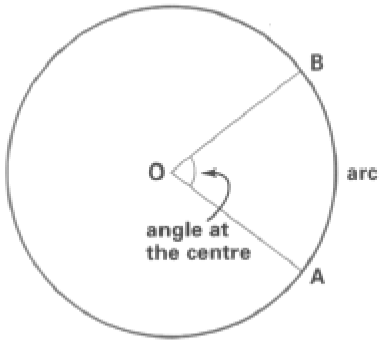
Chapter 5 Trigonometric Functions

5.0 Review

5.0.1 Angle at the Centre

The size of the angle at the centre is proportional to the length of an arc subtending the angle.



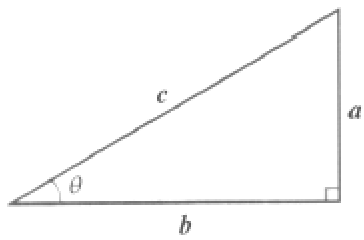
5.0.2 Trigonometric Ratios

In the right-angled triangle,

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$



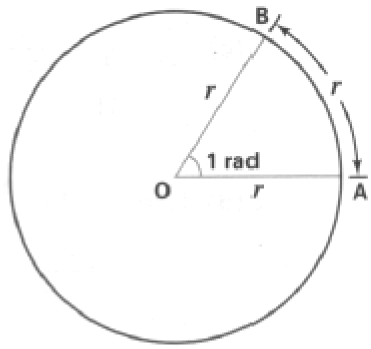
5.0.3 Trigonometric Ratios of Special Angles

	$\theta = 30°$	$\theta = 45°$	$\theta = 60°$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

5.1 Radian Measure

Definition:

A radian is the measure of an angle subtended at the centre of a circle by an arc equal in length to the radius.



$$1\,\text{rad.} = \frac{180^\circ}{\pi} \quad \text{or} \quad \pi\ \text{rad.} = 180^\circ$$

We may write 1 rad. as 1^c or simply 1.

Example 5.1

Convert the following angles into radian measure. Write the answers in terms of π.

(a) 30° (b) 90° (c) 225°

Solution

(a) $30^\circ = 30 \times 1\,\text{rad.} = 30 \times \frac{\pi}{180}\,\text{rad.} = \frac{\pi}{6}\,\text{rad.}$

(b) $90^\circ = 90 \times 1\,\text{rad.} = 90 \times \frac{\pi}{180}\,\text{rad.} = \frac{\pi}{2}\,\text{rad.}$

(c) $225^\circ = 225 \times 1\,\text{rad.} = 225 \times \frac{\pi}{180}\,\text{rad.} = \frac{5\pi}{4}\,\text{rad.}$

Example 5.2

Convert the following angles into degree measure, giving the answers correct to 3 significant figures where necessary.

(a) 3 rad. (b) $\frac{3}{4}\pi$ rad. (c) $\frac{7\pi}{3}$ rad.

Solution

(a) $3\,\text{rad.} = 3 \times 1\,\text{rad.} = 3 \times \frac{180^\circ}{\pi} = 172^\circ$ (corr. to 3 sig. fig.)

(b) $\frac{3}{4}\pi\ \text{rad.} = \frac{3}{4}\pi \times 1\,\text{rad.} = \frac{3}{4}\pi \times \frac{180^\circ}{\pi} = 135^\circ$

(c) $\frac{7\pi}{3}\,\text{rad.} = \frac{7\pi}{3} \times 1\,\text{rad.} = \frac{7\pi}{3} \times \frac{180^\circ}{\pi} = 420^\circ$

Checkpoint 5.1

Complete the following table. (Give the answers correct to 1 decimal place for degree measure or in terms of π for radian measure where necessary.)

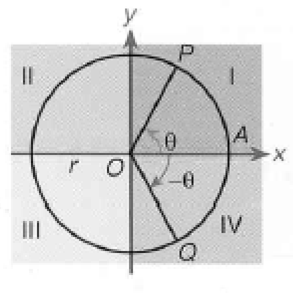
Degree measure	0^o	60^o	240^o	315^o				
Radian measure					$\frac{2\pi}{3}$	$\frac{\pi}{8}$	2.2	2π

5.2 Trigonometric Functions

5.2.1 Angles and Quadrants

In a rectangular coordinate plane, the angle measured from the positive x-axis in an anti-clockwise direction is called a positive angle (θ).

The angle measured from the positive x-axis in a clockwise direction is called a negative angle (– θ).

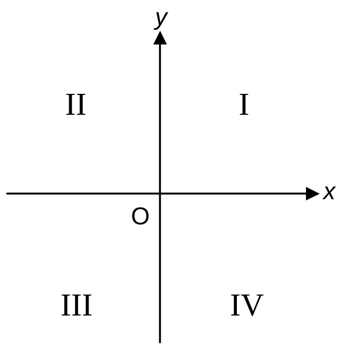


An angle θ is said to lie in Quadrant I when $0° < \theta < 90°$ or $0 < \theta < \frac{\pi}{2}$.

An angle θ is said to lie in Quadrant II when $90° < \theta < 180°$ or $\frac{\pi}{2} < \theta < \pi$.

An angle θ is said to lie in Quadrant III when $180° < \theta < 270°$ or $\pi < \theta < \frac{3\pi}{2}$.

An angle θ is said to lie in Quadrant IV when $270° < \theta < 360°$ or $\frac{3\pi}{2} < \theta < 2\pi$.



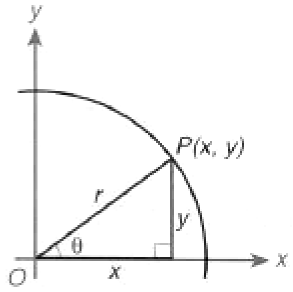
The angles 0^o, 90^o, 180^o, 270^o and 360^o do not belong to any one of the quadrants.

5.2.2 Definitions of Six Trigonometric Functions

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

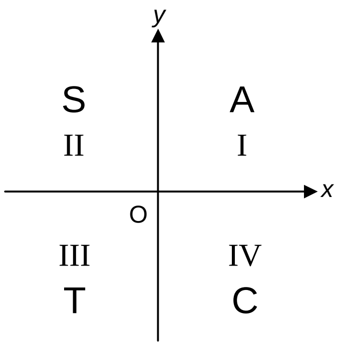
$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$



5.2.3 Signs of Trigonometric Functions

Quadrant	**sin and csc**	**cos and sec**	**tan and cot**
I	+	+	+
II	+	–	–
III	–	–	+
IV	–	+	–

We can use the following chart to remember the signs of the six trigonometric functions in different quadrants.



A: All positive
S: Sine positive (cosecant also positive)
T: Tangent positive (cotangent also positive)
C: Cosine positive (secant also positive)

We may memorize them as **CAST** or **A**ll **S**tudents **T**ake **C**hinese.

Example 5.3

Find the value of each of the following, giving the answers correct to 4 significant figures.

(a) $\sin 146° + \csc 34°$

(b) $\cot 74° \times \cos 136°$

(c) $\dfrac{\tan 224°}{\sec 56°}$

Solution

(a) $\sin 146° + \csc 34°$

$= \sin 146° + \dfrac{1}{\sin 34°}$

$= 2.347$ (corr. to 4 sig. fig.)

(b) $\cot 74° \times \cos 136°$

$= \dfrac{1}{\tan 74°} \times \cos 136°$

$= -0.2063$ (corr. to 4 sig. fig.)

(c) $\dfrac{\tan 224°}{\sec 56°}$

$= \dfrac{\tan 224°}{\dfrac{1}{\cos 56°}}$

$= \tan 224° \times \cos 56°$

$= 0.5400$ (corr. to 4 sig. fig.)

Checkpoint 5.2

By the use of a calculator, find the values of the following. (*Give the answers correct to 3 significant figures.*)

(a) $\cos 46° - \sec 91°$ = ______________

(b) $\sin 211° + \cot 114°$ = ______________

(c) $\dfrac{\tan 62°}{\cot 139°}$ = ______________

(d) $\dfrac{\sec 126° + \csc 18°}{\cot 55°}$ = ______________

(e) $\dfrac{\sin 42° + \sec 32°}{\sin 48° - \sec 148°}$ = ______________

5.2.4 Trigonometric Relations

(1) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$; $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

(2) $\csc\theta = \dfrac{1}{\sin\theta}$; $\sec\theta = \dfrac{1}{\cos\theta}$; $\cot\theta = \dfrac{1}{\tan\theta}$ (Reciprocal Relations)

(3) $\sin^2\theta + \cos^2\theta = 1$; $1 + \tan^2\theta = \sec^2\theta$; $1 + \cot^2\theta = \csc^2\theta$ (Square Relations)

Example 5.4

If $\cot\theta = -\dfrac{4}{3}$ and $\dfrac{\pi}{2} < \theta < \pi$, find the values of csc θ and cos θ.

Solution

$$\csc^2\theta = 1 + \cot^2\theta$$
$$= 1 + \left(-\frac{4}{3}\right)^2$$
$$= \frac{25}{9}$$

$\because$ $\dfrac{\pi}{2} < \theta < \pi$, θ is in quadrant II and csc θ > 0,

$\therefore$ $\csc\theta = \dfrac{5}{3}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\cos\theta = \frac{\sin\theta}{\tan\theta}$$
$$= \sin\theta\cot\theta$$
$$= \frac{\cot\theta}{\csc\theta}$$
$$= \frac{-\frac{4}{3}}{\frac{5}{3}}$$
$$= -\frac{4}{5}$$

Example 5.5

If $\sin\theta = a$, find the values of cos θ and tan θ in terms of a.

Solution

$$\cos^2\theta = 1 - \sin^2\theta$$
$$= 1 - a^2$$
$$\therefore \quad \cos\theta = \pm\sqrt{1-a^2}$$

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$= \pm\frac{a}{\sqrt{1-a^2}}$$

Checkpoint 5.3

If $\sin\theta = \frac{3}{5}$ and $\frac{\pi}{2} < \theta < \pi$, find the values of sec θ and tan θ. Hence evaluate $\frac{\sec\theta + \cot\theta}{\sin\theta + 2\tan\theta}$.

Example 5.6

Simplify $\tan A + \cot A - \csc A \sec A$.

Solution

$$\begin{aligned}
\tan A + \cot A - \csc A \sec A &= \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} - \frac{1}{\sin A} \cdot \frac{1}{\cos A} \\
&= \frac{\sin^2 A + \cos^2 A}{\sin A \cos A} - \frac{1}{\sin A \cos A} \\
&= \frac{1}{\sin A \cos A} - \frac{1}{\sin A \cos A} \\
&= 0
\end{aligned}$$

Example 5.7

Prove the identity $\dfrac{1+\cot\theta+\csc\theta}{1-\cot\theta+\csc\theta} = \csc\theta + \cot\theta$.

Solution

$$\begin{aligned}
\text{L.H.S.} &= \frac{1+\cot\theta+\csc\theta}{1-\cot\theta+\csc\theta} \\
&= \frac{(\csc^2\theta - \cot^2\theta) + \cot\theta + \csc\theta}{1-\cot\theta+\csc\theta} \\
&= \frac{(\csc\theta+\cot\theta)(\csc\theta-\cot\theta) + (\cot\theta+\csc\theta)}{1-\cot\theta+\csc\theta} \\
&= \frac{(\csc\theta+\cot\theta)(\csc\theta-\cot\theta+1)}{1-\cot\theta+\csc\theta} \\
&= \csc\theta + \cot\theta \\
&= \text{R.H.S.}
\end{aligned}$$

$$\therefore \quad \frac{1+\cot\theta+\csc\theta}{1-\cot\theta+\csc\theta} = \csc\theta + \cot\theta$$

Alternative Solution

$$\begin{aligned}
\text{L.H.S.} &= \frac{1+\cot\theta+\csc\theta}{1-\cot\theta+\csc\theta} \\
&= \frac{(1+\cot\theta+\csc\theta)(1-\cot\theta-\csc\theta)}{(1-\cot\theta+\csc\theta)(1-\cot\theta-\csc\theta)} \\
&= \frac{1-(\cot\theta+\csc\theta)^2}{(1-\cot\theta)^2-\csc^2\theta} \\
&= \frac{1-(\cot^2\theta+2\cot\theta\csc\theta+\csc^2\theta)}{1-2\cot\theta+\cot^2\theta-\csc^2\theta} \\
&= \frac{-\cot^2\theta-\cot^2\theta-2\cot\theta\csc\theta}{-2\cot\theta+\csc^2\theta-\csc^2\theta} \\
&= \frac{-2\cot^2\theta-2\cot\theta\csc\theta}{-2\cot\theta} \\
&= \frac{-2\cot\theta(\cot\theta+\csc\theta)}{-2\cot\theta} \\
&= \csc\theta+\cot\theta \\
&= \text{R.H.S.}
\end{aligned}$$

$$\therefore \quad \frac{1+\cot\theta+\csc\theta}{1-\cot\theta+\csc\theta} = \csc\theta+\cot\theta$$

Checkpoint 5.4

Simplify $\dfrac{\cos\theta}{\csc\theta\sec\theta-\tan\theta}$.

Checkpoint 5.5

Prove the identity $\sec^2\theta - \csc^2\theta = \tan^2\theta - \cot^2\theta$.

5.3 The Trigonometric Functions of $2n\pi \pm \theta$, $\frac{\pi}{2} \pm \theta$, $\pi \pm \theta$ and $-\theta$

	$2n\pi + \theta$	$\pi - \theta$	$\pi + \theta$	$2n\pi - \theta$	$-\theta$
sin	$\sin\theta$	$\sin\theta$	$-\sin\theta$	$-\sin\theta$	$-\sin\theta$
cos	$\cos\theta$	$-\cos\theta$	$-\cos\theta$	$\cos\theta$	$\cos\theta$
tan	$\tan\theta$	$-\tan\theta$	$\tan\theta$	$-\tan\theta$	$-\tan\theta$

	$\frac{\pi}{2} - \theta$	$\frac{\pi}{2} + \theta$	$\frac{3\pi}{2} - \theta$	$\frac{3\pi}{2} + \theta$
sin	$\cos\theta$	$\cos\theta$	$-\cos\theta$	$-\cos\theta$
cos	$\sin\theta$	$-\sin\theta$	$-\sin\theta$	$\sin\theta$
tan	$\cot\theta$	$-\cot\theta$	$\cot\theta$	$-\cot\theta$

Note that we call θ the *reference angle*, which is the angle measured from either positive or negative x-axis in clockwise or anti-clockwise direction. We assume $0 < \theta < \frac{\pi}{2}$.

Example 5.8

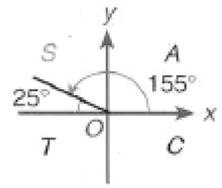
Reduce the trigonometric ratios with obtuse / reflex angles into the ones with acute angles.

(a) cos 155^o (b) sin 230^o (c) tan 320^o

Solution

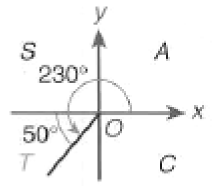
(a) $\cos 155° = \cos(180° - 25°)$

$= -\cos 25°$



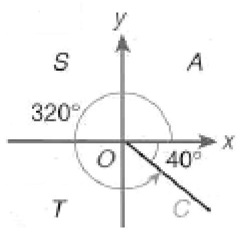
(b) $\sin 230° = \sin(180° + 50°)$

$= -\sin 50°$



(c) $\tan 320° = \tan(360° - 40°)$

$= -\tan 40°$



Example 5.9

Without using a calculator, find the values of the following:

(a) sin 240^o (b) cos (–30^o) (c) tan 150^o

Solution

(a) $\sin 240° = \sin(180° + 60°)$ $\quad$ * $\sin(180° + \theta) = -\sin\theta$

$= -\sin 60°$

$= -\dfrac{\sqrt{3}}{2}$

(b) $\cos(-30°) = \cos 30°$ $\quad$ * $\cos(-\theta) = \cos\theta$

$= \dfrac{\sqrt{3}}{2}$

(c) $\tan 150° = \tan(180° - 30°)$ $\quad$ * $\tan(180° - \theta) = -\tan\theta$

$= -\tan 30°$

$= -\dfrac{\sqrt{3}}{3}$

Example 5.10

Without using a calculator, find the values of the following:

(a) $\sin\frac{3\pi}{4}$ (b) $\sec\frac{5\pi}{6}$ (c) $\tan\frac{5\pi}{4}$

Solution

(a) $\sin\frac{3\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right)$ $\quad$ $*\sin(\pi - \theta) = \sin\theta$

$= \sin\frac{\pi}{4}$

$= \frac{1}{\sqrt{2}}$

(b) $\sec\frac{5\pi}{6} = \sec\left(\pi - \frac{\pi}{6}\right)$ $\quad$ $*\sec(\pi - \theta) = -\sec\theta$

$= -\sec\frac{\pi}{6}$ $\quad$ $*\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$= -\frac{2}{\sqrt{3}}$ $\quad$ $\sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$

(c) $\tan\frac{5\pi}{4} = \tan\left(\pi + \frac{\pi}{4}\right)$ $\quad$ $*\tan(\pi + \theta) = -\tan\theta$

$= \tan\frac{\pi}{4}$

$= 1$

Checkpoint 5.6

Without using a calculator, find the values of the following:

(a) $\sin 330^{o}$ (b) $\sec(-300^{o})$ (c) $\cot 240^{o}$

Checkpoint 5.7

Without using a calculator, find the values of the following:

(a) $\sec\dfrac{7\pi}{4}$ (b) $\cot\dfrac{2\pi}{3}$ (c) $\sin\dfrac{5\pi}{3}$

Example 5.11

Simplify:

(a) $\sin(900° - \theta)$ (b) $\tan(\theta - 720°)$ (c) $\sec(-90° - \theta)$

Solution

(a)
$$\begin{aligned}\sin(900° - \theta) &= \sin(360° \times 2 + 180° - \theta)\\ &= \sin(180° - \theta)\\ &= \sin\theta\end{aligned}$$

(b)
$$\begin{aligned}\tan(\theta - 720°) &= \tan[-(720° - \theta)]\\ &= -\tan(360° \times 2 - \theta)\\ &= -\tan(-\theta)\\ &= \tan\theta\end{aligned}$$

Alternatively,

$$\begin{aligned}\tan(\theta - 720°) &= \tan[360° \times (-2) + \theta]\\ &= \tan\theta\end{aligned}$$

(c)
$$\begin{aligned}\sec(-90° - \theta) &= \sec[-(90° + \theta)]\\ &= \sec(90° + \theta)\\ &= -\csc\theta\end{aligned}$$

Example 5.12

Simplify:

(a) $\sec(3\pi+\theta)$ (b) $\cot\left(\frac{5\pi}{2}-\theta\right)$ (c) $\csc(7\pi-\theta)$

Solution

(a)
$$\begin{aligned}\sec(3\pi+\theta)&=\sec(2\pi+\pi+\theta)\\&=\sec(\pi+\theta)\\&=-\sec\theta\end{aligned}$$

(b)
$$\begin{aligned}\cot\left(\frac{5\pi}{2}-\theta\right)&=\cot\left(2\pi+\frac{\pi}{2}-\theta\right)\\&=\cot\left(\frac{\pi}{2}-\theta\right)\\&=\tan\theta\end{aligned}$$

(c)
$$\begin{aligned}\csc(7\pi-\theta)&=\csc(2\pi\times3+\pi-\theta)\\&=\csc(\pi-\theta)\\&=\csc\theta\end{aligned}$$

Checkpoint 5.8

Simplify the following:

(a) $\sin(360°-\theta)$ (b) $\cos(6\pi+\theta)$

(c) $\cot(\theta-1170°)$ (d) $\sec(630°+\theta)$

Example 5.13

Simplify $\dfrac{\cot A\sin(\pi + A)\cos(\pi - A)\tan\left(\dfrac{\pi}{2} + A\right)\csc\left(A - \dfrac{\pi}{2}\right)}{\sec\left(\dfrac{3\pi}{2} + A\right)}$.

Solution

$$\frac{\cot A\sin(\pi + A)\cos(\pi - A)\tan\left(\frac{\pi}{2} + A\right)\csc\left(A - \frac{\pi}{2}\right)}{\sec\left(\frac{3\pi}{2} + A\right)}$$

$$= \frac{\cot A(-\sin A)(-\cos A)(-\cot A)(-\sec A)}{\csc A}$$

$$= \frac{\frac{\cos A}{\sin A}\cdot\sin A\cdot\cos A\cdot\frac{\cos A}{\sin A}\cdot\frac{1}{\cos A}}{\frac{1}{\sin A}}$$

$$= \cos^2 A$$

Checkpoint 5.9

Simplify $\csc(\theta - \pi)\sec(-\theta)\tan\left(\dfrac{\pi}{2} - \theta\right)$.

5.4 Graphs of Trigonometric Functions

Characteristics of graphs of trigonometric functions:

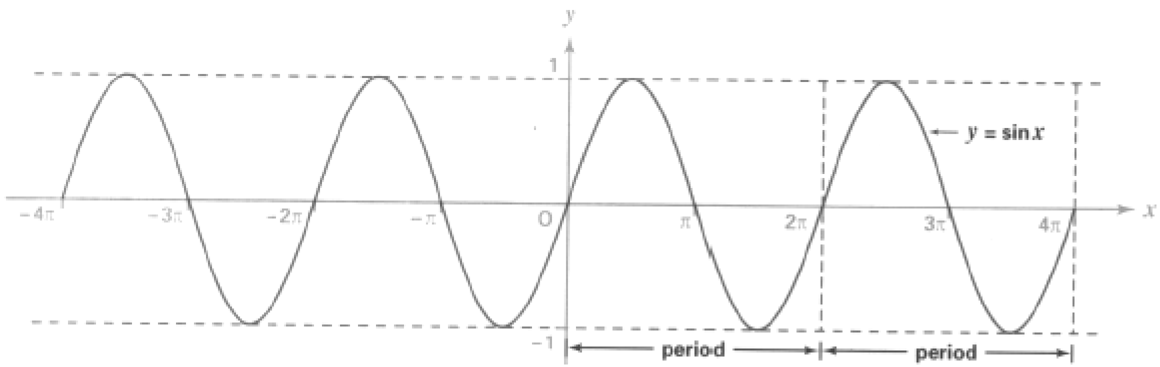
(1) Continuous / Discontinuous

(2) Range

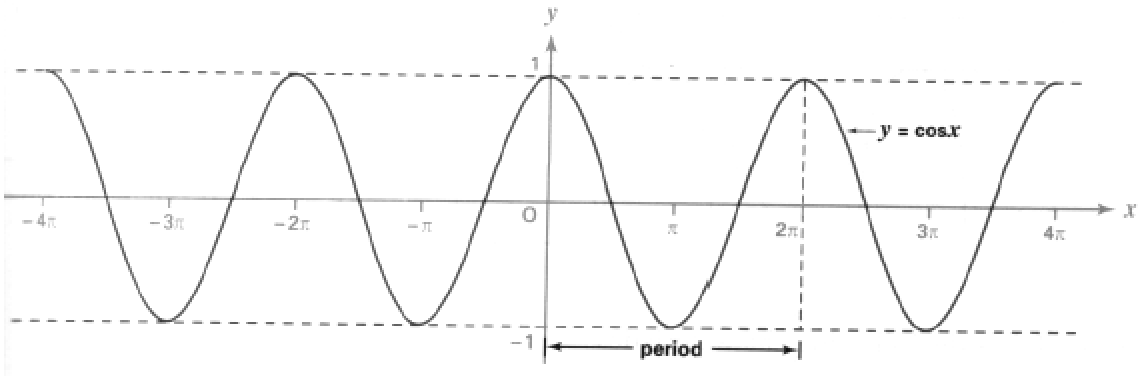
(3) Period

Graphs of the 6 trigonometric functions:

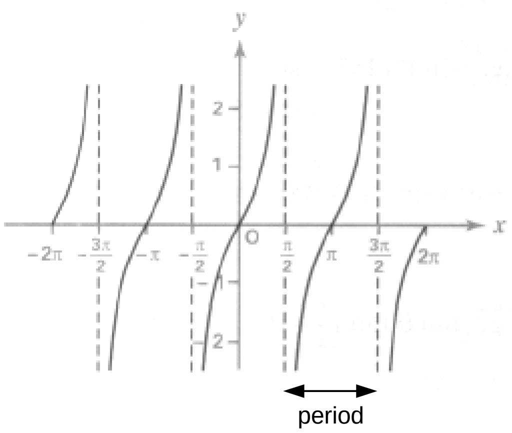
(1) Graph of $y = \sin x$



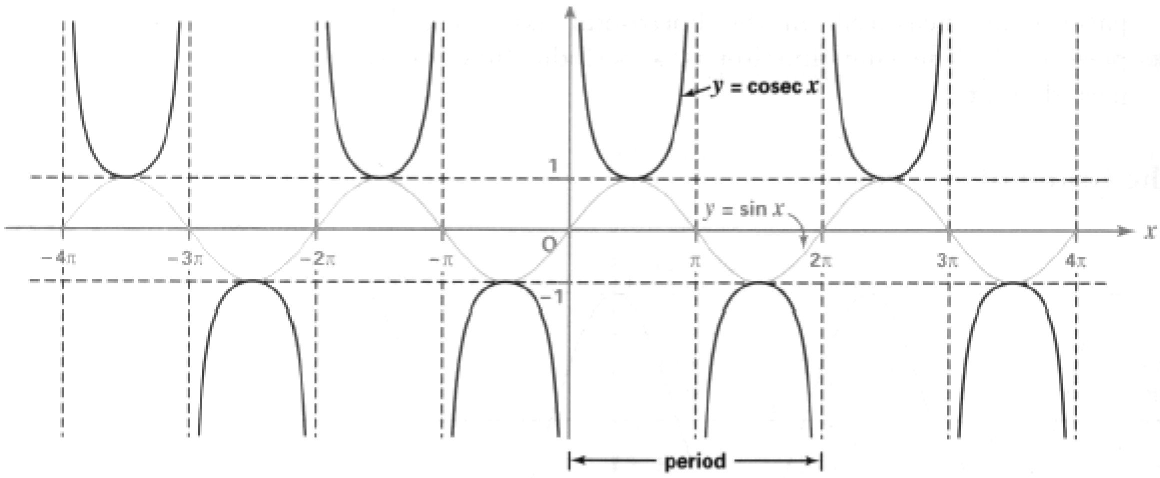
(2) Graph of $y = \cos x$



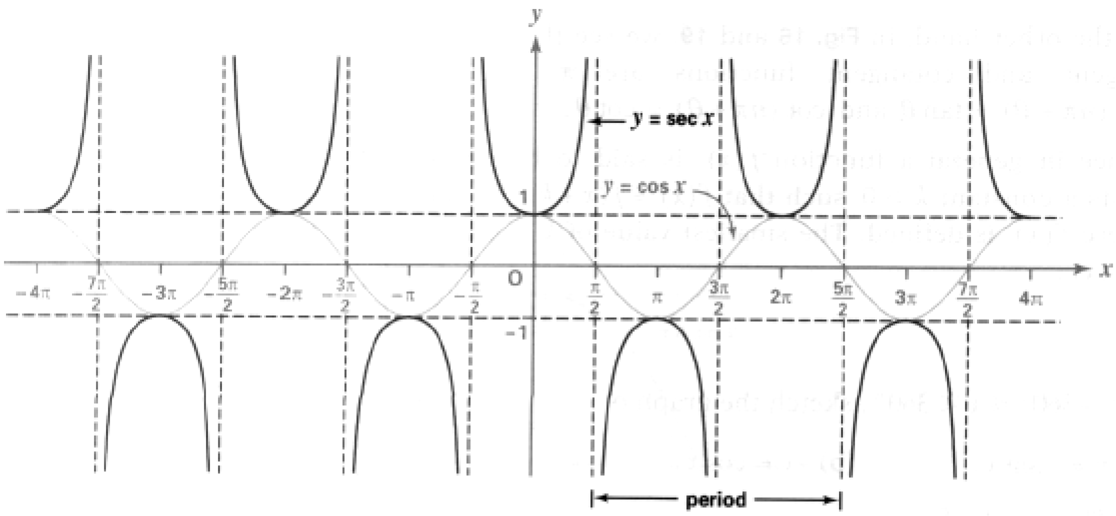
(3) Graph of $y = \tan x$



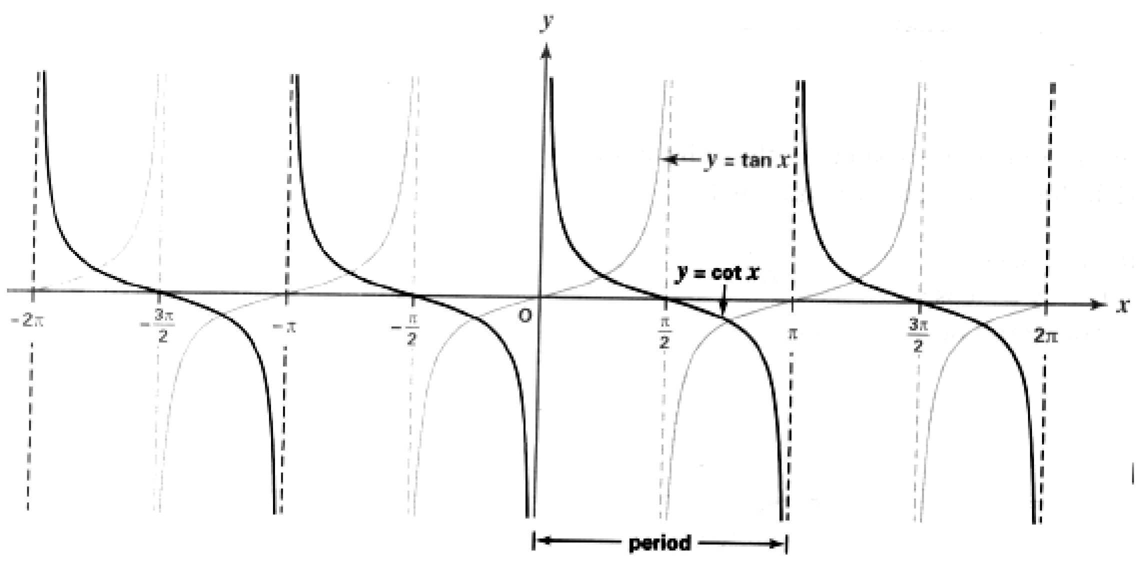
(4) Graph of $y = \csc x$



(5) Graph of $y = \sec x$



(6) Graph of $y = \cot x$

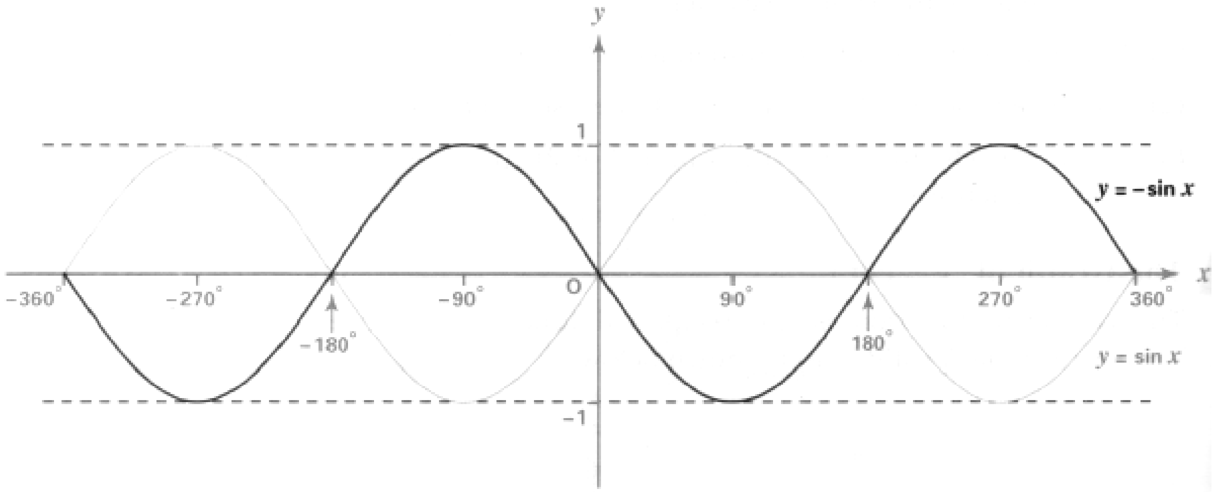


Example 5.14

Sketch the graph of $y = -\sin x$ for $-360° \le x \le 360°$.

Solution

The graph of $y = -\sin x$:



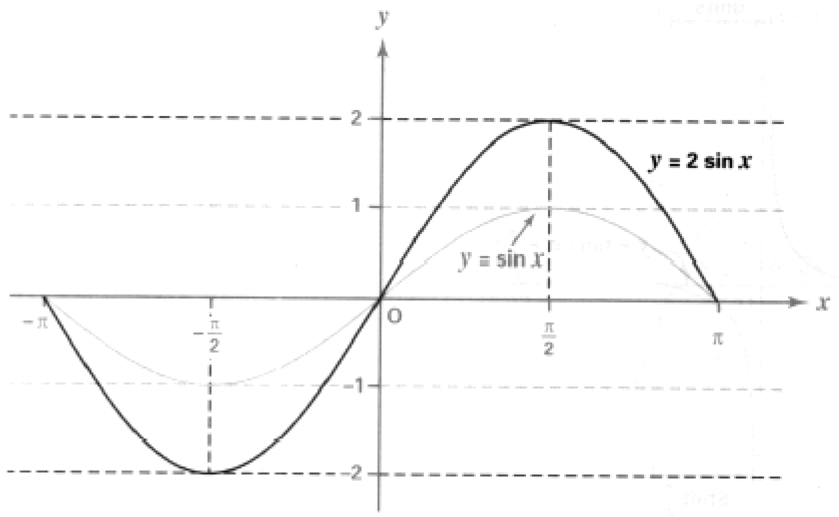
Example 5.15

Sketch the graph of $y = 2\sin x$ for $-\pi \le x \le \pi$.

Solution

[Since $-1 \le \sin x \le 1$, we have $-2 \le 2\sin x \le 2$, i.e. the maximum value and minimum value of y = 2 sin x are respectively –2 and 2.]

The graph of $y = 2\sin x$:

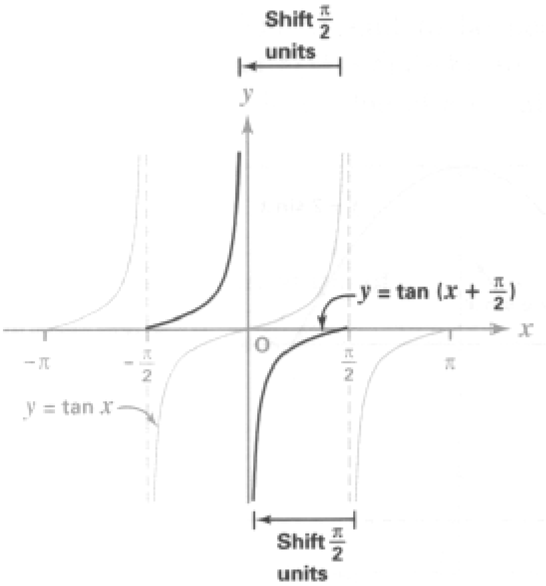


Example 5.16

Sketch the graph of $y = \tan\left(x + \frac{\pi}{2}\right)$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

Solution

The graph of $y = \tan\left(x + \frac{\pi}{2}\right)$:



Example 5.17

For $0 \le x \le 2\pi$, sketch the graphs of

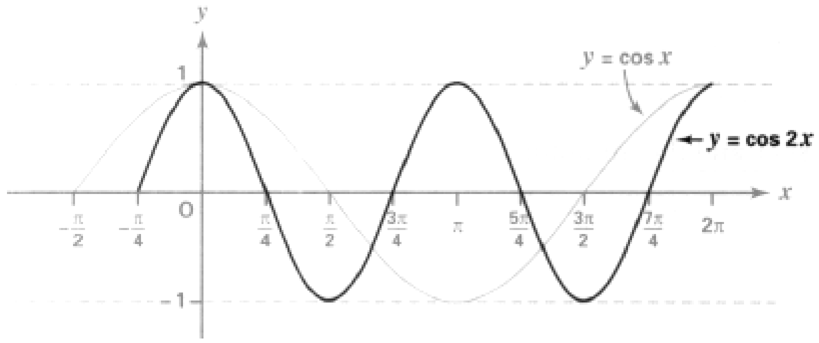
(a) $y = \cos 2x$

(b) $y = \cos\left(2x - \frac{\pi}{4}\right)$

Solution

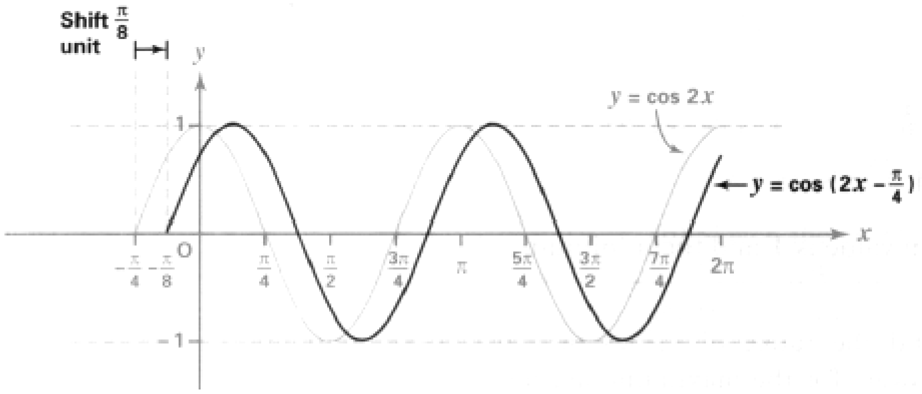
(a) [Since the period of $y = \cos x$ is π, the period of $y = \cos 2x$ is $\frac{1}{2}$ of the period of $y = \cos x$, i.e. $\frac{\pi}{2}$.]

The graph of $y = \cos 2x$:



(b) [To obtain the graph of $y=\cos\left(2x-\frac{\pi}{4}\right)=\cos 2\left(x-\frac{\pi}{8}\right)$, shift the graph of $y=\cos 2x$ by $\frac{\pi}{8}$ units to the right.]

The graph of $y=\cos\left(2x-\frac{\pi}{4}\right)$:

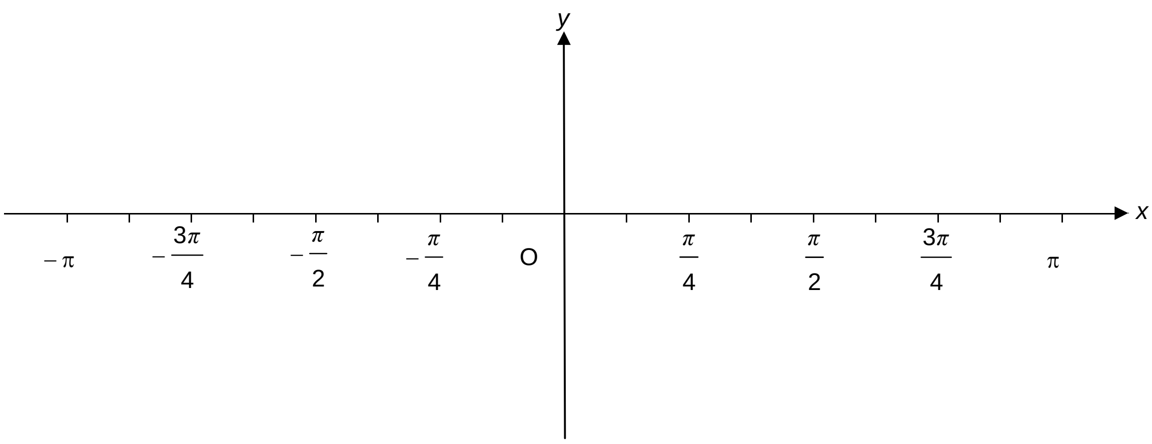


Checkpoint 5.10

For $-\pi \le x \le \pi$,

(a) Sketch the graph of $y=\tan\frac{1}{2}x$.

(b) On the same graph, draw the graph of $y=\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$.



Example 5.18

Find the maximum and minimum values of the following functions.

(a) $y = 2\cos x - 3$

(b) $y = 5\sin\frac{x}{2} + 3$

(c) $y = \frac{1}{\cos x + 2}$

Solution

(a)

$$-1 \le \cos x \le 1$$
$$-2 \le 2\cos x \le 2$$
$$-5 \le 2\cos x - 3 \le -1$$

∴ The maximum value is –1 and the minimum value is –5.

(b)

$$-1 \le \sin\frac{x}{2} \le 1$$
$$-5 \le 5\sin\frac{x}{2} \le 5$$
$$-2 \le 5\sin\frac{x}{2} + 3 \le 8$$

∴ The maximum value is 8 and the minimum value is –2.

(c)

$$-1 \le \cos x \le 1$$
$$1 \le \cos x + 2 \le 3$$
$$\frac{1}{3} \le \frac{1}{\cos x + 2} \le 1$$

∴ The maximum value is 1 and the minimum value is $-\frac{1}{3}$.

Checkpoint 5.11

Find the maximum and minimum values of $y = 3\sin^2 2x + 2\cos^2 2x$.

5.5 Simple Trigonometric Equations

We are going to find the solutions of simple trigonometric equations in a certain range, usually from 0 to 2π (or $0°$ to $360°$). Below are some examples.

General procedure for finding the solutions of a trigonometric equation in *x*:

(1) Find the solution in Quadrant I. The angle obtained is the reference angle.

(2) Determine which quadrants x can lie in.

(3) Find the other solution by the use of the reference angle and the results obtain in (2).

Example 5.19

Solve the equation $\sqrt{3}\csc x = 2\cot x$ for $0° \le x < 360°$.

Solution

$$\sqrt{3}\csc x = 2\cot x$$

$$\frac{\sqrt{3}}{\sin x} = \frac{2\cos x}{\sin x} \qquad \text{for } \sin x \ne 0$$

$$\cos x = \frac{\sqrt{3}}{2}$$

$$\therefore \quad x = 30° \quad \text{or} \quad 360° - 30°$$

$$= 30° \quad \text{or} \quad 330°$$

Note: (1) The solution in Quadrant I is 30^{o}, which is also the reference angle.

(2) Since $\cos x > 0$, x may lie in Quadrant IV.

(3) Then by the use of the reference angle, we can find the other solution in Quadrant IV, which is $360^{o} – 30^{o} = 330^{o}$.

Example 5.20

Solve the equation $\sin x = -\cos 80°$ for $0° \le x < 360°$.

Solution

$$\sin x = -\cos 80°$$

$$= -\sin(90° - 80°)$$

$$= -\sin 10°$$

$$= \sin(180° + 10°) \quad \text{or} \quad \sin(360° - 10°)$$

$$= \sin 190° \quad \text{or} \quad \sin 350°$$

$$\therefore \quad x = 190° \quad \text{or} \quad 350°$$

Checkpoint 5.12

Solve the equation $2\sin x = \sqrt{12}\cos x$ for $0° \le x < 360°$.

Checkpoint 5.13

Solve the equation $\tan x = \cot 26°$ for $0° \le x < 360°$.

Example 5.21

Solve the equation $2\sin^2 x - 3\sin x - 2 = 0$ for $0 \le x < 2\pi$.

Solution

$$2\sin^2 x - 3\sin x - 2 = 0$$

$$(2\sin x + 1)(\sin x - 2) = 0$$

$$\sin x = -\frac{1}{2} \quad \text{or} \quad \sin x = 2 \quad \text{(rejected)}$$

$$x = \pi + \frac{\pi}{6} \quad \text{or} \quad 2\pi - \frac{\pi}{6}$$

$$= \frac{7\pi}{6} \quad \text{or} \quad \frac{11\pi}{6}$$

Example 5.22

Solve the equation $\sin x + 3\cos x = \sqrt{10}$ for $0° \le x < 360°$.

(*Give the answers correct to the nearest 0.1^o.*)

Solution

$$\sin x + 3\cos x = \sqrt{10}$$

$$(\sin x + 3\cos x)^2 = 10$$

$$\sin^2 x + 6\sin x\cos x + 9\cos^2 x = 10(\sin^2 x + \cos^2 x)$$

$$\cos^2 x - 6\sin x\cos x + 9\sin^2 x = 0$$

$$(\cos x - 3\sin x)^2 = 0$$

$$\cos x = 3\sin x$$

$$\tan x = \frac{1}{3}$$

$$x = 18.4° \quad \text{or} \quad 180° + 18.4°$$

$$= 18.4° \quad \text{or} \quad 198.4° \quad \text{(corr. to the nearest } 0.1°\text{)}$$

After substituting $x = 198.4°$ into $\sin x + 3\cos x = \sqrt{10}$, we reject $198.4°$ as a solution.

Checkpoint 5.14

Solve the equation $2\sec^2 x - \tan x = 3$ for $0° \le x < 360°$.

Example 5.23

Solve the equation $\tan\left(2x - \frac{\pi}{6}\right) + \sqrt{3} = 0$ for $0 \le x < 2\pi$.

Solution

Note that $\quad 0 \le x < 2\pi$

$$0 \le 2x < 4\pi$$

Therefore $\quad -\frac{\pi}{6} \le 2x - \frac{\pi}{6} < \frac{23\pi}{6}$

$$\tan\left(2x - \frac{\pi}{6}\right) + \sqrt{3} = 0$$

$$\tan\left(2x - \frac{\pi}{6}\right) = -\sqrt{3}$$

$$2x - \frac{\pi}{6} = \pi - \frac{\pi}{3}, \quad 2\pi - \frac{\pi}{3}, \quad 2\pi + \left(\pi - \frac{\pi}{3}\right), \quad 2\pi + \left(\pi - \frac{\pi}{3}\right)$$

$$= \frac{2\pi}{3}, \quad \frac{5\pi}{3}, \quad \frac{8\pi}{3}, \quad \frac{11\pi}{3}$$

$$2x = \frac{5\pi}{6}, \quad \frac{11\pi}{6}, \quad \frac{17\pi}{6}, \quad \frac{23\pi}{6}$$

$$x = \frac{5\pi}{12}, \quad \frac{11\pi}{12}, \quad \frac{17\pi}{12}, \quad \frac{23\pi}{12}$$

Checkpoint 5.15

Solve the equation $\cos^2\left(2x+\frac{\pi}{3}\right)-2\cos\left(2x+\frac{\pi}{3}\right)+1=0$ for $0 \le x < 2\pi$.

Exercise 5 Trigonometric Functions

5.1

1. Convert the following angles into degree measure.
   (*Give the answers correct to 3 significant figures where necessary.*)
   (a) $\dfrac{\pi}{2}^c$ (b) 5^c

2. Convert the following angles into radian measure.
   (*Give the answers in terms of* π.)
   (a) $180°$ (b) $330°$

3. Convert the following angles into radian measure.
   (*Give the answers correct to 0.1^c where necessary.*)
   (a) $121°$ (b) $305°$

5.2

4. Simplify:
   (a) $\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}$
   (b) $\cot\theta\sec\theta$
   (c) $(\sec\theta-\cos\theta)(\csc\theta-\sin\theta)$
   (d) $\dfrac{1}{1+\dfrac{1}{\cot^2\theta}}+\dfrac{1}{1+\dfrac{1}{\tan^2\theta}}$

5. Prove the following identities:
   (a) $\dfrac{\sin^2\theta}{1-\cos\theta}=1+\cos\theta$
   (b) $\sin\alpha(\tan\alpha+\cot\alpha)=\dfrac{1}{\cos\alpha}$
   (c) $\csc\theta+\sec\theta\tan\theta=\sec^2\theta\csc\theta$
   (d) $\dfrac{\cot A+\tan B}{\cot B+\tan A}=\cot A\tan B$

6. If $\cos\theta=0.8$, find the values of the following expressions.
   (a) $2\sin^2\theta+3\cos\theta$
   (b) $\tan^2\theta-\cos\theta$

7. If $\tan\theta = \dfrac{2u}{1-u^2}$, find $\sec\theta$ and $\csc\theta$ in terms of *u*.

8. If $\tan^2 A - 2\tan^2 B = 1$, find the possible values of $\dfrac{\cos A}{\cos B}$.

9. If $a\sin\theta + b\cos\theta = p$ and $b\sin\theta - a\cos\theta = q$, show that $a^2 + b^2 = p^2 + q^2$.

5.3

10. Find the value of each of the following without using a calculator.

(a) $\sin 150°$ (b) $\sec 300°$

(c) $\cos(-480°)$ (d) $\csc 585°$

(e) $\sin\dfrac{4\pi}{3}$ (f) $\cos\dfrac{13\pi}{4}$

(g) $\tan\left(-\dfrac{3\pi}{4}\right)$ (h) $\csc\dfrac{11\pi}{3}$

11. Without using a calculators , evaluate $\csc^2\dfrac{5\pi}{3} - \cot^2\dfrac{4\pi}{3}$.

12. Simplify:

(a) $\tan(3\pi + \theta)$ (b) $\cot(\theta - 7\pi)$

13. If $\tan(180° + \theta) = \dfrac{\sqrt{3}}{2}$, find $\cot(90° + \theta)$.

14. Simplify:

(a) $$\frac{\tan(180° + \theta) + \tan(180° - \theta) - \sec(90° + \theta)}{\csc(720° + \theta)}$$

(b) $$\frac{\sin(\pi - A)\cot\left(\frac{\pi}{2} - A\right)\cos(2\pi - A)\sin(\pi + A)}{\tan A\tan\left(\frac{\pi}{2} + A\right)\sin(-A)}$$

15. If A, B and C are angles of a triangle, show that

(a) $\sin\left(\frac{A+B}{2}\right)=\cos\frac{C}{2}$

(b) $\frac{\sin A\sin B+\sin(B+C)\sin(A+C)}{\cos A\cos B+\cos(B+C)\cos(A+C)}=\tan A\tan B$

5.4

16. In each of the following, find the range of values of y.

(a) $y=-3\cos\left(2x+\frac{\pi}{6}\right)$

(b) $y=3\sin\frac{x}{2}-5$

17. In each of the following, find the maximum and minimum values of the function.

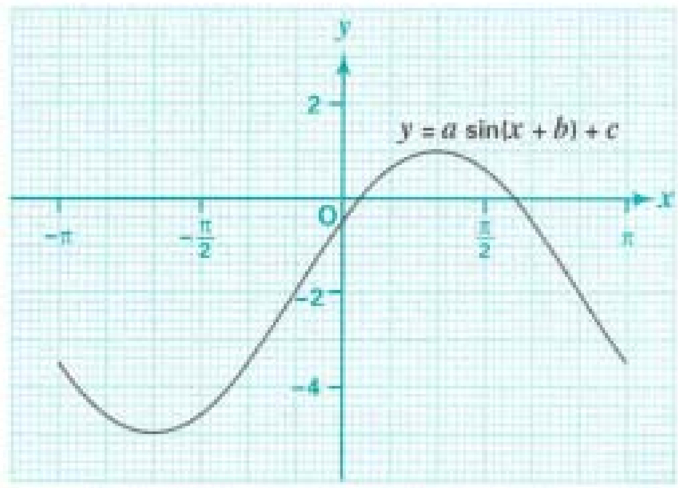
(a) $y=-2\sin x+3$

(b) $y=\frac{6}{7-\sin x}$

18. Sketch the graph of $y=3\cos\frac{x}{2}$, where $0°\le x\le 360°$.

19. Sketch the graph of $y=\tan x+2$, where $-90°<x<90°$.

20. The figure below shows the graph of $y=a\sin(x+b)+c$, where $a>0$ and $0\le b\le\frac{\pi}{2}$, for $-\pi\le x\le\pi$. Determine the values of a, b and c.



5.5

21. Solve $\cos(x+30°)=\sin 30°$ for $0° \le x \le 360°$.

22. Solve the following equations for $0 \le x \le 2\pi$.
(*Give the answers in terms of* π.)
(a) $4\cos^2 x = 3$
(b) $2\cos\left(x-\frac{\pi}{4}\right)+\sqrt{3}=0$

23. Solve the following equations for $0° \le x \le 360°$.
(*Give the answers correct to 1 decimal place where necessary.*)
(a) $8\sin^4 x - 10\sin^2 x + 3 = 0$
(b) $(2\tan x - 1)^2 = 3(\sec^2 x - 2)$
(c) $5\sin^2 x + \sin x\cos x + 2\cos^2 x - 3 = 0$
(d) $2\tan x + 3\sec x = 4\cos x$

24. Solve $\tan^2 x\sec^2 x + \tan^2 x = \sec^2 x - 1$ for $0 \le x \le 2\pi$.
(*Give the answers in terms of* π.)

25. Solve $\sin 2x = \cos 3x$ for $0 \le x \le 2\pi$
(*Give the answers in terms of* π.)