Chapter 7 General Solutions of Trigonometric Equations

7.1 Inverse of Trigonometric Functions

Definitions:

(1) For $-1 \le a \le 1$, $x = \sin^{-1} a$ if $\sin x = a$ and $-\frac{\pi}{2} \le a \le \frac{\pi}{2}$.

(2) For $-1 \le b \le 1$, $x = \cos^{-1} b$ if $\cos x = a$ and $0 \le b \le \pi$.

(3) For any real number c, $x = \tan^{-1} c$ if $\tan x = c$ and $-\frac{\pi}{2} < c < \frac{\pi}{2}$.

7.2 Formulae for General Solutions of Trigonometric Equations

(1) For $\sin\theta = a$, where $-1 \le a \le 1$,

$$\theta = 180°n + (-1)^n \phi \quad \text{or} \quad \theta = n\pi + (-1)^n \phi ,$$

where n is an integer and $\phi = \sin^{-1} a$.

(2) For $\cos\theta = b$, where $-1 \le b \le 1$,

$$\theta = 360°n \pm \phi \quad \text{or} \quad \theta = 2n\pi \pm \phi ,$$

where n is an integer and $\phi = \cos^{-1} b$.

(3) For $\tan\theta = c$,

$$\theta = 180°n + \phi \quad \text{or} \quad \theta = n\pi + \phi ,$$

where n is an integer and $\phi = \tan^{-1} c$.

Example 7.1

Find the general solution of $\sin\theta = 0$.

Solution

$$\sin\theta = 0$$

$$\theta = 180n° + (-1)^n 0°$$

$$= 180n°, \text{ where } n \text{ is an integer.}$$

Example 7.2

Find the general solution of each of the following trigonometric equations, giving the answers correct to the nearest $0.01°$ where necessary.

(a) $\cos\theta = -\frac{\sqrt{3}}{2}$ (b) $\tan\theta = 2.4$

Solution

(a) $\cos\theta = -\frac{\sqrt{3}}{2}$

$\theta = 360n° \pm 150°$, where n is an integer.

(b) $\tan\theta = 2.4$

$\theta = 180n° + 67.38°$, corr. to the nearest $0.01°$, where n is an integer.

Checkpoint 7.1

Find the general solution of each of the following trigonometric equations, giving the answers correct to the nearest $0.01°$ where necessary.

(a) $\cos\theta = -\frac{\sqrt{2}}{2}$ (b) $\sin\theta = -\frac{\sqrt{3}}{2}$

(c) $\tan\theta = \frac{\sqrt{3}}{2}$

Example 7.3

Find the general solution (in degree) of $\tan 3\theta = -1$.

Solution

$$\begin{aligned} \tan 3\theta &= -1 \\ 3\theta &= 180n° + (-45°) \\ \theta &= 60n° - 15°, \text{ where } n \text{ is an integer.} \end{aligned}$$

Checkpoint 7.2

Find the general solution (in radian) of $\cos\left(2x - \frac{2\pi}{3}\right) = 1$.

Example 7.4

Find the general solution of the equation $\cos^2\theta = \frac{1}{2}$, giving your answer in radians.

Solution

$$\begin{aligned} \cos^2\theta &= \frac{1}{2} \\ \cos\theta &= \frac{1}{\sqrt{2}} \quad \text{or} \quad -\frac{1}{\sqrt{2}} \\ \theta &= 2n\pi \pm \frac{\pi}{4} \quad \text{or} \quad 2n\pi \pm \frac{3\pi}{4} \\ \theta &= n\pi \pm \frac{\pi}{4}, \text{ where } n \text{ is an integer.} \end{aligned}$$

Checkpoint 7.3

Find the general solution of the equation $2\sin^2 x - 3\sin x - 2 = 0$, giving your answer in degrees.

7.3 Further Examples

Example 7.5

Find the general solution of the equation $7\sin^2 x - 4\sin x\cos x = 3$, giving your answer correct the nearest $0.01°$.

Solution

$$7\sin^2 x - 4\sin x\cos x = 3$$
$$7\sin^2 x - 4\sin x\cos x = 3\sin^2 x + 3\cos^2 x$$
$$4\sin^2 x - 4\sin x\cos x - 3\cos^2 x = 0$$
$$4\tan^2 x - 4\tan x - 3 = 0$$
$$(2\tan x + 1)(2\tan x - 3) = 0$$
$$\tan x = \frac{3}{2} \quad \text{or} \quad -\frac{1}{2}$$
$$x = 180n° + 56.31° \quad \text{or} \quad 180n° - 26.57°, \text{ corr. to the nearest } 0.01°,$$

where n is an integer.

Example 7.6

Find the general solution of the equation $\cos(2\theta + 30°)\cos\theta - \sin(2\theta + 30°)\sin\theta = \frac{1}{2}$.

Solution

$$\begin{aligned}
\cos(2\theta + 30°)\cos\theta - \sin(2\theta + 30°)\sin\theta &= \frac{1}{2} \\
\cos[(2\theta + 30°) + \theta] &= \frac{1}{2} \\
\cos(3\theta + 30°) &= \frac{1}{2} \\
3\theta + 30° &= 360n° \pm 60° \\
3\theta &= 360n° + 30° \quad \text{or} \quad 110n° - 90° \\
\theta &= 110n° + 10° \quad \text{or} \quad 110n° - 30°, \text{ where } n \text{ is an integer.}
\end{aligned}$$

Checkpoint 7.4

Find the general solution of $11 - 6\sec^2 x = 13\tan x$, giving your answer correct the nearest $0.01°$.

Checkpoint 7.5

Find the general solution of the equation $\tan(\theta - 20°) + \tan 4\theta = 1 - \tan(\theta - 20°)\tan 4\theta$.

Example 7.7

Find the general solution of the equation $\sqrt{3}\cos\theta - \sin\theta = 1$, where θ is measured in radians.

Solution

Let $\sqrt{3}\cos\theta - \sin\theta = R\cos(\theta + \phi) = R\cos\theta\cos\phi - R\sin\theta\sin\phi$.

Then $R = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$$\tan\phi = \frac{1}{\sqrt{3}}$$

$$\phi = \frac{\pi}{6}$$

$$\therefore \quad \sqrt{3}\cos\theta - \sin\theta = 2\cos\left(\theta + \frac{\pi}{6}\right)$$

$$\therefore \quad 2\cos\left(\theta + \frac{\pi}{6}\right) = 1$$

$$\cos\left(\theta + \frac{\pi}{6}\right) = \frac{1}{2}$$

$$\theta + \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3}$$

$\therefore$ $\theta = 2n\pi + \frac{\pi}{6}$ *or* $2n\pi - \frac{\pi}{2}$, where n is an integer.

Checkpoint 7.6

By expressing $\sqrt{3}\sin\theta + \cos\theta$ as $R\sin(\theta + \phi)$, find the general solution of the equation $\sqrt{3}\sin\theta + \cos\theta = 1$, where θ is measured in radians.

Example 7.8

Find the general solution of the equation $2\sin\theta + \cos\theta = 1$, giving the answers correct to 2 decimal places.

Solution

Put $t = \tan\frac{\theta}{2}$. Then $\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$.

$$2\sin\theta + \cos\theta = 1$$

$$\therefore \quad 2\left(\frac{2t}{1+t^2}\right) + \frac{1-t^2}{1+t^2} = 1$$

$$4t + 1 - t^2 = 1 + t^2$$

$$2t^2 - 4t = 0$$

$$2t(t-2) = 0$$

$$t = 0 \quad \text{or} \quad t = 2$$

$$\tan\frac{\theta}{2} = 0 \quad \text{or} \quad \tan\frac{\theta}{2} = 2$$

$$\frac{\theta}{2} = 180n° \quad \text{or} \quad 180n° + 63.4349°$$

$\therefore \quad \theta = 360n° \quad$ or $\quad 360n° + 126.87°$, corr. to 2 d.p., where n is an integer.

Example 7.9

Find the general solution of the equation $\cos 2x + 2\cos^2 \frac{x}{2} = 1$, giving the answers in radians.

Solution

$$\begin{aligned}
\cos 2x + 2\cos^2 \frac{x}{2} &= 1 \\
2\cos^2 x - 1 + 2\left(\frac{1+\cos x}{2}\right) &= 1 \\
2\cos^2 x + \cos x - 1 &= 0 \\
(2\cos x - 1)(\cos x + 1) &= 0 \\
\cos x &= \frac{1}{2} \quad \text{or} \quad -1 \\
x &= 2n\pi \pm \frac{\pi}{3} \quad \textit{or} \quad 2n\pi \pm \pi, \text{ where } n \text{ is an integer.}
\end{aligned}$$

Checkpoint 7.7

Find the general solution of the equation $\sin 2\theta - \cos 2\theta = \sqrt{\frac{3}{2}}$.

Checkpoint 7.8

Find the general solution of the equation $\sin 2x + 2\sin^2 \dfrac{x}{2} = 1$, where θ is measured in radians.

Example 7.10

Find the general solution of the equation $\sin 2\theta + \sin 4\theta + \sin 6\theta = 0$, where θ is measured in radians.

Solution

$$\begin{aligned}
\sin 2\theta + \sin 4\theta + \sin 6\theta &= 0 \\
(\sin 2\theta + \sin 6\theta) + \sin 4\theta &= 0 \\
2\sin\frac{2\theta + 6\theta}{2}\cos\frac{2\theta - 6\theta}{2} + \sin 4\theta &= 0 \\
2\sin 4\theta\cos 2\theta + \sin 4\theta &= 0 \\
\sin 4\theta(2\cos 2\theta + 1) &= 0
\end{aligned}$$

$$\sin 4\theta = 0 \quad \text{or} \quad \cos 2\theta = -\frac{1}{2}$$

$$4\theta = n\pi \quad \text{or} \quad 2\theta = 2n\pi \pm \frac{2\pi}{3}$$

$$\theta = \frac{n\pi}{4} \quad \text{or} \quad n\pi \pm \frac{\pi}{3}\text{, where } n \text{ is an integer.}$$

Checkpoint 7.9

Find the general solution of the equation $\cos 2\theta + \cos 4\theta + \cos 6\theta = 0$, where θ is measured in radians.

Example 7.11

Find the general solution of the equation $2\cos\frac{x}{2}\cos\frac{3x}{2} + 1 = 0$, giving your answers in radians.

Solution

$$2\cos\frac{x}{2}\cos\frac{3x}{2} + 1 = 0$$

$$\cos\left(\frac{x}{2} + \frac{3x}{2}\right) + \cos\left(\frac{x}{2} - \frac{3x}{2}\right) + 1 = 0$$

$$\cos 2x + \cos x + 1 = 0$$

$$2\cos^2 x + \cos x = 0$$

$$\cos x(2\cos x + 1) = 0$$

$$\cos x = 0 \quad \text{or} \quad -\frac{1}{2}$$

$$x = 2n\pi \pm \frac{\pi}{2} \quad \text{or} \quad 2n\pi \pm \frac{2\pi}{3}\text{, where } n \text{ is an integer.}$$

Example 7.12

Find the general solution of the equation $\cos 2x \cos 6x = \sin 4x \sin 8x$, giving your answers in radians.

Solution

$$\cos 2x \cos 6x = \sin 4x \sin 8x$$
$$\frac{1}{2}(\cos 8x + \cos 4x) = \frac{1}{2}(-\cos 12x + \cos 4x)$$
$$\cos 8x + \cos 12x = 0$$
$$2\cos\frac{8x+12x}{2}\cos\frac{8x-12x}{2} = 0$$
$$\cos 10x \cos 2x = 0$$
$$\cos 10x = 0 \quad \text{or} \quad \cos 2x = 0$$
$$10x = 2n\pi \pm \frac{\pi}{2} \quad \text{or} \quad 2x = 2n\pi \pm \frac{\pi}{2}$$
$$x = \frac{n\pi}{5} \pm \frac{\pi}{20} \quad \text{or} \quad n\pi \pm \frac{\pi}{4}\text{, where } n \text{ is an integer.}$$

Checkpoint 7.10

Find the general solution of the equation $\sin 5x \cos 7x = \cos 3x \sin 9x$, giving the answer in radians.

Example 7.13

Find the general solutions of the following equations

(a) $\tan 4x = \cot 2x$, where x is in radians.

(b) $\cos 3x = \sin 2x$, where x is in degrees.

Solution

(a)
$$\tan 4x = \cot 2x$$
$$\tan 4x = \tan\left(\frac{\pi}{2} - 2x\right)$$
$$4x = n\pi + \frac{\pi}{2} - 2x$$
$$6x = n\pi + \frac{\pi}{2}$$
$$x = \frac{n\pi}{6} + \frac{\pi}{12}, \text{ where } n \text{ is an integer.}$$

(b)
$$\cos 3x = \sin 2x$$
$$\cos 3x = \cos(90^\circ - 2x)$$
$$3x = 360n^\circ \pm (90^\circ - 2x)$$
$$3x = 360n^\circ + 90^\circ - 2x \quad \text{or} \quad 360n^\circ - 90^\circ + 2x$$
$$x = 72n^\circ + 18^\circ \quad \text{or} \quad 360n^\circ - 90^\circ, \text{ where } n \text{ is an integer.}$$

Alternatively,
$$\cos 3x = \sin 2x$$
$$\sin(90^\circ - 3x) = \sin 2x$$
$$90^\circ - 3x = 180n^\circ + (-1)^n 2x, \text{ where } n \text{ is an integer.}$$

When n is even, let $n = 2m$,

$\therefore \quad 90^\circ - 3x = 180(2m)^\circ + (-1)^{2m} 2x$
$$90^\circ - 3x = 360m^\circ + 2x$$
$$5x = 90^\circ - 360m^\circ$$
$$x = 72k^\circ + 18^\circ, \text{ where } k \text{ is an integer.} \qquad (\text{Putting } k = -m)$$

When n is odd, let $n = 2m + 1$,

$\therefore \quad 90^\circ - 3x = 180(2m+1)^\circ + (-1)^{2m+1} 2x$
$$90^\circ - 3x = 360m^\circ + 180^\circ - 2x$$
$$x = -360m^\circ - 90^\circ$$
$$x = 360k^\circ - 90^\circ, \text{ where } k \text{ is an integer.} \qquad (\text{Putting } k = -m)$$

Checkpoint 7.11

Find the general solutions of the following equations

(a) $\sin 5x = \cos 3x$, where x is in radians.

(b) $\tan 3\theta \tan 7\theta = -1$, where θ is in degrees.

Exercise 7 General Solutions of Trigonometric Equations

7.2

1. Find the general solution of the equation $\tan x = \frac{1}{\sqrt{3}}$, giving the answer in degrees.

2. Find the general solution of the equation $\sin(2x - 15°) = 0.25$, giving the answer correct to the nearest $0.1°$.

3. Find the general solution of the equation $\tan\theta = \sin\theta$, giving the answer in terms of π.

4. Find the general solution of the equation $\sqrt{3}\sin^2 x + 2\sin x\cos x - \sqrt{3}\cos^2 x = 0$, giving the answer in terms of π.

5. Find the general solution of the equation $\tan\theta + \cot\theta = \frac{4}{\sqrt{3}}$, giving the answer in terms of π.

6. Find the general solution of the equation $25^{\cos^2 x} + 5^{2\sin^2 x} = 10$, giving the answer in terms of π.

7.3

7. Find the general solution of the equation $2\tan x - 2\sin x = 3\sin^2 x\tan x$, giving the answer correct to the nearest $0.1°$.

8. Find the general solution of the equation $\csc^2 x = 25\sin^2 x - 20\cot^2 x$, giving the answer correct to the nearest $0.1°$.

9. Find the general solution of the equation $3\cos^4\theta + 5\sin^2\theta - 3 = 0$, giving the answer correct to the nearest $0.1°$.

10. By expressing $\sin\theta - \cos\theta = R\sin(\theta - \alpha)$, find the general solution of the equation $\sqrt{2}(\sin\theta - \cos\theta) = 2$.

11. (a) Find the general solution of the equation $\sqrt{3}\sin x + \cos x = \sqrt{2}$.

(b) Using the result of (a), find the points of intersection of the curves

$$C_1: \quad y = 2\sqrt{3}\sin x - \cos x + 4\sqrt{2}$$
$$C_2: \quad y = \sqrt{3}\sin x - 2\cos x + 5\sqrt{2}$$

for $-2\pi \le x \le 2\pi$.

(*Give the answers in terms of* π *and correct to 2 decimal places where necessary.*)

12. Find the general solution of the equation $\sin 2x \cos x + \cos 2x \sin x = 1$, giving the answer in radians.

13. Find the general solution of the equation $\cos 6x - \cos 2x = \cos 8x - \cos 4x$, giving the answer in radians.

14. Find the general solution of the equation $\sin 6x \cos 9x - \cos 5x \sin 10x = 0$, giving the answer in radians.

15. Find the general solution of the equation $\sin(\theta + 45°) = 2\cos(\theta - 30°)$, giving the answer correct to the nearest $0.1°$.

16. Find the general solution of the equation $3\cos^2 \frac{x}{2} = 1 + \sin^2 x$, giving the answer correct to the nearest $0.1°$.

17. Using the identity $\cos 3A = 4\cos^3 A - 3\cos A$, find the general solution of $8\cos^3 x = \sqrt{2} + 6\cos x$, giving the answers in radians.

18. Find the general solution of the equation $\cos 2x = \cos x - \sin x$, giving the answer in degrees.

19. (a) Show that $\sin 3x + \sin 5x + \sin 7x + \sin 9x = 2\sin 6x(\cos 3x + \cos x)$.

(b) Find the general solution of $\sin 3x + \sin 5x + \sin 7x + \sin 9x = 0$.

(c) Hence, find the general solution of $-\cos 3x + \cos 5x - \cos 7x + \cos 9x = 0$.

(*Give the answers in radians.*)