

Chapter 6 Trigonometric Formulae Involving Compound Angles

6.1 Sum and Difference Formulae

$$(1) \quad \cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$(2) \quad \cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$(3) \quad \sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$(4) \quad \sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$(5) \quad \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$(6) \quad \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Example 6.1

Without using a calculator, find the values of

(a) $\sin 75^\circ$

(b) $\sin 15^\circ$

(c) $\tan 15^\circ$

Solution

$$\begin{aligned} (a) \quad \sin 75^\circ &= \sin(30^\circ + 45^\circ) \\ &= \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ \\ &= \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{2} + \sqrt{6}}{4} \end{aligned}$$

$$\begin{aligned} (b) \quad \sin 15^\circ &= \sin(45^\circ - 30^\circ) \\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ \\ &= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$$

$$\begin{aligned} (c) \quad \tan 15^\circ &= \tan(60^\circ - 45^\circ) \\ &= \frac{\tan 60^\circ - \tan 45^\circ}{1 + \tan 60^\circ \tan 45^\circ} \\ &= \frac{\sqrt{3} - 1}{1 + \sqrt{3} \times 1} \\ &= \frac{\sqrt{3} - 1}{1 + \sqrt{3}} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1} \\ &= 2 - \sqrt{3} \end{aligned}$$

Checkpoint 6.1

Without using a calculator, find the values of

(a) $\cos 105^\circ$

(b) $\cot 255^\circ$

Example 6.2

Prove that $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B$.

Solution

$$\begin{aligned}\text{L.H.S.} &= \cos(A + B)\cos(A - B) \\ &= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B) \\ &= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B \\ &= \cos^2 A(1 - \sin^2 B) - (1 - \cos^2 A)\sin^2 B \\ &= \cos^2 A - \cos^2 A \sin^2 B - (\sin^2 B - \cos^2 A \sin^2 B) \\ &= \cos^2 A - \sin^2 B \\ &= \text{R.H.S.}\end{aligned}$$

$$\therefore \cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B$$

Checkpoint 6.2

Prove that $\tan A + \tan B = \frac{\sin(A+B)}{\cos A \cos B}$.

Example 6.3

If $\tan A = \frac{7}{4}$ and $\tan(A-B) = \frac{1}{2}$, find the value of $\tan B$ without using a calculator.

Solution

$$\therefore \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\therefore \frac{1}{2} = \frac{\frac{7}{4} - \tan B}{1 + \frac{7}{4} \tan B}$$

$$\frac{1}{2} = \frac{7 - 4 \tan B}{4 + 7 \tan B}$$

$$4 + 7 \tan B = 14 - 8 \tan B$$

$$\tan B = \frac{2}{3}$$

Example 6.4

If A, B, C are the angles of a triangle, prove that $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$.

Solution

$$\begin{aligned} \therefore \quad & A + B + C = 180^\circ \\ \therefore \quad & \frac{A}{2} + \frac{B}{2} + \frac{C}{2} = 90^\circ \\ & \frac{A}{2} + \frac{B}{2} = 90^\circ - \frac{C}{2} \\ & \tan\left(\frac{A}{2} + \frac{B}{2}\right) = \tan\left(90^\circ - \frac{C}{2}\right) \\ \text{i.e.} \quad & \frac{\tan \frac{A}{2} + \tan \frac{B}{2}}{1 - \tan \frac{A}{2} \tan \frac{B}{2}} = \frac{1}{\tan \frac{C}{2}} \\ & \tan \frac{A}{2} \tan \frac{C}{2} + \tan \frac{B}{2} \tan \frac{C}{2} = 1 - \tan \frac{A}{2} \tan \frac{B}{2} \\ \therefore \quad & \tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1 \end{aligned}$$

Checkpoint 6.3

If $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$, find the value of $\tan C$ without using a calculator, where A, B, C are the angles of a triangle.

Example 6.5

Solve the equation $\sin(3\theta + 30^\circ)\cos 30^\circ - \cos(3\theta + 30^\circ)\sin 30^\circ = \frac{\sqrt{2}}{2}$ for $0^\circ \leq \theta \leq 180^\circ$.

Solution

$$\sin(3\theta + 30^\circ)\cos 30^\circ - \cos(3\theta + 30^\circ)\sin 30^\circ = \frac{\sqrt{2}}{2}$$

$$\sin[(3\theta + 30^\circ) - 30^\circ] = \frac{\sqrt{2}}{2}$$

$$\sin 3\theta = \frac{\sqrt{2}}{2}$$

$$3\theta = 45^\circ, 135^\circ, 405^\circ, 495^\circ \quad \text{for } 0^\circ \leq 3\theta \leq 540^\circ$$

$$\theta = 15^\circ, 45^\circ, 135^\circ, 165^\circ$$

Example 6.6

Solve the equation $\cos \frac{\theta}{2} \cos \frac{\pi}{6} - \sin \frac{\theta}{2} \sin \frac{\pi}{6} = \frac{1}{2}$ for $0 \leq \theta \leq \pi$.

Solution

$$\cos \frac{\theta}{2} \cos \frac{\pi}{6} - \sin \frac{\theta}{2} \sin \frac{\pi}{6} = \frac{1}{2}$$

$$\cos\left(\frac{\theta}{2} + \frac{\pi}{6}\right) = \frac{1}{2}$$

$$\frac{\theta}{2} + \frac{\pi}{6} = \frac{\pi}{3}$$

$$\theta = \frac{\pi}{3}$$

Checkpoint 6.4

Solve the equation $\cos 15^\circ \cos \theta + \sin 15^\circ \sin \theta = 1$ for $0^\circ \leq \theta \leq 360^\circ$.

Checkpoint 6.5

Solve the equation $\cos 2\theta \sin \theta = \cos \theta \sin 2\theta$ for $0 \leq \theta \leq \pi$.

6.2 Subsidiary Angle Form

Assume $a, b, R > 0$ and ϕ is an acute angle. We may write

$$(1) \quad a \cos \theta \pm b \sin \theta = R \cos(\theta \mp \phi), \text{ where } R = \sqrt{a^2 + b^2}, a = R \cos \phi \text{ and } b = R \sin \phi;$$

$$(2) \quad a \sin \theta \pm b \cos \theta = R \sin(\theta \pm \phi), \text{ where } R = \sqrt{a^2 + b^2}, a = R \cos \phi \text{ and } b = R \sin \phi.$$

Example 6.7

For each of the two given subsidiary angle forms, R is a positive constant and ϕ is an acute angle.

$$(i) \quad \sqrt{3} \sin \theta + \cos \theta = R \cos(\theta - \phi)$$

$$(ii) \quad \sqrt{3} \sin \theta + \cos \theta = R \sin(\theta + \phi)$$

(a) Find the values of R and ϕ .

(b) Solve the equation $\sqrt{3} \sin \theta + \cos \theta = 1$, where $0^\circ < \theta < 180^\circ$.

Solution

$$(a) (i) \quad \begin{aligned} \sqrt{3} \sin \theta + \cos \theta &= R \cos(\theta - \phi) \\ &= R \sin \phi \sin \theta + R \cos \phi \cos \theta \end{aligned}$$

$$R = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

$$\therefore \quad \sqrt{3} = 2 \sin \phi \quad \text{and} \quad 1 = 2 \cos \phi$$

$$\sin \phi = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos \phi = \frac{1}{2}$$

$$\therefore \quad \phi = 60^\circ$$

$$(ii) \quad \begin{aligned} \sqrt{3} \sin \theta + \cos \theta &= R \sin(\theta + \phi) \\ &= R \cos \phi \sin \theta + R \sin \phi \cos \theta \end{aligned}$$

$$R = \sqrt{(\sqrt{3})^2 + 1^2} = 2$$

$$\therefore \quad \sqrt{3} = 2 \cos \phi \quad \text{and} \quad 1 = 2 \sin \phi$$

$$\cos \phi = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin \phi = \frac{1}{2}$$

$$\therefore \quad \phi = 30^\circ$$

$$(b) (i) \quad \sqrt{3} \sin \theta + \cos \theta = 1$$

$$\therefore \quad 2 \cos(\theta - 60^\circ) = 1$$

$$\cos(\theta - 60^\circ) = \frac{1}{2}$$

$$\theta - 60^\circ = 60^\circ \quad (-60^\circ < \theta - 60^\circ < 120^\circ)$$

$$\theta = 120^\circ$$

$$\begin{aligned}
\text{(b) (ii)} \quad & \because 0^\circ < \theta < 180^\circ \\
& \therefore 30^\circ < \theta + 30^\circ < 210^\circ \\
& \quad \sqrt{3} \sin \theta + \cos \theta = 1 \\
& \therefore 2 \sin(\theta + 30^\circ) = 1 \\
& \quad \sin(\theta + 30^\circ) = \frac{1}{2} \\
& \quad \theta + 30^\circ = 30^\circ \text{ (rejected) or } 150^\circ \\
& \therefore \quad \theta = 120^\circ
\end{aligned}$$

Example 6.8

Solve the equation $4 \cos \theta - 3 \sin \theta = 2$ for $0^\circ \leq \theta \leq 360^\circ$.

(Give the answers correct to 2 decimal places where necessary.)

Solution

$$\begin{aligned}
\text{Let} \quad & 4 \cos \theta - 3 \sin \theta = R \cos(\theta + \phi) \\
& \quad = R \cos \phi \cos \theta - R \sin \phi \sin \theta
\end{aligned}$$

$$\text{where} \quad R = \sqrt{3^2 + 4^2} = 5$$

$$\text{Then} \quad 4 = 5 \cos \phi \quad \text{and} \quad 3 = 5 \sin \phi$$

$$\cos \phi = \frac{4}{5} \quad \text{and} \quad \sin \phi = \frac{3}{5}$$

$$\phi = 36.8699^\circ$$

$$4 \cos \theta - 3 \sin \theta = 2$$

$$\therefore 5 \cos(\theta + 36.8699^\circ) = 2$$

$$\cos(\theta + 36.8699^\circ) = \frac{2}{5}$$

$$\theta + 36.8699^\circ = 66.4218^\circ \quad \text{or} \quad 293.5782^\circ$$

$$\theta = 29.55^\circ \quad \text{or} \quad 256.71^\circ \quad (\text{corr. to 2 d.p.})$$

Checkpoint 6.6

Solve the equation $2 \sin \theta + 3 \cos \theta = -1$ for $0^\circ \leq \theta \leq 360^\circ$.

(Give the answers correct to 2 decimal places where necessary.)

Checkpoint 6.7

Solve the equation $\sqrt{2} \cos \theta - \sqrt{7} \sin \theta = 3$ for $-\pi \leq \theta \leq \pi$.

(Give the answers correct to 2 decimal places where necessary.)

Example 6.9

Find the maximum and minimum values of the expression $-5 \cos \theta - 12 \sin \theta$.

Solution

$$\begin{aligned}\text{Let } -5 \cos \theta - 12 \sin \theta &= -R \cos(\theta - \phi) \\ &= -R \cos \phi \cos \theta - R \sin \phi \sin \theta\end{aligned}$$

$$\text{where } R = \sqrt{5^2 + 12^2} = 13$$

$$\text{Then } 5 = 13 \cos \phi \quad \text{and} \quad 12 = 13 \sin \phi$$

$$\cos \phi = \frac{5}{13} \quad \text{and} \quad \sin \phi = \frac{12}{13}$$

$$\phi = 67.4^\circ \quad (\text{corr. to 3 sig. fig.})$$

$$\therefore -5 \cos \theta - 12 \sin \theta = -13 \cos(\theta - 67.4^\circ)$$

$$\therefore -1 \leq -\cos x \leq 1 \quad \text{for all values of } x$$

$$\therefore -13 \leq -13 \cos(\theta - 67.4^\circ) \leq 13 \quad \text{for all values of } \theta$$

\therefore The maximum and minimum values of $-5 \cos \theta - 12 \sin \theta$ are -13 and 13 respectively.

Checkpoint 6.8

Find the maximum and minimum values of the expression $\frac{1}{7 \cos \theta - \sin \theta}$.

6.3 Multiple Angle Formulae

6.3.1 Double Angle Formulae

$$(1) \quad \sin 2A = 2 \sin A \cos A$$

$$(2) \quad \begin{aligned} \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2 \cos^2 A - 1 \\ &= 1 - 2 \sin^2 A \end{aligned}$$

$$(3) \quad \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Example 6.10

Prove that $(\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta$.

Solution

$$\begin{aligned} \text{L.H.S.} &= (\sin \theta + \cos \theta)^2 \\ &= \sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cos \theta \\ &= 1 + \sin 2\theta \\ &= \text{R.H.S.} \end{aligned}$$

$$\therefore (\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta$$

Example 6.11

Prove that $\frac{1 - \cos 2\theta}{1 + \cos 2\theta} = \tan^2 \theta$.

Solution

$$\begin{aligned} \text{L.H.S.} &= \frac{1 - \cos 2\theta}{1 + \cos 2\theta} \\ &= \frac{1 - (1 - 2 \sin^2 \theta)}{1 + (2 \cos^2 \theta - 1)} \\ &= \frac{\sin^2 \theta}{\cos^2 \theta} \\ &= \tan^2 \theta \\ &= \text{R.H.S.} \end{aligned}$$

$$\therefore \frac{1 - \cos 2\theta}{1 + \cos 2\theta} = \tan^2 \theta$$

Checkpoint 6.9

Prove that $\csc 2\theta - \cot 2\theta = \tan \theta$.

Checkpoint 6.10

Given that $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$, find the value of $\cos 36^\circ$ in surd form.

6.3.2 Triple Angle Formulae

$$(1) \quad \sin 3A = 3 \sin A - 4 \sin^3 A$$

$$(2) \quad \cos 3A = 4 \cos^3 A - 3 \cos A$$

$$(3) \quad \tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

Example 6.12

Prove that $\frac{\sin 3\theta}{\sin \theta} - \frac{\cos 3\theta}{\cos \theta} = 2$.

Solution

$$\begin{aligned} \text{L.H.S.} &= \frac{\sin 3\theta}{\sin \theta} - \frac{\cos 3\theta}{\cos \theta} \\ &= \frac{3 \sin \theta - 4 \sin^3 \theta}{\sin \theta} - \frac{4 \cos^3 \theta - 3 \cos \theta}{\cos \theta} \\ &= 3 - 4 \sin^2 \theta - (4 \cos^2 \theta - 3) \\ &= 6 - 4(\sin^2 \theta + \cos^2 \theta) \\ &= 6 - 4 \\ &= 2 \\ &= \text{R.H.S.} \end{aligned}$$

$$\therefore \frac{\sin 3\theta}{\sin \theta} - \frac{\cos 3\theta}{\cos \theta} = 2.$$

Example 6.13

Solve the equation $8 \cos^3 \theta - 1 = 6 \cos \theta$ for $0^\circ \leq \theta \leq 180^\circ$.

Solution

$$\begin{aligned} 8 \cos^3 \theta - 1 &= 6 \cos \theta \\ 8 \cos^3 \theta - 6 \cos \theta &= 1 \\ 2(4 \cos^3 \theta - 3 \cos \theta) &= 1 \\ \cos 3\theta &= \frac{1}{2} \\ 3\theta &= 60^\circ, \quad 300^\circ, \quad 420^\circ \\ \theta &= 20^\circ, \quad 100^\circ, \quad 140^\circ \end{aligned}$$

Checkpoint 6.11

If $\tan A = \frac{1}{2}$, find the values of $\tan 3A$ and $\tan 6A$.

Checkpoint 6.12

Solve the equation $\cos 3x - \cos 2x = 1$ for $0^\circ \leq \theta \leq 360^\circ$.

(Give the answers correct to 2 decimal places where necessary.)

6.3.3 t -Formulae

Let $t = \tan \frac{\theta}{2}$, then

$$(1) \quad \cos \theta = \frac{1-t^2}{1+t^2}$$

$$(2) \quad \sin \theta = \frac{2t}{1+t^2}$$

$$(3) \quad \tan \theta = \frac{2t}{1-t^2}$$

Example 6.14

Prove that $\frac{1 + \tan \frac{\theta}{2}}{1 - \tan \frac{\theta}{2}} = \frac{1 + \sin \theta}{\cos \theta}$.

Solution

Put $t = \tan \frac{\theta}{2}$.

$$\begin{aligned} \text{R.H.S.} &= \frac{1 + \sin \theta}{\cos \theta} = \frac{1 + \frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2}} \\ &= \frac{1 + 2t + t^2}{1-t^2} \\ &= \frac{(1+t)^2}{(1-t)(1+t)} \\ &= \frac{1+t}{1-t} \\ &= \frac{1 + \tan \frac{\theta}{2}}{1 - \tan \frac{\theta}{2}} \\ &= \text{L.H.S.} \end{aligned}$$

$$\therefore \frac{1 + \tan \frac{\theta}{2}}{1 - \tan \frac{\theta}{2}} = \frac{1 + \sin \theta}{\cos \theta}$$

Example 6.15

Solve the equation $\sin \theta = \sqrt{3}(1 - \cos \theta)$ for $0^\circ \leq \theta \leq 360^\circ$.

Solution

Put $t = \tan \frac{\theta}{2}$.

$$\sin \theta = \sqrt{3}(1 - \cos \theta)$$

$$\frac{2t}{1+t^2} = \sqrt{3} \left(1 - \frac{1-t^2}{1+t^2} \right)$$

$$2t = \sqrt{3}(2t^2)$$

$$2t(1 - \sqrt{3}t) = 0$$

$$t = 0 \quad \text{or} \quad t = \frac{1}{\sqrt{3}}$$

$$\tan \frac{\theta}{2} = 0 \quad \text{or} \quad \tan \frac{\theta}{2} = \frac{1}{\sqrt{3}}$$

$$\frac{\theta}{2} = 0^\circ, 180^\circ \quad \text{or} \quad \frac{\theta}{2} = 30^\circ$$

$$\theta = 0^\circ, 60^\circ, 360^\circ$$

Checkpoint 6.13

Solve the equation $5(\sin x + \cos x) = 7$ for $0^\circ \leq x \leq 360^\circ$, giving answers correct to 2 decimal places.

Example 6.16

If α and β are two distinct roots of the equation $5 \sin \theta - 3 \cos \theta = 2$, where α and β lie between π and $-\pi$, find the value of $\cot^2 \frac{\alpha}{2} + \cot^2 \frac{\beta}{2}$ without using a calculator.

Solution

Put $t = \tan \frac{\theta}{2}$.

$$5 \sin \theta - 3 \cos \theta = 2$$

$$5 \left(\frac{2t}{1+t^2} \right) - 3 \left(\frac{1-t^2}{1+t^2} \right) = 2$$

$$10t - 3 + 3t^2 = 2(1+t^2)$$

$$t^2 + 10t - 5 = 0 \quad \dots\dots\dots(*)$$

$$\text{sum of roots} = -10$$

$$\text{product of roots} = -5$$

Since $\tan \frac{\alpha}{2}$ and $\tan \frac{\beta}{2}$ are the roots of (*), we have

$$\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} = -10$$

$$\tan \frac{\alpha}{2} \tan \frac{\beta}{2} = -5$$

$$\begin{aligned} \cot^2 \frac{\alpha}{2} + \cot^2 \frac{\beta}{2} &= \frac{1}{\tan^2 \frac{\alpha}{2}} + \frac{1}{\tan^2 \frac{\beta}{2}} \\ &= \frac{\tan^2 \frac{\beta}{2} + \tan^2 \frac{\alpha}{2}}{\tan^2 \frac{\alpha}{2} \tan^2 \frac{\beta}{2}} \\ &= \frac{\left(\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} \right)^2 - 2 \tan \frac{\alpha}{2} \tan \frac{\beta}{2}}{\left(\tan \frac{\alpha}{2} \tan \frac{\beta}{2} \right)^2} \\ &= \frac{(-10)^2 - 2(-5)}{(-5)^2} \\ &= \frac{22}{5} \end{aligned}$$

6.4 Half Angle Formulae

$$(1) \quad \sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}$$

$$(2) \quad \cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$$

$$(3) \quad \begin{aligned} \tan \frac{\theta}{2} &= \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} \\ &= \frac{1 - \cos \theta}{\sin \theta} \\ &= \frac{\sin \theta}{1 + \cos \theta} \end{aligned}$$

Note that the signs of the square roots depends on which quadrant $\frac{\theta}{2}$ lies in.

Example 6.17

Find the value of $\tan 15^\circ$ in surd form by using the half angle formulae.

Solution

$$\begin{aligned} \tan 15^\circ &= \sqrt{\frac{1 - \cos 30^\circ}{1 + \cos 30^\circ}} \\ &= \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{1 + \frac{\sqrt{3}}{2}}} \\ &= \sqrt{\frac{2 - \sqrt{3}}{2 + \sqrt{3}}} \\ &= \sqrt{\frac{(2 - \sqrt{3})(2 - \sqrt{3})}{(2 + \sqrt{3})(2 - \sqrt{3})}} \\ &= \sqrt{\frac{(2 - \sqrt{3})^2}{4 - 3}} \\ &= 2 - \sqrt{3} \end{aligned}$$

Checkpoint 6.14

Without using a calculator, find the value of $\sin \frac{\pi}{8}$ in surd form.

Checkpoint 6.15

Prove that $\frac{1 + \sin \theta}{1 - \sin \theta} = \tan^2 \left(45^\circ + \frac{\theta}{2} \right)$.

6.5 Sum and Product Formulae

6.5.1 Product to Sum Formulae

$$(1) \quad \sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

$$(2) \quad \cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$$

$$(3) \quad \cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$(4) \quad \sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$$

6.5.2 Sum to Product Formulae

$$(1) \quad \sin x + \sin y = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$(2) \quad \sin x - \sin y = 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

$$(3) \quad \cos x + \cos y = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$(4) \quad \cos x - \cos y = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

Example 6.18

Evaluate $\cos 82.5^\circ \cos 37.5^\circ$ without using a calculator.

Solution

$$\begin{aligned} \cos 82.5^\circ \cos 37.5^\circ &= \frac{1}{2}[\cos(82.5^\circ + 37.5^\circ) + \cos(82.5^\circ - 37.5^\circ)] \\ &= \frac{1}{2}[\cos 120^\circ + \cos 45^\circ] \\ &= \frac{1}{2}\left[-\frac{1}{2} + \frac{\sqrt{2}}{2}\right] \\ &= \frac{\sqrt{2}-1}{4} \end{aligned}$$

Checkpoint 6.16

Without using a calculator, evaluate $\cos 15^\circ \sin 75^\circ$.

Example 6.19

Prove that $\cos \theta + \cos(\theta + 120^\circ) + \cos(\theta + 240^\circ) = 0$ for all values of θ .

Solution

$$\begin{aligned}\text{L.H.S.} &= \cos \theta + \cos(\theta + 120^\circ) + \cos(\theta + 240^\circ) \\ &= [\cos \theta + \cos(\theta + 240^\circ)] + \cos(\theta + 120^\circ) \\ &= 2 \cos\left(\frac{\theta + \theta + 240^\circ}{2}\right) \cos\left(\frac{\theta - \theta - 240^\circ}{2}\right) + \cos(\theta + 120^\circ) \\ &= 2 \cos(\theta + 120^\circ) \cos 120^\circ + \cos(\theta + 120^\circ) \\ &= \cos(\theta + 120^\circ)(2 \cos 120^\circ + 1) \\ &= \cos(\theta + 120^\circ) \left[2\left(-\frac{1}{2}\right) + 1 \right] \\ &= 0 \\ &= \text{R.H.S.}\end{aligned}$$

$$\therefore \cos \theta + \cos(\theta + 120^\circ) + \cos(\theta + 240^\circ) = 0$$

Checkpoint 6.17

Prove that $\frac{\sin 3x - \sin 2x}{\cos 3x - \cos 2x} = -\cot \frac{5x}{2}$.

Example 6.20

Solve the equation $2 \sin^3 \theta - \sin^2 \theta - \sin \theta = -\frac{1}{2}$ for $0^\circ \leq \theta \leq 360^\circ$.

Solution

$$2 \sin^3 \theta - \sin^2 \theta - \sin \theta = -\frac{1}{2}$$

$$-4 \sin^3 \theta + 2 \sin^2 \theta + 2 \sin \theta = 1$$

$$-4 \sin^3 \theta + 3 \sin \theta - \sin \theta = 1 - 2 \sin^2 \theta$$

$$\sin 3\theta - \sin \theta = \cos 2\theta$$

$$2 \cos 2\theta \sin \theta = \cos 2\theta$$

$$\cos 2\theta(2 \sin \theta - 1) = 0$$

$$\cos 2\theta = 0$$

$$\text{or } \sin \theta = \frac{1}{2}$$

$$2\theta = 90^\circ, 270^\circ, 450^\circ, 630^\circ \quad \text{or} \quad \theta = 30^\circ, 150^\circ$$

$$\theta = 30^\circ, 45^\circ, 135^\circ, 150^\circ, 225^\circ, 315^\circ$$

Checkpoint 6.18

Solve the equation $\sin \theta + \sin 3\theta + \sin 5\theta = 0$ for $0^\circ \leq \theta \leq 360^\circ$.

Exercise 6 Trigonometric Formulae Involving Compound Angles

6.1

- Find the values of each of the following in surd form:
 - $\cos 195^\circ$
 - $\tan 255^\circ$
- Find, without using a calculator, the value of each of $\frac{1 + \tan 15^\circ}{1 - \tan 15^\circ}$.
- Simplify each of the following:
 - $\frac{\tan 6w - \tan 2w}{1 + \tan 6w \tan 2w}$
 - $\cos(A + 2B) + \cos(A - 2B)$
- Express the following in terms of $\sin x$ and $\cos x$ only:
 - $\sin\left(\frac{\pi}{4} + x\right)$
 - $\cot\left(\frac{\pi}{3} + x\right)$
- Simplify $\sin A \sin(B - C) + \sin B \sin(C - A) + \sin C \sin(A - B)$.
- Prove that $\tan(45^\circ + A) = \frac{\cos A + \sin A}{\cos A - \sin A}$.
- Prove that $(\sin A + \cos A)(\sin B + \cos B) = \sin(A + B) + \cos(A - B)$.
- If $\sin(A + B) = 3 \sin(A - B)$, prove that $\tan A = 2 \tan B$.
- Prove that
$$\begin{aligned}\sin(A + B + C) &= \sin A \cos B \cos C + \cos A \sin B \cos C \\ &\quad + \cos A \cos B \sin C - \sin A \sin B \sin C\end{aligned}$$
- If A, B and C are the angles of a triangle, prove that
$$\cot A \cot B + \cot B \cot C + \cot C \cot A = 1.$$

6.2

11. Find the maximum and minimum values of
- (a) $y = 3 \sin \theta + \cos \theta$
- (b) $y = 7 \cos \theta - \sin \theta$
12. Find the maximum and minimum values of
- (a) $y = 2 \sin \theta + 3 \cos \theta + 4$
- (b) $y = \frac{1}{2 \sin \theta + 3 \cos \theta + 4}$
13. Solve the equation $5 \sin \theta - \cos \theta = 2$ for $0^\circ \leq \theta \leq 180^\circ$.
14. (a) If $\cos \theta - 2 \sin \theta = R \cos(\theta + \alpha)$ where $R > 0$ and $0^\circ < \alpha < 90^\circ$, find the values of R and α .
- (b) $\cos \theta - 2 \sin \theta$ can also be written in the form $R \sin(\theta + \beta)$ where $90^\circ < \beta < 180^\circ$. Find a relation between α and β .

6.3

15. If $\tan \phi = 3$, find the exact value of $\tan 3\phi$.
16. Express $(1 + \sin \theta)(4 \sin \theta + 3 \cos \theta + 5)$ in terms of t if $t = \tan \frac{\theta}{2}$.
17. Simplify $\frac{\sin 2A - (\sin 3A + 4 \sin^3 A)}{3(\cos 2A - 3 \cos A + 1)}$.
18. Solve the following equations for $0^\circ \leq x \leq 360^\circ$, giving the answers correct to the nearest 0.1° where necessary.
- (a) $\sin x \cos x = \frac{1}{2}$
- (b) $\cos 2x + \cos^2 x = 5 \sin^2 x$
- (c) $\tan 2x = \tan x$
19. If $\cos 3A = x$ and $\cos^2 A = \frac{3}{4} - y$, prove that $x^2 + 16y^3 = 12y^2$.
20. If $\tan^2 A - 2 \tan^2 B = 1$, prove that $\cos 2A + \sin^2 B = 0$.

21. Find the value of $\cot^2 \frac{\alpha}{2} + \cot^2 \frac{\beta}{2}$, where α and β are the two distinct roots of the equation $2 \cos \theta - 3 \sin \theta + 1 = 0$, and α, β lie between $-\pi$ and π .
22. (a) Prove that $\csc \theta + \cot \theta = \cot \frac{\theta}{2}$.
- (b) Express $\csc \theta + \csc 2\theta + \csc 4\theta + \csc 8\theta$ as a difference of $\cot \theta$ using the result of (a).
- (c) Hence show that $\csc \frac{4\pi}{5} + \csc \frac{8\pi}{5} + \csc \frac{16\pi}{5} + \csc \frac{32\pi}{5} = 0$.

6.4

23. Prove that $\frac{1 - \cos \theta + \sin \frac{\theta}{2}}{\sin \theta + \cos \frac{\theta}{2}} = \tan \frac{\theta}{2}$.
24. Prove that $2 \cot 2\theta \cot \frac{\theta}{2} = \frac{\cos \theta}{1 - \cos \theta} - \frac{1 + \cos \theta}{\cos \theta}$.
25. Solve the equation $4 \sin^2 \frac{\theta}{2} = 1 + \cos \theta$ for $0^\circ \leq \theta \leq 360^\circ$, using the half angle formulae.
(Give the answers correct to the nearest 0.1° .)
26. If the roots of the quadratic equation $x^2 + ax + b = 0$ are $\sin \theta$ and $\cos \theta$, find the value of $2 \cos^2 \frac{\theta}{2} \left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right)$ in terms of a and b .

6.5

27. Without using a calculator, evaluate $\sin 20^\circ \sin 40^\circ \sin 80^\circ$.
28. Prove that $\sin \theta + \sin(\theta + 120^\circ) + \sin(\theta + 240^\circ) = 0$.
29. Prove that $\frac{\cos(A + 2B) - \cos A}{\sin(A + 2B) - \sin A} = -\tan(A + B)$.

30. Prove that $\frac{\sin x \sin y}{\cos x + \cos y} = \frac{2 \tan \frac{x}{2} \tan \frac{y}{2}}{1 - \tan^2 \frac{x}{2} \tan^2 \frac{y}{2}}$.
31. Prove that $\sin 3\theta + 2 \sin 5\theta \sin^2 \theta + \sin 7\theta = \cos \theta (\sin 6\theta + \sin 4\theta)$.
32. Solve the following equations for $0^\circ \leq x \leq 360^\circ$, giving the answers correct to the nearest 0.1° where necessary.
- (a) $\cos x + \cos 5x = 0$
- (b) $\sin x - \sin 2x + \sin 3x = 0$
- (c) $\cos 2x \cos 7x + \sin 4x \sin 5x = \frac{1}{2} \cos 3x$
33. Prove that $\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$. Hence solve the equation $\sin^2 4\theta - \sin^2 \theta = \sin 3\theta$ for $0^\circ \leq \theta \leq 360^\circ$.
34. (a) Expand $\cos A \sin \frac{\theta}{2}$ into the difference of two sines.
- (b) Hence prove that $\cos \theta + \cos 2\theta + \cos 3\theta + \dots + \cos 8\theta = \frac{\sin \frac{17\theta}{2} - \sin \frac{\theta}{2}}{2 \sin \frac{\theta}{2}}$.
- (c) Deduce from (b) that $\cos 20^\circ + \cos 40^\circ + \cos 60^\circ + \dots + \cos 160^\circ = 0$.
35. (a) Show that $\cos \theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = \frac{\sin 8\theta}{2 \sin \theta}$.
- (b) Hence, solve the equation $\cos \theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$ for $0^\circ \leq \theta \leq 360^\circ$.
(Give the answers correct to the nearest 0.1° .)