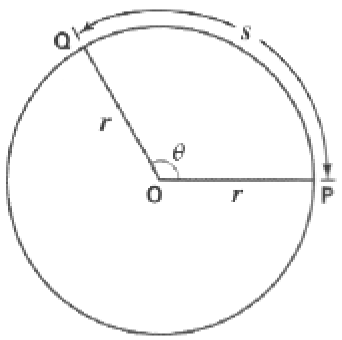
Chapter 8 Applications of Trigonometry

8.1 Arc Length and Area of Sector

8.1.1 Arc Length of a Sector



Refer to the figure, s denotes the arc length of the sector.

$$s = r\theta$$

8.1.2 Area of a Sector

Refer to the figure again, A denotes the area of the sector.

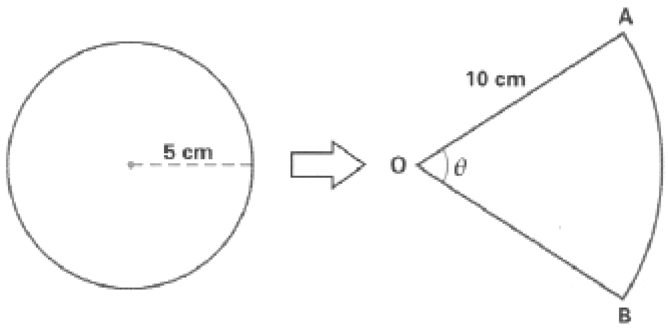
$$A = \frac{1}{2}rs = \frac{1}{2}r^2\theta$$

Example 8.1

A wire in the form of a circle of radius 5 cm is bent to form a sector OAB of a circle of radius 10 cm. Find

(a) the angle of the sector in radians correct to 3 significant figures.

(b) the area of the sector in terms of π.



Solution

(a) Let θ be the angle of the sector in radians.

$$\begin{aligned} \text{Length of wire} &= 2\pi(5) \\ &= 10\pi \text{ cm} \end{aligned}$$

$$\overset{\frown}{\text{AB}} = 10\theta \text{ cm}$$

$$\therefore \quad 10\pi = 2(10) + 10\theta$$

$$\begin{aligned} \theta &= \frac{10\pi - 20}{10} \\ &= 1.14, \text{ corr. to 3 sig. fig.} \end{aligned}$$

(b)

$$\begin{aligned} \text{Area of the sector} &= \frac{1}{2}(10)^2\left(\frac{10\pi - 20}{10}\right)\text{cm}^2 \\ &= (50\pi - 100)\text{ cm}^2 \end{aligned}$$

Checkpoint 8.1

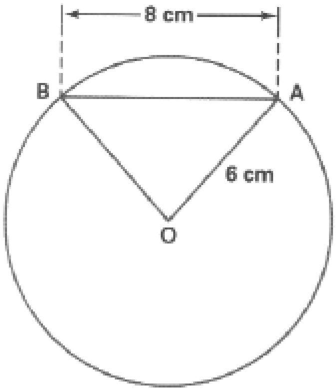
In the figure, O is the centre of the circle of radius 6 cm. AB is a chord of the circle of length 8 cm. Find

(a) $\angle AOB$ in radians,

(b) the length of $\overset{\frown}{AB}$,

(c) the area of minor sector OAB.

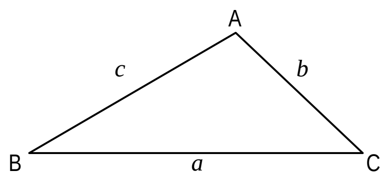
(*Give the answers correct to 3 significant figures.*)



8.2 Area of Triangles

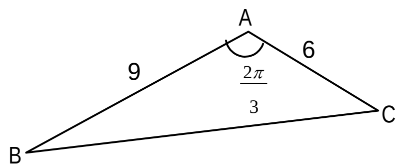
Refer to the figure,

Area of $\Delta ABC = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$



Example 8.2

Find the area of ΔABC if $b = 6$, $c = 9$ and $A = \frac{2\pi}{3}$.

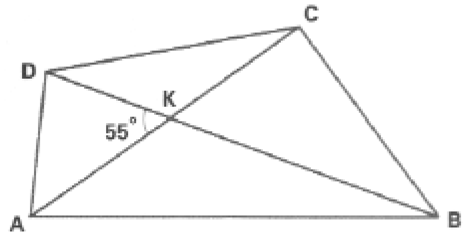


Solution

$$\text{Area of } \Delta ABC = \frac{1}{2}(6)(9)\sin\frac{2\pi}{3}$$
$$= \frac{27\sqrt{3}}{2}$$

Example 8.3

In the figure, ABCD is a quadrilateral with AC = 14 cm, BD = 18 cm and $\angle AKD = 55°$. Find the area of quadrilateral ABCD correct to the nearest $0.1\,\text{cm}^2$.



Solution

Let AK = x cm and DK = y cm.
Then KC = (14 – x) cm and KB = (18 – y) cm.
$\angle BKC = \angle AKD = 55°$ and $\angle AKB = \angle CKD = 180° - 55° = 125°$

$$\text{Area of } \Delta AKD = \frac{1}{2}xy\sin 55°$$

$$\text{Area of } \Delta AKD = \frac{1}{2}x(18-y)\sin 125°$$

$$= \frac{1}{2}x(18-y)\sin 55°$$

$$\text{Area of } \Delta AKD = \frac{1}{2}(14-x)(14-y)\sin 55°$$

$$\text{Area of } \Delta AKD = \frac{1}{2}(14-x)y\sin 125°$$

$$= \frac{1}{2}(14-x)y\sin 55°$$

$$\therefore \quad \text{Area of quadrilateral ABCD} = \left[\frac{1}{2}xy\sin 55° + \frac{1}{2}x(18-y)\sin 55° + \frac{1}{2}(14-x)(18-y)\sin 55° + \frac{1}{2}(14-x)y\sin 55°\right]$$

$$= \frac{1}{2}\sin 55°[xy + x(18-y) + (14-x)(18-y) + (14-x)y]$$

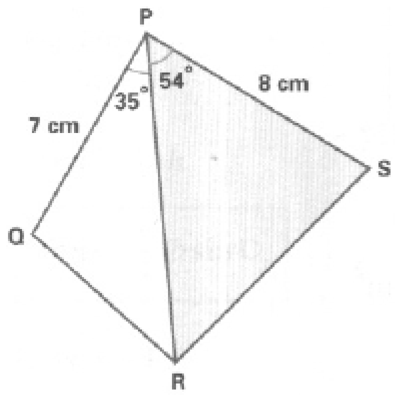
$$= \frac{1}{2}\sin 55°(xy + 18x - xy + 252 - 14y - 18x + 14y + xy - xy)$$

$$= \frac{1}{2}\sin 55° \times 252$$

$$= 103.2\,\text{cm}^2, \text{ corr. to the nearest } 0.1\,\text{cm}^2.$$

Checkpoint 8.2

In the figure, if the area of ΔPQR is $20\,\text{cm}^2$, find the area of ΔPRS.



8.3 Solution of Triangles

8.3.1 The Sine Formula

For any ΔABC,

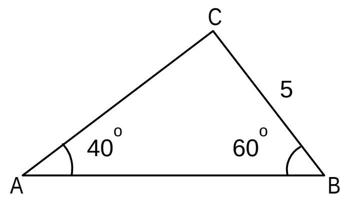
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

where R is the radius of the circumcircle of ΔABC.

Example 8.4

In ΔABC, $A = 40°$, $B = 60°$, $a = 5°$. Solve the triangle, giving your answers correct to 3 significant figures where necessary.

Solution



$$\begin{aligned} C &= 180° - A - B \\ &= 180° - 40° - 60° \\ &= 80° \end{aligned}$$

By the sine formula,

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\begin{aligned} b &= \frac{a \sin B}{\sin A} \\ &= \frac{5 \sin 60°}{\sin 40°} \\ &= 6.74, \text{ corr. to 3 sig. fig.} \end{aligned}$$

Also,

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\begin{aligned} c &= \frac{a \sin C}{\sin A} \\ &= \frac{5 \sin 80°}{\sin 40°} \\ &= 7.66, \text{ corr. to 3 sig. fig.} \end{aligned}$$

Example 8.5

Solve ΔABC if

(a) $A = 40°$, $a = 6$ and $b = 8$;

(b) $A = 40°$, $a = 5$ and $b = 8$.

(*Give the answers correct to 3 significant figures if necessary.*)

Solution

(a) By the sine formula,

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\sin B = \frac{8\sin 40°}{6}$$

$$B = 59.0° \quad \text{or} \quad 121°, \text{corr. to 3 sig. fig.}$$

If $B = 58.987°$,

$$\begin{aligned} C &= 180° - A - B \\ &= 180° - 40° - 58.987° \\ &= 81.0°, \text{corr. to 3 sig. fig.} \end{aligned}$$

$$\because \quad \frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\therefore \quad c = \frac{6\sin 81.013°}{\sin 40°}$$

$$B = 9.22, \text{corr. to 3 sig. fig.}$$

If $B = 121.013°$,

$$\begin{aligned} C &= 180° - A - B \\ &= 180° - 40° - 121.013° \\ &= 19.0°, \text{corr. to 3 sig. fig.} \end{aligned}$$

$$\because \quad \frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\therefore \quad c = \frac{6\sin 18.987°}{\sin 40°}$$

$$B = 3.04, \text{corr. to 3 sig. fig.}$$

$$\text{Hence, } \begin{cases} B = 59.0° \\ C = 81.0° \\ c = 9.22 \end{cases} \quad \text{or} \quad \begin{cases} \text{B} = 121° \\ C = 19.0° \\ c = 3.04 \end{cases}$$

(b) By the sine formula,

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\sin B = \frac{8\sin 40°}{5}$$

$$= 1.0285$$

$$> 1$$

$\therefore$ There is no solution.

Checkpoint 8.3

Consider ΔABC with $a = 3.2$, $B = 55°$ and $C = 65°$.

(a) Solve ΔABC.

(b) Hence, find the area of ΔABC.

(*Give the answers correct to 3 significant figures if necessary*.)

Checkpoint 8.4

In each of the following, solve ΔABC.

(a) $a = 3$, $b = 4$ and $A = 45°$.

(b) $b = 5$, $c = 4$ and $B = 85°$.

(*Give the answers correct to 3 significant figures if necessary.*)

8.3.2 The Cosine Formula

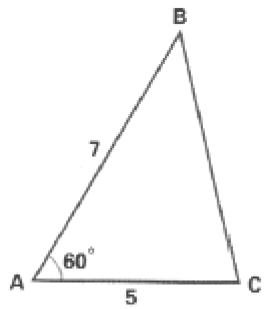
For any ΔABC,

$$\begin{cases} a^2 = b^2 + c^2 - 2bc\cos A \\ b^2 = a^2 + c^2 - 2ac\cos B \\ c^2 = a^2 + b^2 - 2ab\cos C \end{cases} \quad \text{or} \quad \begin{cases} \cos A = \dfrac{b^2 + c^2 - a^2}{2bc} \\ \cos B = \dfrac{a^2 + c^2 - b^2}{2ac} \\ \cos C = \dfrac{a^2 + b^2 - c^2}{2ab} \end{cases}$$

Example 8.6

In the figure, $A = 60°$, $b = 5$, $c = 7$. Solve ΔABC.

(*Give the answers correct to 3 significant figures if necessary.*)



Solution

By the cosine formula,

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$a = \sqrt{5^2 + 7^2 - 2(5)(7)\cos 60°}$$

$$= 6.24, \text{corr. to } 3 \text{ sig. fig.}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{6.2450^2 + 7^2 - 5^2}{2(6.2450)(7)}$$

$$\therefore \quad B = 43.9°, \text{corr. to } 3 \text{ sig. fig.}$$

$$C = 180° - 60° - 43.898°$$

$$= 76.1°, \text{corr. to } 3 \text{ sig. fig.}$$

Checkpoint 8.5

Solve ΔABC if $B = 42.5°$, $a = 8$, $c = 11$.

Example 8.7

In ΔABC, find $\cos A : \cos B : \cos C$ if $\sin A : \sin B : \sin C = 2 : 3 : 4$.

Solution

By the sine formula,

$$a : b : c = \sin A : \sin B : \sin C = 2 : 3 : 4$$

$\therefore \quad \frac{a}{2} = \frac{b}{3} = \frac{c}{4} = k$, where k is a non-zero constant.

We have $a = 2k$, $b = 3k$, $c = 4k$.

By the cosine formula,

$$\cos A = \frac{(3k)^2 + (4k)^2 - (2k)^2}{2(3k)(4k)}$$

$$= \frac{7}{8}$$

$$\cos B = \frac{(2k)^2 + (4k)^2 - (3k)^2}{2(2k)(4k)}$$

$$= \frac{11}{16}$$

$$\cos C = \frac{(2k)^2 + (3k)^2 - (4k)^2}{2(2k)(3k)}$$

$$= -\frac{1}{4}$$

$\therefore$ $\cos A : \cos B : \cos C = \frac{7}{8} : \frac{11}{16} : -\frac{1}{4}$

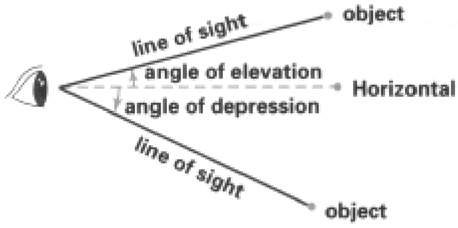
$= 14 : 11 : -4$

Checkpoint 8.6

In ΔABC, find the ratio of $\cos A : \cos B : \cos C$ if $a = 7$, $b = 9$, $c = 12$.

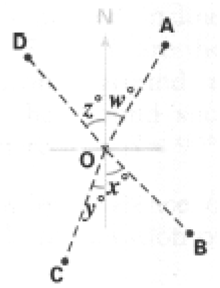
8.4 Problems in Two Dimensions

8.4.1 Angles of Elevation and Depression



8.4.2 Bearings

(1) Compass Bearing



The compass bearing of

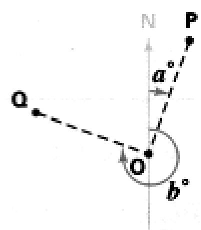
A from O is Nw°E;

B from O is Sx°E.

C from O is Sy°W.

D from O is Nz°EW

(2) True Bearing



The true bearing of

P from O is a°,

Q from O is b°.

(The integral part of a and b is written in 3 digits.)

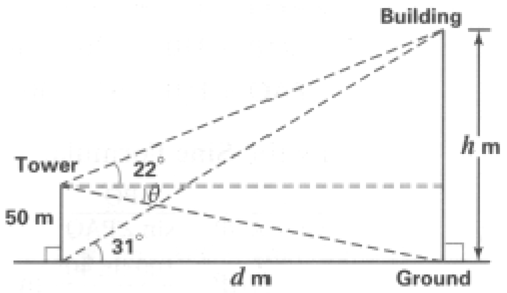
Example 8.8

From the top and the bottom of a vertical tower, the angles of elevation of the top of a vertical building are $22°$ and $31°$ respectively. If the height of the building is 50 metres, find

(a) the height of the building;

(b) the angle of depression of the bottom of the building from the top of the tower.

(*Give the answers correct to 3 significant figures.*)



Solution

(a) Let the height of the building be *h* metres, and the distance between the building and the tower be *d* metres.

$$\tan 31° = \frac{h}{d}$$

$\therefore$ $$d = h\cot 31°$$

$$\tan 22° = \frac{h-50}{d}$$

$\therefore$ $$d = (h-50)\cot 22°$$

$$h = \frac{50\cot 22°}{\cot 22° - \cot 31°}$$

$$= 153, \text{ corr. to } 3 \text{ sig.fig.}$$

$\therefore$ The height of the building is 153 metres.

(b) Let θ be the angle of depression of the bottom of the building from the top of the tower.

$$\text{Then } \tan\theta = \frac{50}{d}$$

$$= \frac{50}{h\cot 31°}$$

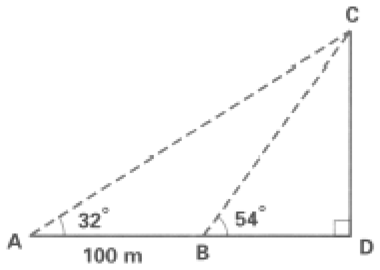
$$= \frac{50\tan 31°}{152.63}$$

$$\theta = 11.1°, \text{ corr. to } 3 \text{ sig. fig.}$$

$\therefore$ The angle of depression of the bottom of the building from the top of the tower is $11.1°$.

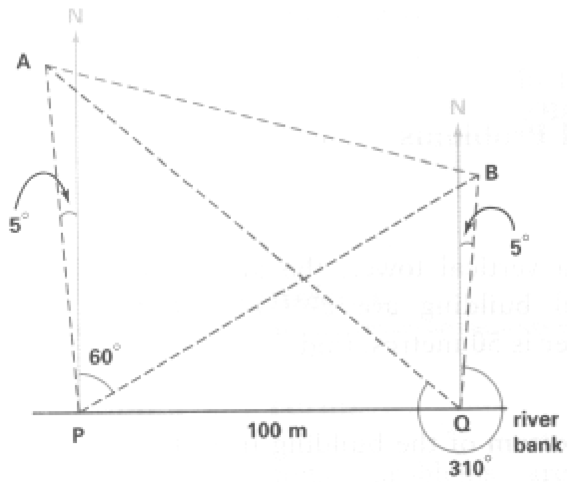
Checkpoint 8.7

The angles of elevation of the top C of a tower CD from two points A and B are $32°$ and $54°$ respectively. A and B are 100 m apart and in line with D. Find the height of the tower.



Example 8.9

Two buoys A and B in a river are observed from two points P and Q on the river bank running in east-west direction. The compass bearings of A and B as observed from P are N5°W and N60°E respectively. The true bearings of A and B as observed from Q are 310° and 005° respectively. If Q is 100 m east of P, find the distance between A and B correct to 3 significant figures.



Solution

In ΔAPQ,

$$\angle APQ = 5° + 90° = 95°$$
$$\angle AQP = 310° - 270° = 40°$$
$$\angle PAQ = 180° - 95° - 40° = 45°$$

By the sine formula,

$$\frac{AP}{\sin\angle AQP} = \frac{PQ}{\sin\angle PAQ}$$
$$AP = \frac{100\sin 40°}{\sin 45°}$$
$$= 90.904 \text{ m}$$

In ΔBPQ,

$$\angle BQP = 5° + 90° = 95°$$
$$\angle BPQ = 90° - 60° = 30°$$
$$\angle PBQ = 180° - 95° - 30° = 55°$$

By the sine formula,

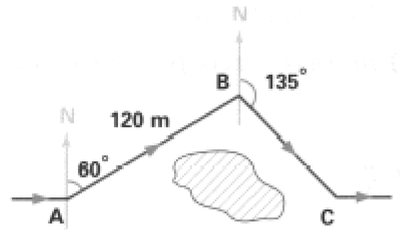
$$\frac{BP}{\sin\angle BQP} = \frac{PQ}{\sin\angle PBQ}$$
$$AP = \frac{100\sin 95°}{\sin 55°}$$
$$= 121.61 \text{ m}$$

In ΔAPB, by the cosine formula,

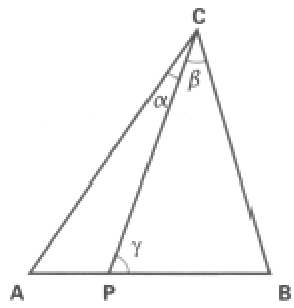
$$AB^2 = AP^2 + BP^2 - 2(AP)(BP)\cos\angle APB$$
$$AB = \sqrt{90.904^2 + 121.61^2 - 2(90.904)(121.61)\cos 65°}$$
$$= 117 \text{ m, corr. to 3 sig. fig.}$$

Checkpoint 8.8

A scout moving due east turns at A (see the figure) to avoid an obstacle and walks 120 m to B and then turns and walks to C. The true bearing of B from A is $060°$ and the true bearing of C from B is $135°$. Find the length of BC if C is due east of A.



Example 8.10



In the figure, P is a point on AB of ΔABC such that $\frac{\text{AP}}{\text{PB}} = \frac{m}{n}$. Let $\angle\text{ACP} = \alpha$, $\angle\text{BCP} = \beta$ and $\angle\text{BPC} = \gamma$. Prove that

(a) $\frac{\text{AC}}{\text{BC}} = \frac{\sin(\gamma+\beta)}{\sin(\gamma-\alpha)}$.

(b) $m\cot\alpha - n\cot\beta = (m+n)\cot\gamma$.

Solution

(a) $\angle\text{ABC} = 180° - \gamma - \beta$

$\angle\text{CAB} = \gamma - \alpha$

In ΔABC, by the sine formula,

$$\frac{\text{AC}}{\sin\angle\text{ABC}} = \frac{\text{BC}}{\sin\angle\text{CAB}}$$

$$\frac{\text{AC}}{\sin(180° - \gamma - \beta)} = \frac{\text{BC}}{\sin(\gamma-\alpha)}$$

$$\therefore \quad \frac{\text{AC}}{\text{BC}} = \frac{\sin(\gamma+\beta)}{\sin(\gamma-\alpha)}$$

(b) In ΔACP,

$\angle\text{APC} = 180° - \gamma$

By the sine formula,

$$\frac{\text{AC}}{\sin\angle\text{APC}} = \frac{\text{AP}}{\sin\angle\text{ACP}}$$

$$\text{AP} = \frac{\text{AC}\sin\alpha}{\sin\gamma} \quad ......(1)$$

In ΔBCP, by the sine formula,

$$\frac{\text{BC}}{\sin\angle\text{BPC}} = \frac{\text{BP}}{\sin\angle\text{BCP}}$$

$$\text{BP} = \frac{\text{BC}\sin\beta}{\sin\gamma} \quad ......(2)$$

$\frac{(1)}{(2)}$:

$$\frac{\text{AP}}{\text{PB}} = \frac{\text{AC}\sin\alpha}{\text{BC}\sin\beta}$$

$$\frac{\text{AP}}{\text{PB}} = \frac{m}{n} \quad \textit{and} \quad \frac{\text{AC}}{\text{BC}} = \frac{\sin(\gamma+\beta)}{\sin(\gamma-\alpha)}$$

$$\frac{m}{n} = \frac{\sin(\gamma+\beta)\sin\alpha}{\sin(\gamma-\alpha)\sin\beta}$$

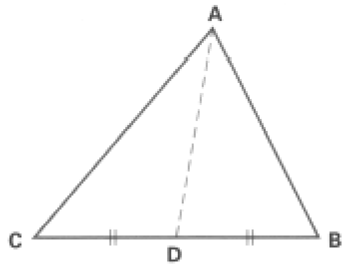
$$m[\sin\gamma\cos\alpha - \cos\gamma\sin\alpha]\sin\beta = n[\sin\gamma\cos\beta + \cos\gamma\sin\beta]\sin\alpha$$

$$m\cot\alpha - m\cot\gamma = n\cot\beta + n\cot\gamma$$

$$m\cot\alpha - n\cot\beta = (m+n)\cot\gamma$$

Checkpoint 8.9

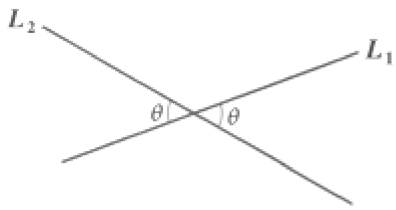
In ΔABC, D is the mid-point of the base BC. Prove that $4AD^2 + BC^2 = 2(AB^2 + AC^2)$.



8.5 Problems in Three Dimensions

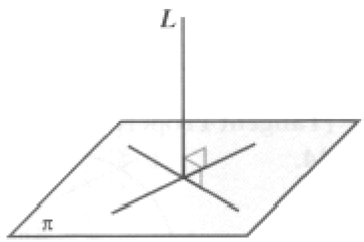
7.5.1 Angle Between Two Intersecting Straight Lines

The angle between two intersecting straight lines L_1 and L_2, in general, means the *acute* angle between them.



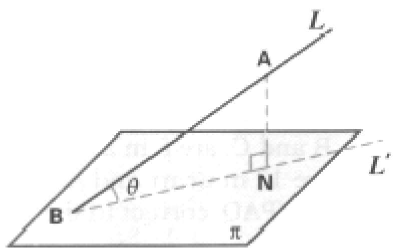
7.5.2 Perpendicular Line to a Plane

A straight line L is perpendicular to a plane π if it is perpendicular to every straight line lying on that plane.



7.5.3 Angle Between A Straight Line and a Plane

In the figure, L is a straight line intersecting the plane π at B and AN is the perpendicular line drawn from point A to the plane.

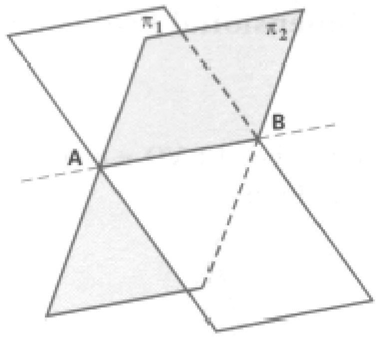


N is the point of ***projection*** of the point A on the plane π. The line L' is the projection of the line L on the plane.

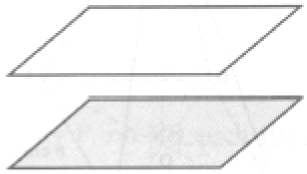
The angle between the line L and the plane π is the angle between the line L and its projection L' on the plane, i.e. θ.

7.5.4 Line of Intersection of Two Planes

When two lines intersect, they must meet in a line and this line is called the line of intersection of two planes.

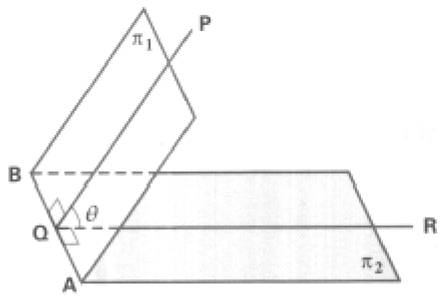


If two planes are parallel, there is no line of intersection.



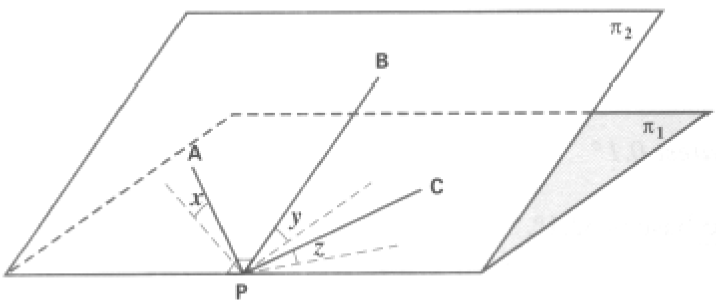
7.5.5 Angle Between Two Intersecting Planes

The angle between two intersecting planes in general refers to the acute angle between two straight lines, one in each plane, drawn perpendicular to and from a point on the line of intersection.



7.5.6 Line of the Greatest Slope

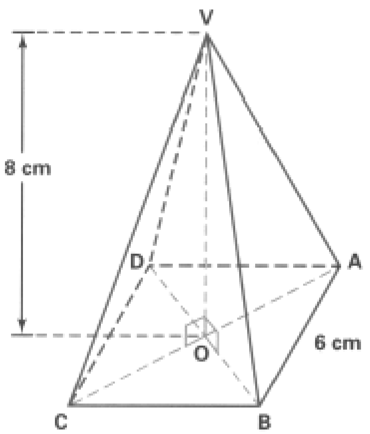
The figure shows two intersecting planes π_1 and π_2, one is horizontal and the other inclined. Different lines on the inclined plane π_2 make different angles of inclination with the horizontal plane π_1.



The line BP, perpendicular to the line of intersection, makes the greatest angle of inclination with the horizontal plane is thus is called the line of the greatest slope.

Example 8.11

VABCD is a right pyramid of height VO = 8 cm. Its base ABCD is a square with side 6 cm long. Find



(a) the angle between the slant edge VA and the base,

(b) the angle between the slant face VAB and the base,

(c) the angle between the adjacent slant faces VAB and VBC.

(*Give the answers correct to the nearest 0.1^o.*)

Solution

(a) The angle between VA and the base is $\angle VAO$.

$$AC = \sqrt{AB^2 + BC^2}$$
$$= \sqrt{6^2 + 6^2}$$
$$= 6\sqrt{2} \text{ cm}$$
$$AO = \frac{1}{2}AC = 3\sqrt{2} \text{ cm}$$

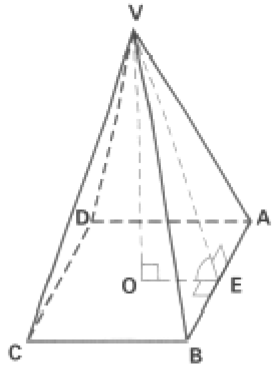
In ΔVAO,

$$\tan \angle VAO = \frac{VO}{AO}$$
$$= \frac{8}{3\sqrt{2}}$$
$$\angle VAO = 62.1°, \text{ corr. to the nearest } 0.1°$$

$\therefore$ The angle between VA and the base is $62.1°$.

(b) With the notation as shown in the figure, the angle between VAB and the base is $\angle VEO$.

Since E is the mid-point of AB,



$$OE = \frac{1}{2}BC = 3\text{ cm}$$

In ΔVEO,

$$\tan\angle VEO = \frac{VO}{OE} = \frac{3}{8}$$

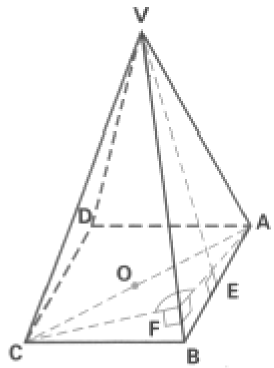
$$\angle VEO = 69.4°\text{, corr. to the nearest }0.1°$$

$\therefore$ The angle between VAB and the base is $69.4°$.

(c) Since $\Delta VAB \cong \Delta VCB$, there is a point, say F, on VB such that $AF \perp VB$ and $CF \perp VB$.

The angle between VAB and VBC is $\angle AFC$.

In ΔVEO,



$$VE^2 = VO^2 + OE^2$$

$$VE = \sqrt{8^2 + 3^2}$$

$$= \sqrt{73}\text{ cm}$$

In ΔVBE,

$$\tan\angle VBE = \frac{VE}{BE}$$

$$= \frac{\sqrt{73}}{\frac{1}{2}\times 6}$$

$$\angle VBE = 70.653°$$

$\therefore$ $AF = AB\sin\angle VBE$

$$= 6\sin 70.653°$$

$$= 5.661\text{ cm}$$

$\therefore$ $CF = 5.661\text{ cm}$

In ΔAFC,

$$AC = 6\sqrt{2}\text{ cm}$$

By the cosine formula,

$$\cos\angle AFC = \frac{AF^2 + CF^2 - AC^2}{2(AF)(CF)}$$

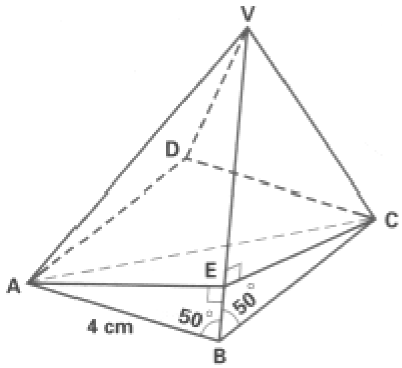
$$= \frac{2(5.661)^2 - (6\sqrt{2})^2}{2(5.661)^2}$$

$$\angle AFC = 97.1°\text{, corr. to the nearest }0.1°$$

$\therefore$ The angle between VAB and VBC is $97.1°$.

Checkpoint 8.10

In the figure, VABCD is a right pyramid with a square base ABCD. If AB = 4 cm, $\angle VBA = \angle VBC = 50°$, find

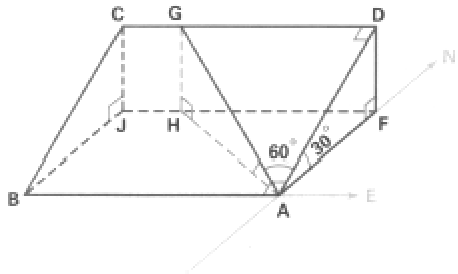


(a) AE and AC;

(b) the angle between the faces VAB and VBC.

Example 8.12

In the figure, a plane hillside ABCD facing due north inclines at an angle of $30°$ with the horizontal. A path AG on its face makes an angle of $60°$ with the line of greatest slope AD. Find



(a) the angle of inclination of the path AG to the horizontal,

(b) the compass bearing of G from A.

(*Give the answers correct to the nearest 0.1°*.)

Solution

The angle of inclination of AG to the horizontal is $\angle GAH$.

Let DF = GH = *h*.

(a) In ΔADF,

$$\sin\angle DAF = \frac{DF}{AD}$$

$$\sin 30° = \frac{h}{AD}$$

$$AD = 2h$$

In ΔADG,

$$\sin\angle GAD = \frac{AD}{AG}$$

$$\cos 60° = \frac{2h}{AG}$$

$$AG = 4h$$

In ΔAGH,

$$\sin\angle GAH = \frac{GH}{AG}$$

$$\cos 60° = \frac{h}{4h}$$

$$= \frac{1}{4}$$

$\therefore\ \angle GAH = 14.5°$, corr. to the nearest $0.1°$

(b) The compass bearing of G from A is given by $\angle HAF$.

In ΔADF,

$$\tan \angle DAF = \frac{DF}{AF}$$
$$\cos 60^\circ = \frac{h}{AF}$$
$$AF = \sqrt{3}h$$

In ΔAGH,

$$AH^2 + GH^2 = AG^2$$
$$AH = \sqrt{AG^2 - GH^2}$$
$$= \sqrt{(4h)^2 - h^2}$$
$$= \sqrt{15}h$$

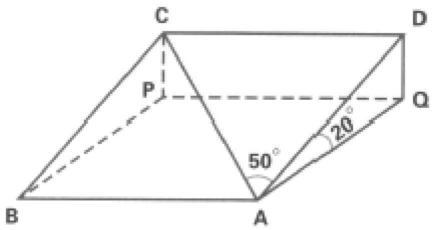
In ΔAFH,

$$\cos \angle HAF = \frac{AF}{AH}$$
$$AH = \frac{\sqrt{3}h}{\sqrt{15}h}$$
$$= \frac{1}{\sqrt{5}}$$
$$\angle HAF = 63.4^\circ, \text{ corr. to the nearest } 0.1^\circ$$

$\therefore$ The compass bearing of G from A is N63.4°W.

Checkpoint 8.11

In the wedge ABCDQP, the rectangle CDQP is perpendicular to the rectangle ABPQ. The angle between the line of the greatest slope AD and the horizontal plane ABPQ is $20°$. If $\angle CAD = 50°$ and CP = 32 cm, find

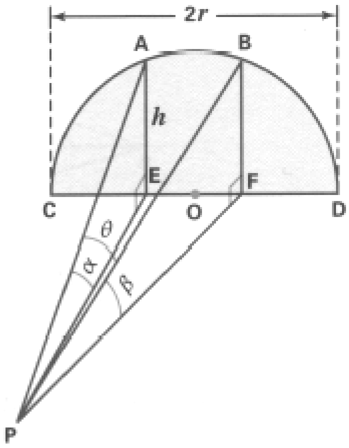


(a) the length of AC,

(b) the angle between AC and the plane ABPQ.

Example 8.13

A vertical wall is in the shape of a semi-circle of diameter $2r$, centre O. Two points A and B lie on the top of the wall. The projections of A and B on the horizontal ground are E and F respectively such that they trisect the base of the wall CD. From an external point P on the horizontal ground, the angles of elevation of A and B are α and β respectively. Let h be the height of A above the ground and $\angle \text{APB} = \theta$.



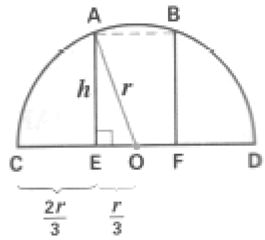
(c) Express h in terms of r.

(d) Show that $\cos\theta = \dfrac{2\sin^2\alpha + 2\sin^2\beta - \sin^2\alpha\sin^2\beta}{4\sin\alpha\sin\beta}$.

(e) If $\alpha = \beta$, find the least value of cos θ.

Solution

(a) $\because$ E and F trisect CD,



$\therefore$ $\text{CE} = \dfrac{2r}{3}$ and $\text{EO} = r - \dfrac{2r}{3} = \dfrac{r}{3}$.

In ΔAEO, $\text{AE}^2 + \text{EO}^2 = \text{AO}^2$

$\therefore$ $$h^2 + \left(\frac{r}{3}\right)^2 = r^2$$

$$h^2 = \frac{8}{9}r^2$$

$\therefore$ $$h = \frac{2\sqrt{2}}{3}r$$

(b) By symmetry, BF = AE = h.

$$\text{AB} = \text{EF} = \frac{2r}{3} = \frac{h}{\sqrt{2}}$$

In ΔAEP, $\sin\alpha = \frac{\text{AE}}{\text{AP}}$

$$\text{AP} = \frac{h}{\sin\alpha}$$

In ΔBFP, $\sin\beta = \frac{\text{BF}}{\text{BP}}$

$$\text{BP} = \frac{h}{\sin\beta}$$

In ΔABP, by the cosine formula,

$$\cos\angle\text{APB} = \frac{\text{AP}^2 + \text{BP}^2 - \text{AB}^2}{2(\text{AP})(\text{BP})}$$

$$\cos\theta = \frac{\left(\frac{h}{\sin^2\alpha}\right)^2 + \left(\frac{h}{\sin^2\beta}\right)^2 - \left(\frac{h}{\sqrt{2}}\right)^2}{2\left(\frac{h}{\sin\alpha}\right)\left(\frac{h}{\sin\beta}\right)}$$

$$= \frac{\frac{1}{\sin^2\alpha} + \frac{1}{\sin^2\beta} - \frac{1}{2}}{\frac{2}{\sin\alpha\sin\beta}}$$

$$= \frac{2\sin^2\beta + 2\sin^2\alpha - \sin^2\alpha\sin^2\beta}{4\sin\alpha\sin\beta}$$

$\therefore \quad \cos\theta = \frac{2\sin^2\alpha + 2\sin^2\beta - \sin^2\alpha\sin^2\beta}{4\sin\alpha\sin\beta}$

(c) If $\alpha = \beta$, $\cos\theta = \frac{4\sin^2\alpha - \sin^4\alpha}{4\sin^2\alpha} = 1 - \frac{1}{4}\sin^2\alpha$

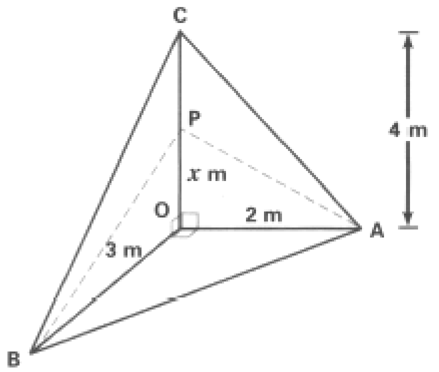
The least value of $\cos\theta$ occurs when the value of $\sin^2\alpha$ is greatest.

i.e. when $\sin^2\alpha = 1$.

$\therefore$ The least value of $\cos\theta = 1 - \frac{1}{4}(1) = \frac{3}{4}$.

Checkpoint 8.12

ΔOAB is a right-angled triangle lying on a horizontal plane with $\angle \text{AOB} = 90°$. C is the point vertically above O and P is a variable point on OC such that OP = *x* m. Given that OA = 2 m, OB = 3 m, OC = 4 m.



(a) If $\angle \text{BPA} = \theta$, express $\cos\theta$ in terms of *x*.

(b) (i) Using the result of (a), or otherwise, show that $S = \frac{1}{2}\sqrt{13x^2 + 36}$.

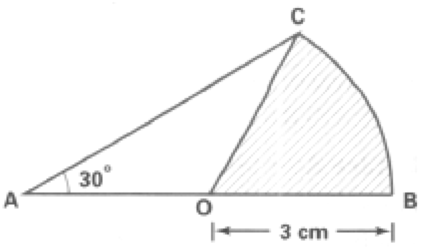
(ii) If $S^2 \le \frac{49}{4}$, find the possible range of values of *x*.

Exercise 8 Applications of Trigonometry

In this exercise, unless otherwise stated, give the answers of lengths correct to 3 significant figures and the answers of angles correct to the nearest 0.1^{o} *where necessary.*

8.1

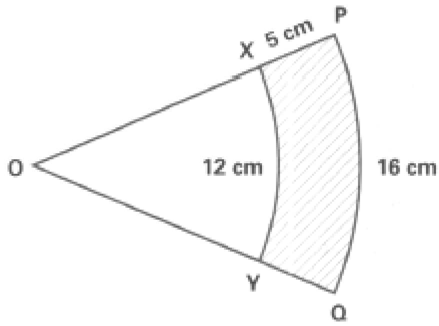
1. AB is a diameter of a circle of radius 3 cm. AC is a chord of the circle making an angle of 30^{o} with AB. Find the area of the shaded sector AOB.



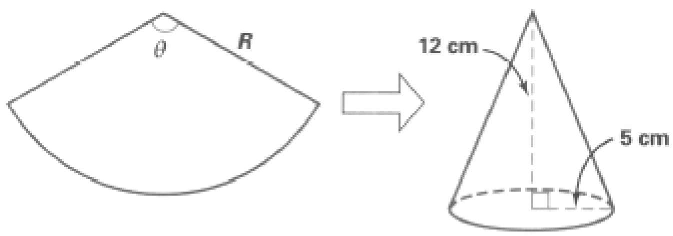
2. In the figure, $\overset{\frown}{XY}$ and $\overset{\frown}{PQ}$ are arcs of two concentric circles of centre O. OXP and OYQ are two straight lines. If $\overset{\frown}{XY} = 12$ cm, $\overset{\frown}{PQ} = 16$ cm and XP = 5 cm, find

   (a) $\angle XOY$,

   (b) the area of the shaded region.



3. A thin metallic plate in the form of a sector of radius *R* and angle θ (in radians) is bent to form a right circular cone of radius 5 cm and height 12 cm. Find *R* and θ.

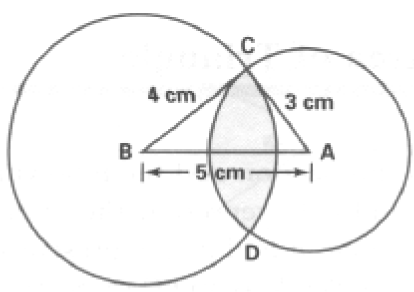


4. Two intersecting circles of radii 3 cm and 4 cm with centres A and B are at a distance of 5 cm apart. Find

   (a) $\angle CAB$,

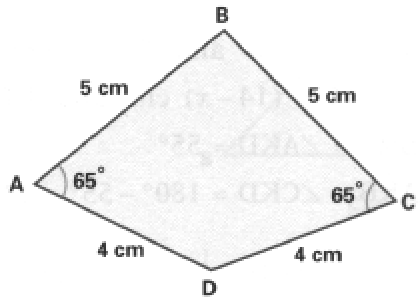
   (*Give the answer correct to the nearest* 0.1^{o}.)

   (b) the area common to both circles.

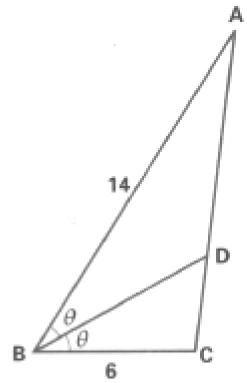


8.2

5. In the figure, given AB = BC = 5 cm, AD = DC = 4 cm, $\angle BAD = \angle BCD = 65°$. Find the area of the quadrilateral ABCD.



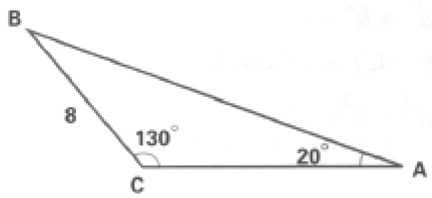
6. In the figure, BD is the angle bisector of $\angle ABC$, $\angle ABD = \angle DBC = \theta$, AB = 14 and BC = 6. By considering the areas of ΔABC, ΔABD and ΔBCD, find the length of BD if $\cos\theta = \frac{7}{8}$.



7. The area of ΔABC is 9 cm^2. If AB = BC = 6 cm, find $\angle BAC$.

8.3

8. In the figure, solve ΔABC.



9. Solve the triangle ABC in each of the following.
   (a) $A = 129°$, $a = 8$, $b = 5$.
   (b) $A = 50°$, $a = 6$, $c = 7$.
   (c) $a = 13$, $b = 8$, $B = 45.5°$.

10. In ΔABC, if $A : B : C = 2 : 3 : 4$ and $a + b + c = 20$, find the values of *a*, *b* and *c* correct to 2 decimal places.

11. Show that for any ΔABC,

$$a[\sin B - \sin(A+B)] + b[\sin(A+B) - \sin A] + c(\sin A - \sin B) = 0 .$$

12. (a) Solve $2\sin 3\theta - 3\sin\theta = 0$ for $0° < \theta < 90°$.

(b) In ΔPQR, if $\angle PQR = 2\angle QPR = 2\theta$ and 2PQ = 3QR, find θ.

13. Solve the triangle ABC in each of the following.

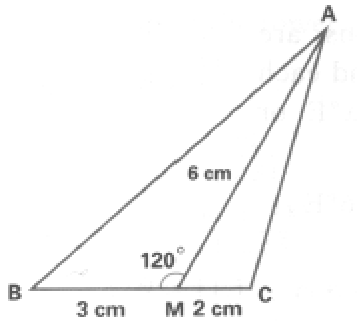
(a) $a = 9$, $b = 7$, $c = 5$.

(b) $a = 3$, $c = 7$ $B = 30°$.

14. In ΔABC, if BM = 3 cm, MC = 2 cm, AM = 6 cm and $\angle AMB = 120°$, find

(a) AB,

(b) AC.



15. In ΔXYZ, YZ = 25, ZX = 21 and XY = 20. Find the length of the perpendicular from Z to XY.

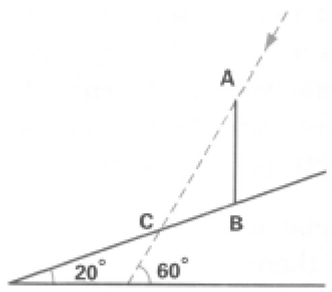
16. In ΔABC, $\sin A : \sin B : \sin C = 8 : 9 : 10$, find $\cos A : \cos B : \cos C$.

17. In ΔABC, $A = 60°$, a = 6 and $b - c$ = 1. Find the values of b and c and the area of ΔABC.

18. Show that for any triangle ABC, $\dfrac{\tan B}{\tan C} = \dfrac{b}{c} \times \left(\dfrac{b - c\cos A}{c - b\cos A}\right)$.

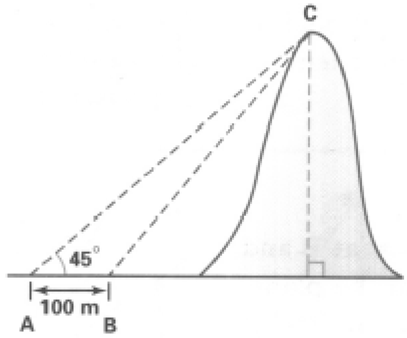
8.4

19. A man AB, 1.73 m high, stands vertically on a hillside inclined at an angle of 20° to the horizontal (see the figure). When the elevation of the sun is 60°, find the length of his shadow BC on the hillside.

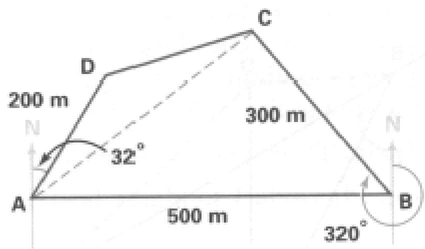


20. (a) Show that $\sin 75° = \frac{\sqrt{6}+\sqrt{2}}{4}$.

(b) In the figure, A and B are two points on the level ground. A, B and the foot of the hill all lie in the same straight line. From the top C of the hill, the difference in the angles of depression of A and B is $30°$. If the angle of elevation of C from A is $45°$ and AB = 100 m, show that the height of the hill is $50(\sqrt{3}+1)$ m.



21. The figure shows a field in the form of a quadrilateral ABCD where AB = 500 m, BC = 300 m and AD = 200 m. B is due east of A. The true bearing of D from A is $032°$ and the true bearing of C from B is $320°$.

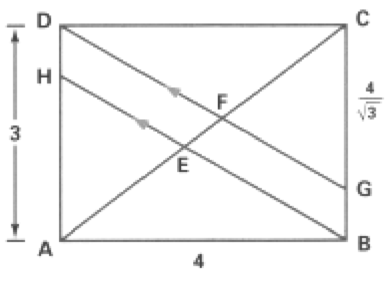


(a) Find CA and the true bearing of C from A.

(b) Find the length of CD.

(c) Find the area of ABCD.

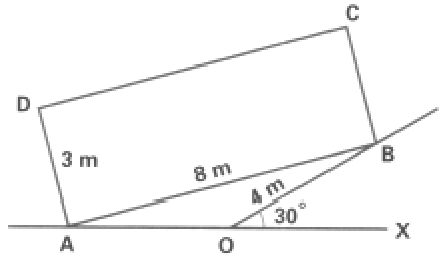
22. In the figure, ABCD is a rectangle. AB = 4, AD = 3, $CG = \frac{4}{\sqrt{3}}$ and GD // BH.



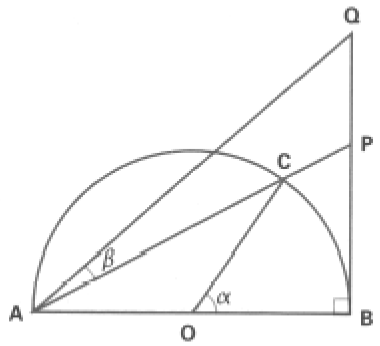
(a) Find $\angle CAB$ and $\angle HBA$.

(b) Find EF.

23. A rectangular block with dimensions 3 m × 8 m rests against an inclined plane OB (see the figure). AOX is the ground and OB = 4 m. Find the heights of C and D above the ground.



24. In the figure, AB is a diameter of a circle, centre O, and BPQ is perpendicular to AB. $\angle PAQ = \beta$, $\angle BOC = \alpha$ and the radius of the circle = r.

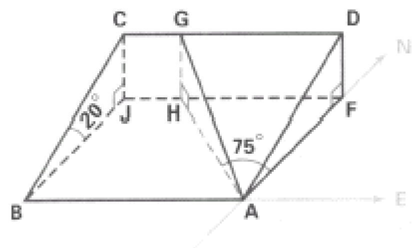


(a) Prove that $PQ = 2r\tan\left(\beta + \frac{\alpha}{2}\right) - 2r\tan\frac{\alpha}{2}$;

(b) Hence, or otherwise, deduce that $\frac{2AB}{PQ} = \cot\beta + \cos(\alpha+\beta)\csc\beta$.

8.5

25. The figure shows a hillside ABCD which is a rectangular plane facing due north. ABCD makes an angle of 20° with the horizontal plane ABJF. A path AG lies on ABCD and the compass bearing of G from A is N75°W . Find the angle of inclination of the path AG with the horizontal.

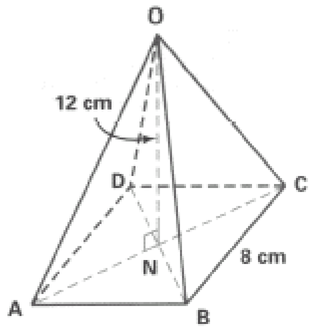


26. The figure shows a right pyramid OABCD on a square base ABCD. If BC = 8 cm and ON 12 cm, find

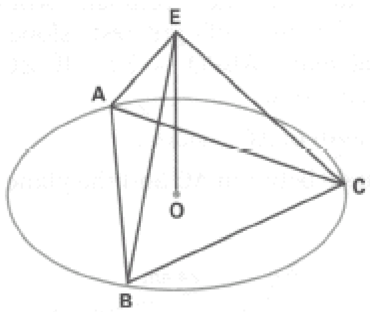
(a) the angle between the plane OAB and the base,

(b) the angle between the planes OAB and OBC,

(c) the angle between the planes OAB and OCD.



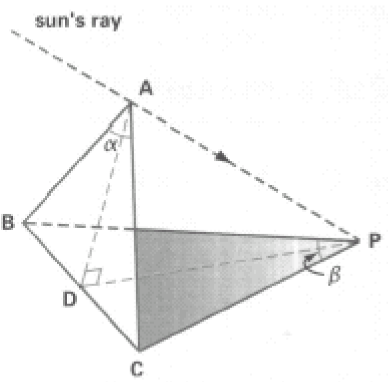
27. In the figure, ABC is an iron framework in the form of an equilateral triangle inscribed in a circular rope, centre O, radius 8 cm. ΔABC lies in a horizontal plane. A vertical pole OE of length 6 m is fixed vertically at O, and suspended by wires from E to A, B, C. Find



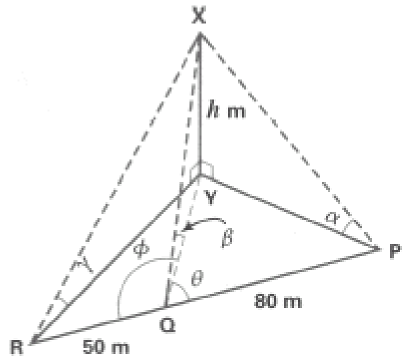
(a) $\angle AEB$,

(b) the angle between the planes BEC and BAC.

28. In the figure, ABC is an isosceles triangle in a vertical plane with AB = AC and its base BC lies in a horizontal plane. The shadow of ΔABC on the horizontal plane is ΔPBC with PB = PC. If $\angle BPC = \beta$, $\angle BAC = \alpha$ and the angle of elevation of the sun from P is θ, show that $\tan\theta = \cot\frac{\alpha}{2}\tan\frac{\beta}{2}$.



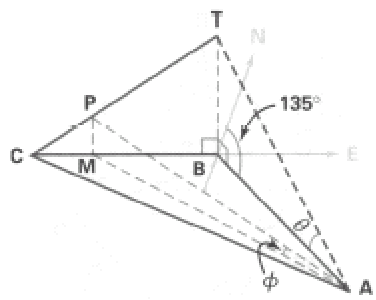
29. P, Q, R are points on a horizontal line such that PQ = 80 m and QR = 50 m. The angles of elevation of a vertical pole XY from P, Q and R are α, β and γ respectively, where $\tan\alpha = \frac{1}{3}$, $\tan\beta = \frac{1}{7}$ and $\tan\gamma = \frac{1}{10}$. The foot of the pole, Y, is at the same level as P, Q and R. Let XY = h m, $\angle YQP = \theta$ and $\angle YQR = \phi$.



(a) Express $\cos\theta$ and $\cos\phi$ in terms of h.

(b) Find the values of h.

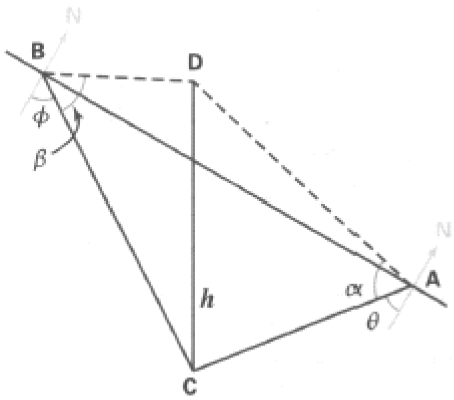
30. In the figure, A, B and C are three points on a horizontal plane with AB = a and BC = $3b$. BC runs in the east-west direction and the true bearing of A from B is $135°$. M is a point on BC such that BM = 2MC. T and P are points vertically above B and M respectively and θ is the angle of elevation of T from A.



(a) Express AM in terms of a and b. Hence, or otherwise, show that if ϕ is the angle of elevation of P from A, then $\tan\phi = \dfrac{a\tan\theta}{3\sqrt{a^2 + 4b^2 + 2\sqrt{2}ab}}$.

(b) If $a = 3b$, find the true bearing of P from A.

31. In the figure, CD is a vertical tower of height h and A, B, C are on the same horizontal ground. The bearings of C from A and B are SθW and SϕE respectively. The angles of elevation of D from A and B are α and β respectively. If B is at a distance of b due west of A, show that



(a) $h\sin(\theta+\phi)=b\cos\phi\tan\alpha$

(b) $\cos\phi=\dfrac{\cos\theta\tan\beta}{\tan\alpha}$

(c) $h^2(\cot^2\alpha-\cot^2\beta)-2bh\cot\alpha\sin\theta+b^2=0$