{a} ,

- (1) \mathbf{a} and $k\mathbf{a}$ are parallel vectors for any non-zero constant k .
- (2) the unit vector of \mathbf{a} is $\frac{\mathbf{a}}{|\mathbf{a}|}$.



For any vectors \mathbf{u} , \mathbf{v} , \mathbf{w} and scalars m , n , we have

- (1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- (2) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
- (3) $m(n\mathbf{u}) = (mn)\mathbf{u} = n(m\mathbf{u})$
- (4) $(m + n)\mathbf{u} = m\mathbf{u} + n\mathbf{u}$
- (5) $m(\mathbf{u} + \mathbf{v}) = m\mathbf{u} + m\mathbf{v}$

Example 10.2

In $\triangle ABC$, M and N are the mid-points of AB and AC respectively. Prove that $MN \parallel BC$ and

$$MN = \frac{1}{2}BC.$$

Solution

Since M and N are the mid-points of AB and AC respectively,

$$\therefore \overrightarrow{AM} = \overrightarrow{MB} \text{ and } \overrightarrow{AN} = \overrightarrow{NC}$$

$$\overrightarrow{AB} = 2\overrightarrow{AM} \text{ and } \overrightarrow{AC} = 2\overrightarrow{AN}$$

$$\begin{aligned} \overrightarrow{BC} &= \overrightarrow{BA} + \overrightarrow{AC} \\ &= 2\overrightarrow{MA} + 2\overrightarrow{AN} \\ &= 2(\overrightarrow{MA} + \overrightarrow{AN}) \\ &= 2\overrightarrow{MN} \end{aligned}$$

$\therefore \overrightarrow{BC}$ and \overrightarrow{MN} have the same direction, i.e. $BC \parallel MN$.

$$\overrightarrow{BC} = 2\overrightarrow{MN}$$

$$|\overrightarrow{BC}| = 2|\overrightarrow{MN}|$$

$$BC = 2MN$$

$$MN = \frac{1}{2}BC$$

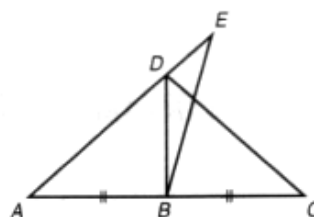
$\therefore MN \parallel BC$ and $MN = \frac{1}{2}BC$.



Checkpoint 10.3

In the figure, B is the mid-point of AC, D is a point on AE.

- (a) Find $\overrightarrow{BD} + \overrightarrow{DC}$.
(b) Show that $\overrightarrow{EA} + \overrightarrow{DC} = \overrightarrow{DB} + \overrightarrow{EB}$.



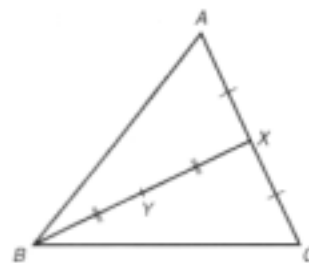
Example 10.3

In $\triangle ABC$, X is the mid-point of AC, and Y is the mid-point of BX. If $\overrightarrow{BA} = r\overrightarrow{BY} + s\overrightarrow{AC}$, find r and s .

Solution

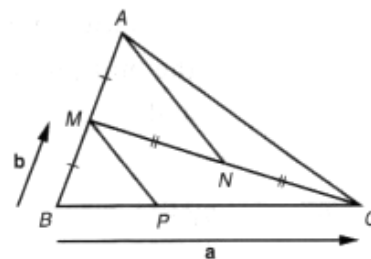
$$\begin{aligned}\overrightarrow{BA} &= \overrightarrow{BX} + \overrightarrow{XA} \\ &= \overrightarrow{BX} + \overrightarrow{XC} + \overrightarrow{CA} \\ &= 2\overrightarrow{BY} + \frac{1}{2}\overrightarrow{AC} - \overrightarrow{AC} \\ &= 2\overrightarrow{BY} - \frac{1}{2}\overrightarrow{AC}\end{aligned}$$

$$\therefore r = 2, s = -\frac{1}{2}$$



Example 10.5

In the figure, M is the mid-point of AB. N is the mid-point of MC. P is a point on BC such that AN // NP. Let $\overrightarrow{BC} = \mathbf{a}$ and $\overrightarrow{BM} = \mathbf{b}$.



- (a) Find \overrightarrow{MN} and \overrightarrow{AN} in terms of \mathbf{a} and \mathbf{b} .
 (b) Let $\overrightarrow{MP} = k\overrightarrow{AN}$. Express \overrightarrow{BP} in terms of \mathbf{a} , \mathbf{b} and k .

Hence find the value of k and show that $BP = \frac{1}{3}BC$.

Solution

$$\begin{aligned} \text{(a)} \quad \overrightarrow{MN} &= \frac{1}{2}\overrightarrow{MC} \\ &= \frac{1}{2}(\overrightarrow{MB} + \overrightarrow{BC}) \\ &= \frac{1}{2}(-\mathbf{b} + \mathbf{a}) \end{aligned}$$

$$\begin{aligned} \overrightarrow{AN} &= \overrightarrow{AM} + \overrightarrow{MN} \\ &= -\mathbf{b} + \frac{1}{2}(-\mathbf{b} + \mathbf{a}) \\ &= \frac{1}{2}\mathbf{a} - \frac{3}{2}\mathbf{b} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \overrightarrow{BP} &= \overrightarrow{BM} + \overrightarrow{MP} \\ &= \mathbf{b} + k\left(\frac{1}{2}\mathbf{a} - \frac{3}{2}\mathbf{b}\right) \\ &= \frac{k}{2}\mathbf{a} + \left(1 - \frac{3k}{2}\right)\mathbf{b} \end{aligned}$$

Since $BP \parallel BC$,

$$\therefore \overrightarrow{BP} = r\overrightarrow{BC}$$

$$\frac{k}{2}\mathbf{a} + \left(1 - \frac{3k}{2}\right)\mathbf{b} = r\mathbf{a}$$

$$\therefore \begin{cases} 1 - \frac{3k}{2} = 0 \\ \frac{k}{2} = r \end{cases}$$

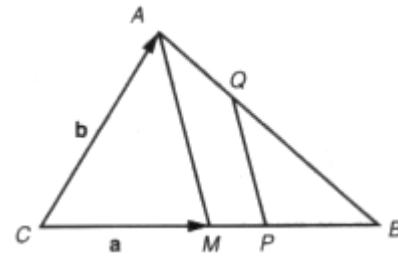
On solving, $k = \frac{2}{3}$ and $r = \frac{1}{3}$.

$$\therefore \overrightarrow{BP} = \frac{1}{3}\overrightarrow{BC} \text{ and } BP = \frac{1}{3}BC$$

Checkpoint 10.5

In the figure, M and P are points on BC with $CM = MB = 3MP$. Q is a point on AB such that $MA \parallel PQ$.

- (a) Find \overrightarrow{MA} in terms of \mathbf{a} and \mathbf{b} .
- (b) Let $\overrightarrow{PQ} = p\overrightarrow{MA}$, $\overrightarrow{BQ} = q\overrightarrow{BA}$.
 - (i) Express \overrightarrow{PQ} in terms of \mathbf{a} , \mathbf{b} and p .
 - (ii) Express \overrightarrow{BQ} in terms of \mathbf{a} , \mathbf{b} and q .
- (c) By considering $\overrightarrow{PQ} = \overrightarrow{PB} + \overrightarrow{BQ}$, find the value of p .



10.3 Vectors in the Rectangular Coordinate System

In the rectangular coordinate system, \mathbf{i} and \mathbf{j} are unit vectors along the x - and y - axes.

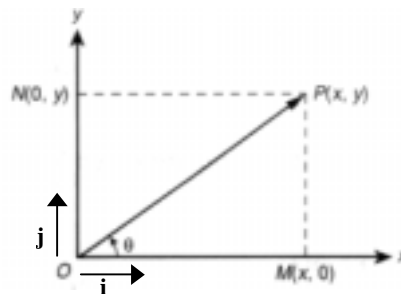
Let $P(x, y)$ be a point on the rectangular coordinate plane and O be the origin. Then

$$(1) \quad \overrightarrow{OP} = x\mathbf{i} + y\mathbf{j};$$

$$(2) \quad |\overrightarrow{OP}| = \sqrt{x^2 + y^2};$$

$$(3) \quad \tan \theta = \frac{y}{x},$$

where θ is angle between OP and the positive x -axis anticlockwise.



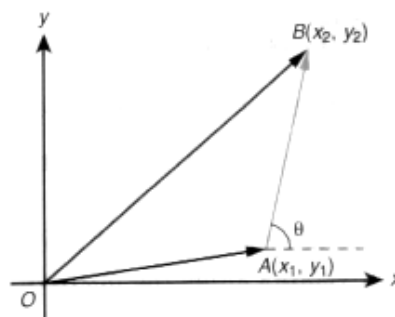
Given two vectors $\overrightarrow{OA} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\overrightarrow{OB} = x_2\mathbf{i} + y_2\mathbf{j}$. Then

$$(1) \quad \overrightarrow{AB} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j};$$

$$(2) \quad |\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2};$$

$$(3) \quad \tan \theta = \frac{y_2 - y_1}{x_2 - x_1};$$

$$(4) \quad \text{if } \overrightarrow{OA} = \overrightarrow{OB}, \text{ then } x_1 = x_2 \text{ and } y_1 = y_2.$$



Example 10.6

If A is $(-2, -1)$ and B is $(1, 3)$, find the magnitude and direction of \overrightarrow{AB} .

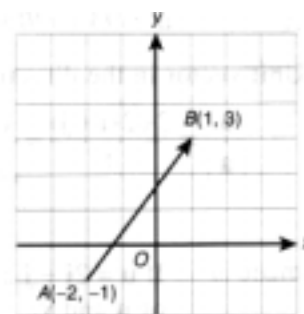
Solution

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (\mathbf{i} + 3\mathbf{j}) - (-2\mathbf{i} - \mathbf{j}) = 3\mathbf{i} + 4\mathbf{j}$$

$$\therefore |\overrightarrow{AB}| = \sqrt{3^2 + 4^2} = 5$$

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) = 53.13^\circ$$

\therefore The magnitude of $\overrightarrow{AB} = 5$ units and \overrightarrow{AB} makes an angle of 53.13° with the positive x -axis.



Example 10.7

If the coordinates of A are (2, -1) and $\overrightarrow{AB} = -\mathbf{i} + 6\mathbf{j}$, find the coordinates of B.

Solution

Let B be (x, y). Then $\overrightarrow{OB} = x\mathbf{i} + y\mathbf{j}$.

$$\begin{aligned}\overrightarrow{OA} + \overrightarrow{AB} &= \overrightarrow{OB} \\ \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= (x\mathbf{i} + y\mathbf{j}) - (-\mathbf{i} + 6\mathbf{j}) \\ &= (x - 2)\mathbf{i} + (y + 1)\mathbf{j}\end{aligned}$$

It is given that $\overrightarrow{AB} = -\mathbf{i} + 6\mathbf{j}$.

$$\begin{aligned}\therefore x - 2 = -1 \quad \text{and} \quad y + 1 = 6 \\ x = 1 \quad \text{and} \quad y = 5\end{aligned}$$

\therefore The coordinates of B are (1, 5).

Checkpoint 10.6

In each of the following, find \overrightarrow{AB} in terms of \mathbf{i} and \mathbf{j} .

- (a) A(1, 3), B(4, 6)
- (b) A(-2, 5), B(-3, -7)

Checkpoint 10.7

If the coordinates of B are $(-1, 4)$ and $\overrightarrow{AB} = 4\mathbf{i} + 6\mathbf{j}$, find the coordinates of A.

Example 10.8

\mathbf{i} and \mathbf{j} are unit base vectors as shown in the figure. \mathbf{a} is a vector making an angle of 45° with \mathbf{i} and $|\mathbf{a}| = 4$. Express \mathbf{a} in terms of \mathbf{i} and \mathbf{j} . Hence find the unit vector in the direction of \mathbf{a} .

Solution

Let $\overrightarrow{OA} = \mathbf{a}$. Then

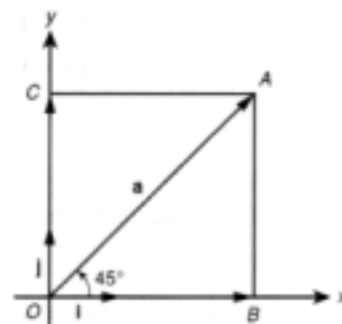
$$|\overrightarrow{OB}| = |\overrightarrow{OA}| \cos 45^\circ = 4 \left(\frac{\sqrt{2}}{2} \right) = 2\sqrt{2}$$

$$|\overrightarrow{OC}| = |\overrightarrow{OA}| \sin 45^\circ = 4 \left(\frac{\sqrt{2}}{2} \right) = 2\sqrt{2}$$

$$\overrightarrow{OB} = 2\sqrt{2}\mathbf{i} \quad \text{and} \quad \overrightarrow{OC} = 2\sqrt{2}\mathbf{j}$$

$$\therefore \mathbf{a} = \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OC} = 2\sqrt{2}(\mathbf{i} + \mathbf{j})$$

$$\text{Unit vector of } \mathbf{a} \text{ is } \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{2\sqrt{2}(\mathbf{i} + \mathbf{j})}{4} = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$$



Checkpoint 10.8

Given $P(7, 1)$ and $Q(3, 4)$. Find the magnitude and direction of \overrightarrow{PQ} . Write down the unit vector in the direction of \overrightarrow{PQ}

Example 10.9

If $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$, express $8\mathbf{i} + 9\mathbf{j}$ in terms of \mathbf{u} and \mathbf{v} .

Solution

Let $8\mathbf{i} + 9\mathbf{j} = m\mathbf{u} + n\mathbf{v}$

$\therefore 8\mathbf{i} + 9\mathbf{j} = m(2\mathbf{i} + \mathbf{j}) + n(\mathbf{i} + 3\mathbf{j})$

$$8 = 2m + n \quad \text{.....(1)}$$

$$9 = m + 3n \quad \text{.....(2)}$$

$$(2) \times 2, \quad 18 = 2m + 6n \quad \text{.....(3)}$$

$$(3) - (1), \quad 10 = 5n$$

$$n = 2$$

Substitute $n = 2$ into (2),

$$9 = m + 3(2)$$

$$m = 3$$

$$\therefore 8\mathbf{i} + 9\mathbf{j} = 3\mathbf{u} + 2\mathbf{v}$$

Example 10.10

Prove that, for any two non-zero vectors $\mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j}$, $\mathbf{a} \perp \mathbf{b}$ if and only if $x_1x_2 + y_1y_2 = 0$.

Solution

Let $\overrightarrow{OA} = \mathbf{a} = x_1\mathbf{i} + y_1\mathbf{j}$ and $\overrightarrow{OB} = \mathbf{b} = x_2\mathbf{i} + y_2\mathbf{j}$.

$$\begin{aligned} \therefore \overrightarrow{AB} &= \mathbf{b} - \mathbf{a} \\ &= (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} \end{aligned}$$

$$\therefore |\overrightarrow{AB}|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

By Pythagoras' theorem and the converse of Pythagoras' theorem, $\mathbf{a} \perp \mathbf{b}$

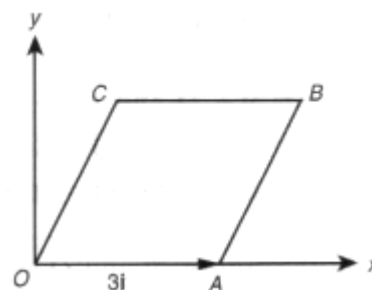
$$\begin{aligned} \text{iff} \quad & |\overrightarrow{AB}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 \\ \text{iff} \quad & (x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_1^2 + y_1^2) + (x_2^2 + y_2^2) \\ \text{iff} \quad & x_2^2 - 2x_1x_2 + x_1^2 + y_2^2 - 2y_1y_2 + y_1^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2 \\ \text{iff} \quad & x_1x_2 + y_1y_2 = 0 \end{aligned}$$

$\therefore \mathbf{a} \perp \mathbf{b}$ if and only if $x_1x_2 + y_1y_2 = 0$.

Checkpoint 10.9

In the figure, OABC is a rhombus with $\overrightarrow{OA} = 3\mathbf{i}$ and $\angle AOC = 60^\circ$.

- Find \overrightarrow{OC} in terms of \mathbf{i} and \mathbf{j} .
- Find \overrightarrow{OB} and \overrightarrow{AC} in terms of \mathbf{i} and \mathbf{j} .
- Show that $\overrightarrow{OB} \perp \overrightarrow{AC}$.



10.4 Point of Division

10.4.1 Position Vectors

In the rectangular coordinate plane, the position vector of A with respect to O is defined as \overrightarrow{OA} and is denoted by \mathbf{a} . Similarly, the position vectors of B, C, D, ... are \overrightarrow{OB} , \overrightarrow{OC} , \overrightarrow{OD} , ... or \mathbf{b} , \mathbf{c} , \mathbf{d} , ...

It can be found that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$.

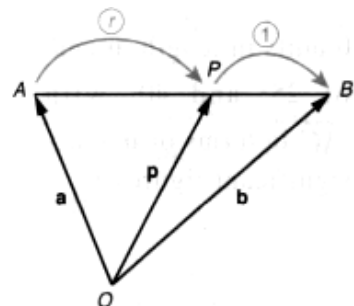
10.4.2 Point of Division

Let A and B be two points with position vectors \mathbf{a} and \mathbf{b} with respect to O. If P is a point on the line AB such that $AP : PB = r : 1$, where r is a real number, then the position vector of P, \mathbf{p} , is given by

$$\mathbf{p} = \frac{\mathbf{a} + r\mathbf{b}}{1 + r}$$

In particular, if P is the mid-point of AB, then

$$r = 1 \text{ and } \mathbf{p} = \frac{\mathbf{a} + \mathbf{b}}{2}$$



Example 10.11

P is a point on AB such that $3\overrightarrow{AP} = 2\overrightarrow{PB}$. If A is $(-1, -3)$ and B is $(4, 7)$, find the coordinates of P.

Solution

$$\mathbf{a} = \overrightarrow{OA} = -\mathbf{i} - 3\mathbf{j}$$

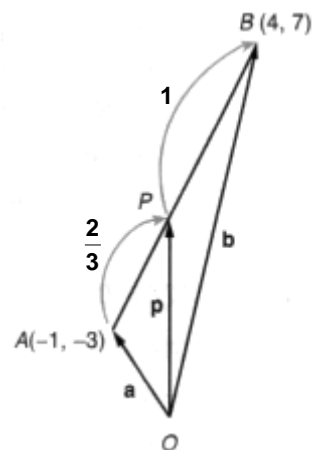
$$\mathbf{b} = \overrightarrow{OB} = 4\mathbf{i} + 7\mathbf{j}$$

$$3\overrightarrow{AP} = 2\overrightarrow{PB}$$

$$\frac{AP}{PB} = \frac{2}{3}$$

$$\begin{aligned} \mathbf{p} &= \frac{(-\mathbf{i} - 3\mathbf{j}) + \frac{2}{3}(4\mathbf{i} + 7\mathbf{j})}{1 + \frac{2}{3}} \\ &= \frac{3(-\mathbf{i} - 3\mathbf{j}) + 2(4\mathbf{i} + 7\mathbf{j})}{3 + 2} \\ &= \frac{5\mathbf{i} + 5\mathbf{j}}{5} \\ &= \mathbf{i} + \mathbf{j} \end{aligned}$$

\therefore The coordinates of P are $(1, 1)$.

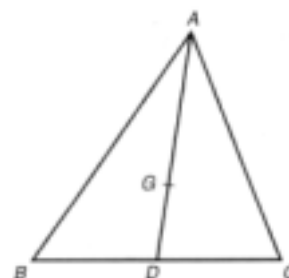
**Checkpoint 10.10**

The position vector of A is denoted by \mathbf{a} . In each of the following, if $\mathbf{a} = \overrightarrow{BC}$, find the coordinates of A.

- (a) $B(-3, 2), C(4, 5)$
- (b) $B(-6, 1), C(-2, 3)$

Example 10.12

Given that AD is the median of $\triangle ABC$, and G is a point on A such that $AG = 2GD$. Find the position vector of G in terms of **a**, **b** and **c**, where **a**, **b** and **c** are the position vectors of A, B and C with respect to O respectively. (G is called the *centroid* of $\triangle ABC$.)

**Solution**

Let **g** and **d** be the position vectors of the points of G and D with respect to O respectively. Since $AG = 2GD$, we have

$$\begin{aligned} \mathbf{g} &= \frac{\mathbf{a} + 2\mathbf{d}}{1 + 2} \\ &= \frac{1}{3}(\mathbf{a} + 2\mathbf{d}) \dots\dots(1) \end{aligned}$$

D is the mid-point of BC,

$$\mathbf{d} = \frac{1}{2}(\mathbf{b} + \mathbf{c}) \dots\dots(2)$$

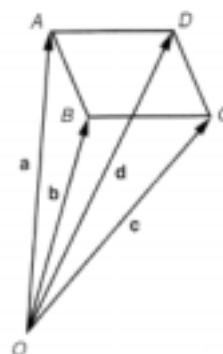
Substituting (2) into (1), we have

$$\begin{aligned} \mathbf{g} &= \frac{1}{3}[\mathbf{a} + 2 \cdot \frac{1}{2}(\mathbf{b} + \mathbf{c})] \\ &= \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c}) \end{aligned}$$

Example 10.13

ABCD is a parallelogram. The position vectors of A, B, C and D with respect to O are **a**, **b**, **c** and **d** respectively. Prove that

- (a) $\mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{d}$
- (b) the diagonals AC and BD bisect each other.

**Solution**

- (a) $\because AD = BC$ and $AD \parallel BC$
- $\therefore \overrightarrow{AD} = \overrightarrow{BC}$
- $\mathbf{d} - \mathbf{a} = \mathbf{c} - \mathbf{b}$
- $\mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{d}$

- (b) Let G_1 and G_2 , with position vectors \mathbf{g}_1 and \mathbf{g}_2 , be the mid-points of AC and BD respectively.

$$\text{Then } \mathbf{g}_1 = \frac{1}{2}(\mathbf{a} + \mathbf{c}) \text{ and } \mathbf{g}_2 = \frac{1}{2}(\mathbf{b} + \mathbf{d})$$

Using the result of (a), $\mathbf{g}_1 = \mathbf{g}_2$.

As G_1 and G_2 coincide, the diagonals AC and BD bisect each other.

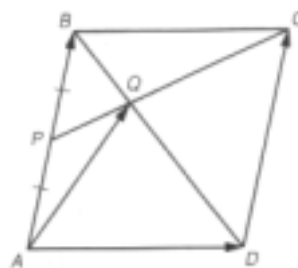
Checkpoint 10.11

ABCD is a quadrilateral. P, Q, R and S divide AB, AD, CD and CB respectively in the ratio 2 : 1.

- (a) Find the position vectors \mathbf{p} , \mathbf{q} , \mathbf{r} , \mathbf{s} , in terms of \mathbf{a} , \mathbf{b} , \mathbf{c} , \mathbf{d} .
(b) Prove that PQRS is a parallelogram.

Example 10.14

In the parallelogram ABCD, P is the mid-point of AB. PC intersects BD at Q. $BQ : QD = 1 : r$, $PQ : QS = 1 : s$; $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$.



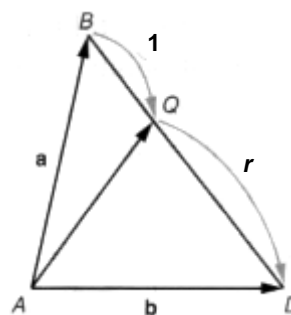
- (a) By considering $\frac{QD}{BQ} = r$, find \overrightarrow{AQ} in terms of \mathbf{a} , \mathbf{b} and r .
- (b) By considering $\frac{QC}{PQ} = s$, find \overrightarrow{AQ} in terms of \mathbf{a} , \mathbf{b} and s .
- (c) Hence find r and s .

Solution

(a) Since $\frac{QD}{BQ} = r$,

$$\therefore \overrightarrow{AQ} = \frac{(1)\mathbf{b} + r\mathbf{a}}{1+r}$$

$$= \frac{r}{1+r}\mathbf{a} + \frac{1}{1+r}\mathbf{b}$$

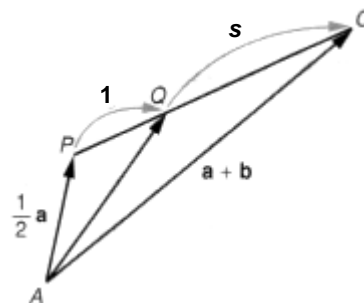


(b) $\overrightarrow{AP} = \frac{1}{2}\mathbf{a}$
 $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
 $= \mathbf{a} + \mathbf{b}$

Since $\frac{QC}{PQ} = s$,

$$\therefore \overrightarrow{AQ} = \frac{(1)(\mathbf{a} + \mathbf{b}) + s\left(\frac{1}{2}\mathbf{a}\right)}{1+s}$$

$$= \frac{2+s}{2(1+s)}\mathbf{a} + \frac{1}{1+s}\mathbf{b}$$



(c) From the results of (a) and (b),

$$\frac{r}{1+r}\mathbf{a} + \frac{1}{1+r}\mathbf{b} = \frac{2+s}{2(1+s)}\mathbf{a} + \frac{1}{1+s}\mathbf{b}$$

Since \mathbf{a} and \mathbf{b} are non-zero and not parallel,

$$\frac{r}{1+r}\mathbf{a} + \frac{1}{1+r}\mathbf{b} = \frac{2+s}{2(1+s)}\mathbf{a} + \frac{1}{1+s}\mathbf{b}$$

$$\therefore \frac{r}{1+r} = \frac{2+s}{2(1+s)} \dots\dots(1)$$

$$\text{and } \frac{1}{1+r} = \frac{1}{1+s} \dots\dots(2)$$

From (2), $r = s$.

Substituting $r = s$ into (1), we have

$$\frac{s}{1+s} = \frac{2+s}{2(1+s)}$$

$$2s = 2 + s$$

$$s = 2$$

$$\therefore r = 2$$

Checkpoint 10.12

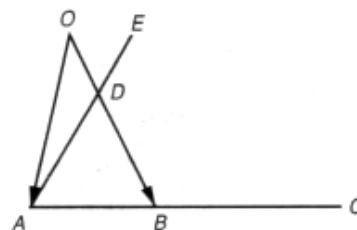
ABC is a triangle. M and N are the points dividing AB and AC respectively in the ratio 3 : 4.

Find, in terms of \mathbf{a} , \mathbf{b} and \mathbf{c} , the vectors \overrightarrow{BC} and \overrightarrow{MN} . Hence, or otherwise, prove that

$MN \parallel BC$ and $|\overrightarrow{MN}| : |\overrightarrow{BC}| = 3 : 7$.

Checkpoint 10.13

In the figure, taking O as the reference point, the position vectors of points A and B are \mathbf{a} and \mathbf{b} respectively. D is a point on OB with $OD : DB = 1 : 2$. E and C are points on AD and AB respectively with $AB : BC = 2 : 3$.

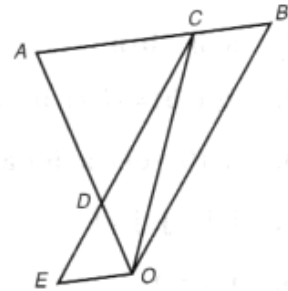


- (a) Find \overrightarrow{AD} and \overrightarrow{AC} in terms of \mathbf{a} and \mathbf{b} .
- (b) Let $\overrightarrow{AE} = k\overrightarrow{AD}$.
 - (i) Find \overrightarrow{OE} and \overrightarrow{OC} in terms of k , \mathbf{a} and \mathbf{b} .
 - (ii) If O , E and C are collinear, find the value of k .

Checkpoint 10.14

In the figure, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. C is a point on AB such that $AC : CB = 2 : 1$. D is a point on OA and E is a point lying on CD produced such that $CD : DE = 2 : 1$.

- (a) Express \overrightarrow{OC} in terms of \mathbf{a} and \mathbf{b} .
- (b) It is given that $\overrightarrow{OD} = m\mathbf{a}$.
 - (i) Express \overrightarrow{CD} and \overrightarrow{CE} in terms of \mathbf{a} , \mathbf{b} and m .
 - (ii) Hence express \overrightarrow{OE} in terms of \mathbf{a} , \mathbf{b} and m .
- (c) Suppose AB is parallel to EO .
 - (i) Find the value of m .
 - (ii) Hence prove that $OBCE$ is a parallelogram.



Example 10.16

Find the angle between the vectors $3\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + 2\mathbf{j}$.

Solution

Let θ be the angle between the vectors $3\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + 2\mathbf{j}$.

$$\begin{aligned}(3\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} + 2\mathbf{j}) &= (3)(1) + (1)(2) \\ &= 5\end{aligned}$$

$$\begin{aligned}(3\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} + 2\mathbf{j}) &= |3\mathbf{i} + \mathbf{j}| |\mathbf{i} + 2\mathbf{j}| \cos \theta \\ &= \sqrt{3^2 + 1^2} \times \sqrt{1^2 + 2^2} \cos \theta \\ &= 5\sqrt{2} \cos \theta\end{aligned}$$

$$\therefore 5\sqrt{2} \cos \theta = 5$$

$$\cos \theta = \frac{1}{\sqrt{2}}$$

$$\theta = 45^\circ$$

Checkpoint 10.15

Find $\mathbf{a} \cdot \mathbf{b}$ and the angle between \mathbf{a} and \mathbf{b} in each of the following.

(a) $\mathbf{a} = 4\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = -\mathbf{i} + \mathbf{j}$

(b) $\mathbf{a} = -2\mathbf{i} + 5\mathbf{j}$, $\mathbf{b} = \mathbf{i} - 4\mathbf{j}$

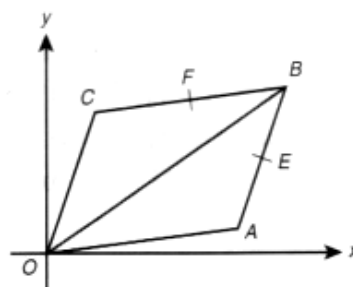
(Give the answers correct to the nearest 0.1° if necessary.)

Checkpoint 10.17

In the figure, OABC is a parallelogram. E and F are the mid-points of AB and BC respectively. $\vec{OA} = 6\mathbf{i} + 2\mathbf{j}$ and $\vec{OB} = 7\mathbf{i} + 5\mathbf{j}$.

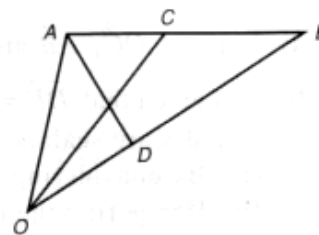
- (a) Find \vec{OC} , \vec{OE} and \vec{OF} .
- (b) Find $\vec{OE} \cdot \vec{OF}$.
- (c) Find $\angle EOF$.

(Give the answers correct to the nearest 0.1° if necessary.)



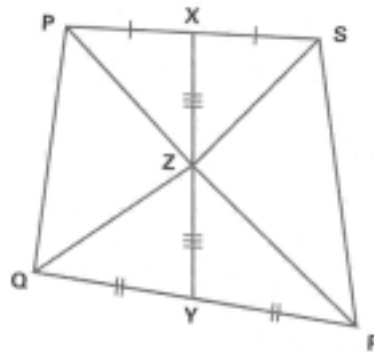
Checkpoint 10.18

In the figure, OAB is a triangle. C lies on AB such that $AC:CB = 1:r$, where $r > 0$. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. D lies on OB such that $\overrightarrow{OD} = \frac{1}{r}\mathbf{b}$.



- (a) Express \overrightarrow{OC} and \overrightarrow{AD} in terms of \mathbf{a} , \mathbf{b} and r .
- (b) Suppose $OA = 1$ and $OB = r$.
 - (i) Find $\mathbf{a} \cdot \mathbf{a}$ and $\mathbf{b} \cdot \mathbf{b}$.
 - (ii) Hence prove that AD is always perpendicular to OC for $r > 0$.

8. In the figure, X, Y are the mid-points of the sides PS and QR of a quadrilateral respectively. Z is the mid-point of XY. Let $\overrightarrow{PS} = \mathbf{a}$, $\overrightarrow{QR} = \mathbf{b}$ and $\overrightarrow{XY} = \mathbf{c}$.



- (a) Express \overrightarrow{ZP} , \overrightarrow{ZQ} , \overrightarrow{ZR} and \overrightarrow{ZS} in terms of \mathbf{a} , \mathbf{b} and \mathbf{c} .
 (b) Hence prove that $\overrightarrow{ZP} + \overrightarrow{ZQ} + \overrightarrow{ZR} + \overrightarrow{ZS} = \mathbf{0}$.

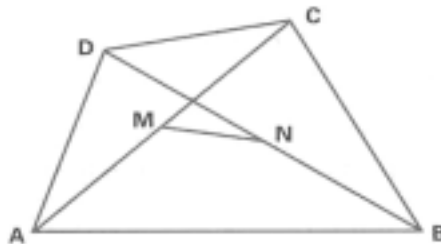
10.3

9. Find the magnitude and the angle θ the vector $-3\mathbf{i} + 4\mathbf{j}$ makes with the positive direction of the x -axis, measured anti-clockwise.
 (Give the value of θ correct to the nearest 0.1° .)
10. Let P be the point $(-8, 3)$. Find the coordinates of Q if $\overrightarrow{PQ} = -12\mathbf{i} + 9\mathbf{j}$.
11. In each of the following, find a unit vector \mathbf{u} in the same direction as the given vector \mathbf{v} .
 (a) $\mathbf{v} = -4\mathbf{i}$ (b) $\mathbf{v} = -8\mathbf{i} + 15\mathbf{j}$
12. Let $\mathbf{a} = 3\mathbf{i} + 5\mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} - 3\mathbf{j}$.
 (a) Express the unit vectors \mathbf{i} and \mathbf{j} in terms of \mathbf{a} and \mathbf{b} .
 (b) Express $\mathbf{p} = -4\mathbf{i} + 2\mathbf{j}$ in terms of \mathbf{a} and \mathbf{b} .
13. Let $\mathbf{u} = 12\mathbf{i} - 7\mathbf{j}$, $\mathbf{v} = -6\mathbf{i} + 14\mathbf{j}$. Find a vector \mathbf{p} which has magnitude 8 and is parallel to $3\mathbf{u} + 2\mathbf{v}$.
14. Given three points A(2, 7), B(-1, 3) and C(2, 1).
 (a) Find \overrightarrow{AB} .
 (b) If D is a point such that \overrightarrow{CD} is parallel to \overrightarrow{AB} and $|\overrightarrow{CD}| = 15$, find
 (i) \overrightarrow{CD} ;
 (ii) the coordinates of D.

15. Given that $\overrightarrow{OA} = 2\mathbf{i} + 6\mathbf{j}$, $\overrightarrow{OB} = 5\mathbf{i} - \mathbf{j}$ and $\overrightarrow{OC} = \mathbf{i} - 5\mathbf{j}$. P is a point such that $\overrightarrow{AP} = k\overrightarrow{AB}$.
- (a) Express \overrightarrow{OP} in terms of k .
- (b) If $|\overrightarrow{CP}| = |\overrightarrow{AB}|$, find the possible values of k .

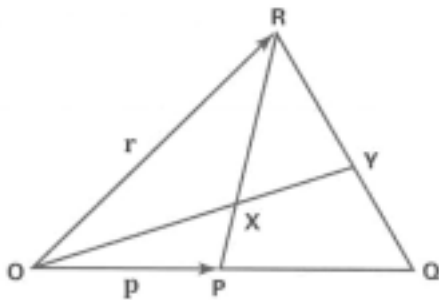
10.4

16. P is point on a line segment AB. If $\overrightarrow{OA} = -3\mathbf{i} + 3\mathbf{j}$, $\overrightarrow{OP} = 1.5\mathbf{j}$ and $AP : PB = 3 : 5$, find \overrightarrow{OB} .
17. \mathbf{h} and \mathbf{k} are two non-zero vectors which are not parallel. The position vectors \mathbf{a} and \mathbf{b} of two points A and B with respect to O are given by $\mathbf{a} = 5\mathbf{h} + 3\mathbf{k}$ and $\mathbf{b} = -2\mathbf{h} - 4\mathbf{k}$ respectively.
If P divides AB in the ratio 4 : 3, express the position vector of P in terms of \mathbf{h} and \mathbf{k} .
18. Given a triangle ABC and a point O. G is a point such that $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \mathbf{0}$.
- (a) Express \overrightarrow{GA} in terms of \overrightarrow{OA} and \overrightarrow{OG} .
- (b) Show that $\overrightarrow{OG} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$.
19. The figure shows a quadrilateral ABCD. The position vectors of A, B, C and D with respect to O are \mathbf{a} , \mathbf{b} , \mathbf{c} and \mathbf{d} respectively. If M and N are the mid-points of AC and BD as shown, express \overrightarrow{MN} in terms of \mathbf{a} , \mathbf{b} , \mathbf{c} and \mathbf{d} .

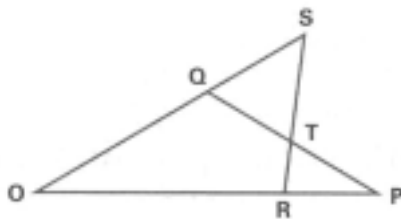


20. The position vectors of the points A, B, C and D with respect to O are \mathbf{a} , \mathbf{b} , \mathbf{c} and \mathbf{d} respectively. A, B and C lie on a straight line such that $AC : CB = 1 : k$.
- (a) Express \mathbf{c} in terms of \mathbf{a} and \mathbf{b} .
- (b) D is a point lying on OC such that $\mathbf{d} = 2\mathbf{a} + 3\mathbf{b}$. Find the value(s) of k .

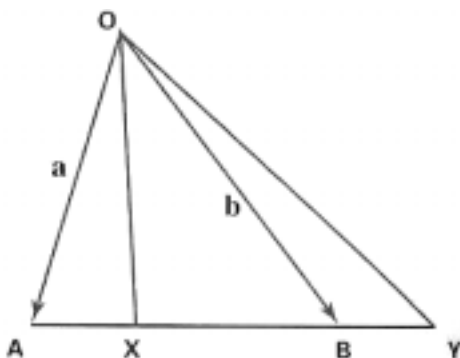
21. In the figure, P is the mid-point of OQ and $PX : XR = 1 : 3$. Let $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OR} = \mathbf{r}$.



- (a) Express \overrightarrow{OX} in terms of \mathbf{p} and \mathbf{r} .
 (b) Let $QY : YR = m : n$. Express \overrightarrow{OY} in terms of \mathbf{p} , \mathbf{r} , m and n .
 (c) Using the results of (a) and (b), find $QY : YR$.
22. In the figure, QRP and OQS are straight lines. The lines PQ and RS intersect at T such that $PT : TQ = 3 : 4$, $RT : TS = 1 : 2$. The position vectors of P, Q, R, S and T with respect to O are \mathbf{p} , \mathbf{q} , \mathbf{r} , \mathbf{s} and \mathbf{t} respectively.

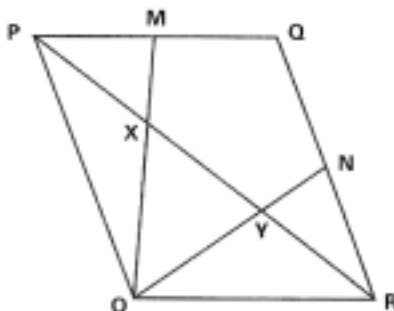


- (a) By considering $\triangle ORS$, express \mathbf{t} in terms of \mathbf{r} and \mathbf{s} .
 (b) Given that $\mathbf{p} = \frac{7}{6}\mathbf{r}$ and $\mathbf{q} = \frac{\mu}{9}\mathbf{s}$, by considering $\triangle OPQ$, show that $\mathbf{t} = \frac{14\mathbf{r} + \mu\mathbf{s}}{21}$.
 Hence, find the value of μ .
23. In the figure, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

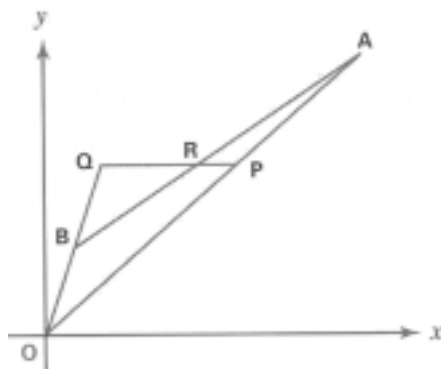


- (a) If X divides AB in the ratio 1 : 2, find \overrightarrow{OX} in terms of \mathbf{a} and \mathbf{b} .
 (b) If B divides AY in the ratio 3 : 1, find \overrightarrow{OY} in terms of \mathbf{a} and \mathbf{b} .
 (c) Show that $\overrightarrow{AB} = \overrightarrow{XY}$ and hence show that $\overrightarrow{AX} = \overrightarrow{BY}$.

24. In the figure, OPQR is a parallelogram. M and N are the mid-points of PQ and QR respectively. OM and ON intersect the diagonal PR at X and Y respectively such that $\frac{OX}{XM} = \lambda$, $\frac{OY}{YN} = \mu$. Let $\vec{OP} = \mathbf{p}$, $\vec{OR} = \mathbf{r}$.



- (a) Find \vec{PX} and \vec{YR} in terms of \mathbf{p} , \mathbf{r} , λ and μ .
 (b) Show that X and Y trisect PR.
25. In the figure, $\vec{OA} = 15\mathbf{i} + 10\mathbf{j}$ and $\vec{OB} = \mathbf{i} + 2\mathbf{j}$. $OP : PA = 3 : 2$ and $OB : BQ = 1 : 2$.

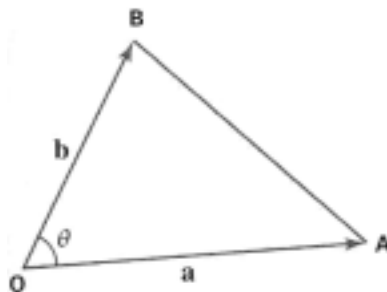


- (a) Write down \vec{AB} , \vec{OP} , \vec{OQ} and \vec{PQ} in terms of \mathbf{i} and \mathbf{j} .
 (b) Given that $\vec{AR} = h\vec{AB}$, express \vec{OR} in terms of h , \mathbf{i} and \mathbf{j} .
 (c) Given that $\vec{PR} = k\vec{PQ}$, express \vec{OR} in terms of k , \mathbf{i} and \mathbf{j} .
 (d) Using the results of (b) and (c), find the values of h and k .

10.5

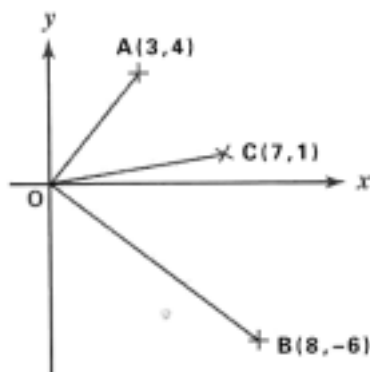
26. Find $\mathbf{u} \cdot \mathbf{v}$ if
- (a) $\mathbf{u} = -\mathbf{i} + \mathbf{j}$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$
 (b) $\mathbf{u} = 6\mathbf{i} - 7\mathbf{j}$, $\mathbf{v} = 10\mathbf{i} - 9\mathbf{j}$
27. In each of the following, find the angle between \mathbf{a} and \mathbf{b} .
 (Give the answers correct to the nearest degree if necessary.)
- (a) $\mathbf{a} = 4\mathbf{i} + \mathbf{j}$, $\mathbf{b} = 3\mathbf{i} + 5\mathbf{j}$
 (b) $\mathbf{a} = -\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = 3\mathbf{i} + 4\mathbf{j}$

28. Given that $|\mathbf{a}| = 2$, $|\mathbf{b}| = 5$ and the angle between \mathbf{a} and \mathbf{b} is 60° .
- Find $\mathbf{a} \cdot \mathbf{b}$.
 - Hence find $(\mathbf{a} + 3\mathbf{b}) \cdot (4\mathbf{b} - 5\mathbf{a})$.
29. Given that $\overrightarrow{OA} = 2\mathbf{i} + \mathbf{j}$, $\overrightarrow{OB} = 4\mathbf{i} + k\mathbf{j}$ and $\overrightarrow{OC} = 3\mathbf{i} + 3\mathbf{j}$. If $\overrightarrow{AC} \cdot \overrightarrow{OB} = \overrightarrow{OA} \cdot \overrightarrow{OC}$, find
- the value of k ,
 - $\angle ABC$ correct to 2 decimal places.
30. Find the unknown constant k if $\mathbf{a} = 16\mathbf{i} + k\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} - k\mathbf{j}$ are perpendicular to each other.
31. Given two vectors $\mathbf{p} = 5\mathbf{i} - 2\mathbf{j}$, $\mathbf{q} = \sqrt{13}\mathbf{i} + 4\mathbf{j}$. Show that $\mathbf{p} + \mathbf{q}$ and $\mathbf{p} - \mathbf{q}$ are perpendicular.
32. Given that $|\mathbf{a}| = 3$, $|\mathbf{b}| = 2$ and $|\mathbf{a} + 2\mathbf{b}| = 5$, show that \mathbf{a} is perpendicular to \mathbf{b} .
33. In $\triangle OAB$, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\angle AOB = \theta$. Show that

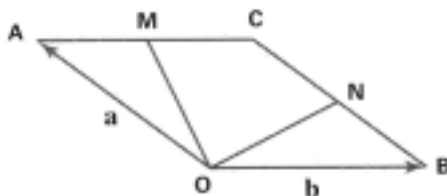


- $$\sin^2 \theta = \frac{|\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2}{|\mathbf{a}|^2 |\mathbf{b}|^2},$$
- $$\text{area of } \triangle OAB = \frac{1}{2} \sqrt{|\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2}.$$

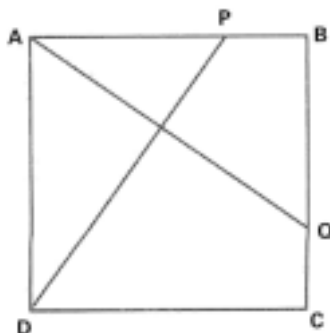
34. The figure shows three points $A(3, 4)$, $B(8, -6)$ and $C(7, 1)$. Prove that OC bisects $\angle AOB$.



35. In the figure, $OACB$ is a rhombus and $OA = 1$. M and N are the mid-points of AC and CB respectively. Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.



- (a) Express \overrightarrow{OM} and \overrightarrow{ON} in terms of \mathbf{a} and \mathbf{b} .
- (b) Prove that $\overrightarrow{ON} \cdot \overrightarrow{OM} = \frac{5}{4} \mathbf{a} \cdot \mathbf{b} + 1$.
- (c) If $\angle NOM$ is a right angle, find $\angle AOB$.
(Give the answer correct to the nearest 0.1° .)
36. In the figure, $ABCD$ is a square and $AB = 1$. P and Q are points on AB and BC respectively such that $PB = QC = k$, $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{BC} = \mathbf{v}$.



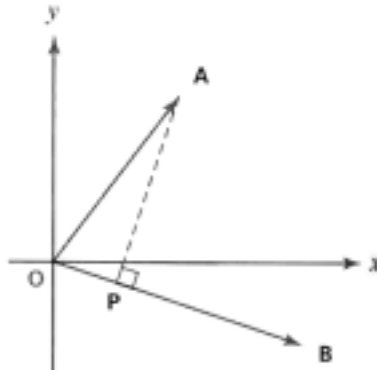
- (a) Express \overrightarrow{AQ} and \overrightarrow{DP} in terms of \mathbf{u} , \mathbf{v} and k .
- (b) Show that $AQ \perp DP$.

37. Given that P, Q and R are three points (5, -8), (2, 3) and (8, 7) respectively.

- (a) Find \overrightarrow{PQ} and \overrightarrow{PR} .
- (b) S is a point on PR such that $SQ \perp PR$.
 - (i) Express \overrightarrow{SQ} in terms of \overrightarrow{PQ} and \overrightarrow{PS} .
 - (ii) Find PS : SR.

38. \mathbf{u} and \mathbf{v} are unit vectors and \mathbf{a} , \mathbf{b} are vectors such that $\mathbf{a} = \mathbf{u} - 3\mathbf{v}$ and $\mathbf{b} = \mathbf{v} - 5\mathbf{u}$. Find the angle between \mathbf{a} and \mathbf{b} if $\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}$.

39. In the figure, $AP \perp OB$.



- (a) Show that $OP = \frac{\overrightarrow{OA} \cdot \overrightarrow{OB}}{|\overrightarrow{OB}|}$.
- (b) Let $\overrightarrow{OA} = 3\mathbf{i} + 4\mathbf{j}$ and $\overrightarrow{OB} = 6\mathbf{i} - 2\mathbf{j}$.
 - (i) Find OP : OB.
 - (ii) Find \overrightarrow{OP} and \overrightarrow{PA} .