Chapter 4 Binomial Theorem

4.1 Factorials

The product of the first n positive integers is denoted by $n!$,
i.e. $$n! = n \times (n-1) \times \ldots \times 2 \times 1$$
In particular, we define 0! = 1.

Example 4.1

Evaluate each of the following without using a calculator:

(a) $6!$

(b) $\frac{7!}{4!}$

(c) $\frac{8!}{2!6!}$

(d) $\frac{n!}{(n-1)!}$

Solution

(a) $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$
$= 720$

(b) $\frac{7!}{4!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1}$
$= 7 \times 6 \times 5$
$= 210$

(c) $\frac{8!}{2!6!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(2 \times 1) \times (6 \times 5 \times 4 \times 3 \times 2 \times 1)}$
$= 28$

(d) $\frac{n!}{(n-1)!} = \frac{n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1}{(n-1)!}$
$= \frac{n \times (n-1)!}{(n-1)!}$
$= n$

Checkpoint 4.1

Evaluate each of the following without using a calculator:

(a) $4!$ (b) $7!2!$

(c) $\frac{12!}{9!}$ (d) $\frac{10!}{10\times 9!}$

4.2 Pascal Triangle

The Pascal Triangle is the triangular number pattern.

							1								← row 0
						1		1							← row 1
					1		2		1						← row 2
				1		3		3		1					← row 3
			1		4		6		4		1				← row 4
		1		5		10		10		5		1			← row 5
	1		6		15		20		15		6		1		← row 6
1		7		21		35		35		21		7		1	← row 7
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

The $(r + 1)$st number in row n of the Pascal Triangle is denoted by C_r^n, where r is an integer such that $0 \le r \le n$.

In factorials,

$$C_r^n = \frac{n!}{r!(n-r)!}$$

e.g. $$C_3^5 = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = 10$$

Note: C_3^5 is the 4th number in row 5.

Example 4.2

Without using a calculator, evaluate

(a) C_2^8

(b) C_7^{10}

(c) C_0^n

(d) C_n^n

Solution

(a) $$C_2^8 = \frac{8!}{2!(8-2)!} = \frac{8!}{2!6!} = 28$$

(b) $$C_7^{10} = \frac{10!}{7!(10-7)!} = \frac{10!}{7!3!} = 120$$

(c) $$C_0^n = \frac{n!}{0!(n-0)!} = \frac{n!}{1(n!)} = 1$$

(d) $$C_n^n = \frac{n!}{n!(n-n)!} = \frac{n!}{n!(1)} = 1$$

Checkpoint 4.2

Without using a calculator, evaluate

(a) C_4^7

(b) C_{11}^{14}

(c) C_{30}^{30}

(d) $C_3^{10} + C_4^{10}$

Properties

(1) $C_r^{n+1} = C_r^n + C_{r-1}^n$

e.g. $C_2^{4+1} = C_2^4 + C_{2-1}^4$

$C_2^5 = C_2^4 + C_1^4$

(2) $C_r^n = C_{n-r}^n$

e.g. ${}_5C_2 = {}_5C_3$

Example 4.3

Solve for n if $C_{n+1}^{12} = C_{2n-1}^{12}$.

Solution

Since $C_{n+1}^{12} = C_{2n-1}^{12}$, we have

$$
\begin{aligned}
n+1 = 2n-1 \quad &\text{or} \quad n+1 = 12-(2n-1) \\
n = 2 \quad &\text{or} \quad n+1 = 13-2n \\
n = 2 \quad &\text{or} \quad 3n = 12 \\
n = 2 \quad &\text{or} \quad n = 4
\end{aligned}
$$

Example 4.4

Simplify $C_3^{n+1} - C_3^n$.

Solution

$$
\begin{aligned}
C_3^{n+1} - C_3^n &= (C_3^n + C_2^n) - C_3^n \\
&= C_2^n
\end{aligned}
$$

Checkpoint 4.3

Solve for n if $C_{14-n}^{23} = C_{2n+3}^{23}$.

4.3 Binomial Theorem

Binomial theorem states that:

For any positive integer n,

$$(a+b)^n = C_0^n a^n + C_1^n a^{n-1}b + C_2^n a^{n-2}b^2 + C_3^n a^{n-3}b^3 \ldots + C_r^n a^{n-r}b^r \ldots + C_n^n b^n$$
$$= \sum_{k=0}^{n} C_k^n a^{n-k} b^k$$

Note: $C_r^n a^{n-r} b^r$ is called the general term.

In particular, by letting $a = 1$ and $b = x$, we have

$$(1+x)^n = 1 + C_1^n x + C_2^n x^2 + C_3^n x^3 \ldots + C_r^n x^r \ldots + C_n^n x^n$$
$$= \sum_{k=0}^{n} C_k^n x^k$$

Example 4.5

Expand

(a) $(2+3x)^4$ in descending powers of x;

(b) $(7x-1)^5$ in ascending powers of x.

Solution

(a) $$(2+3x)^4 = C_0^4 2^4 + C_1^4 2^3(3x) + C_2^4 2^2(3x)^2 + C_3^4 2(3x)^3 + C_4^4 (3x)^4$$
$$= 16 + 4(8)(3x) + 6(4)(9x^2) + 4(2)(27x^3) + 81x^4$$
$$= 81x^4 + 216x^3 + 216x^2 + 96x + 16$$

(b) $$(7x-1)^5 = C_0^5 (7x)^5 + C_1^5 (7x)^4(-1) + C_2^5 (7x)^3(-1)^2 + C_3^5 (7x)^2(-1)^3 + C_4^5 (7x)(-1)^4 + C_5^5 (-1)^5$$
$$= (7x)^5 - 5(7x)^4 + 10(7x)^3 - 10(7x)^2 + 5(7x) - 1$$
$$= -1 + 35x - 490x^2 + 3430x^3 - 12005x^4 + 16807x^5$$

Checkpoint 4.4

Expand

(a) $(1+x)^3$

(b) $(2-x)^4$

(c) $(2x+1)^6$

(d) $\left(\frac{1}{2}-x\right)^5$

Example 4.6

Expand

(a) $(2x-3y)^3$

(b) $(3x^2+y)^5$

(c) $\left(2x-\frac{3}{x}\right)^4$

Solution

(a)
$$\begin{aligned}(2x-3y)^3 &= C_0^3(2x)^3+C_1^3(2x)^2(-3y)+C_2^3(2x)(-3y)^2+C_3^3(-3y)^3\\ &= 8x^3+3(4x^2)(-3y)+3(2x)(9y^2)-27y^3\\ &= 8x^3-36x^2y+54xy^2-27y^3\end{aligned}$$

(b)
$$\begin{aligned}(3x^2+y)^5 &= C_0^5(3x^2)^5+C_1^5(3x^2)^4y+C_2^5(3x^2)^3y^2+C_3^5(3x^2)^2y^3+C_4^5(3x^2)y^4+C_5^5y^5\\ &= 243x^{10}+5(81x^8)y+10(27x^6)y^2+10(9x^4)y^3+5(3x^2)y^4+y^5\\ &= 243x^{10}+405x^8y+270x^6y^2+90x^4y^3+15x^2y^4+y^5\end{aligned}$$

(c)
$$\begin{aligned}\left(2x-\frac{3}{x}\right)^4 &= C_0^4(2x)^4+C_1^4(2x)^3\left(-\frac{3}{x}\right)+C_2^4(2x)^2\left(-\frac{3}{x}\right)^2+C_3^4(2x)\left(-\frac{3}{x}\right)^3+C_4^4\left(-\frac{3}{x}\right)^4\\ &= 16x^4+4(8x^3)\left(-\frac{3}{x}\right)+6(4x^2)\left(\frac{9}{x^2}\right)+4(2x)\left(-\frac{27}{x^3}\right)+\frac{81}{x^4}\\ &= 16x^4-96x^2+216-\frac{216}{x^2}+\frac{81}{x^4}\end{aligned}$$

Checkpoint 4.5

Expand $(x-2y)^5$.

Checkpoint 4.6

Expand

(a) $(3a+2b^2)^3$

(b) $\left(\frac{2}{x}-5x\right)^4$

Example 4.7

Find the coefficient of x^5 in the expansion of $(1+3x)^7$.

Solution

The general term is $C_7^n(1)^{7-r}(3x)^r = C_7^n 3^r x^r$

$\therefore$ The coefficient of $x^5 = C_5^7 3^5 = 5103$

Example 4.8

Find the constant term in the expansion of $\left(2x^2+\dfrac{1}{3x^2}\right)^6$.

Solution

The general term is $C_r^6(2x^2)^{6-r}\left(\dfrac{1}{3x^2}\right)^r = C_r^6(2)^{6-r}\left(\dfrac{1}{3}\right)^r x^{12-4r}$

In the constant term, the power of x is 0, i.e. $12-4r=0$ or $r=3$.

$$\therefore \quad \text{The constant term} = C_3^6(2)^{6-3}\left(\frac{1}{3}\right)^3$$

$$= 20\times 8\times \frac{1}{27}$$

$$= \frac{160}{27}$$

Checkpoint 4.7

Find the first three terms in the expansion of $(x-2y)^7$ in descending powers of x.

Checkpoint 4.8

In the expansion of $\left(\frac{2}{x}-\frac{1}{4x^3}\right)^8$ in ascending powers of x,

(a) find the coefficient of the first term;

(b) find the coefficient of the last term;

(c) find the ratio of the coefficient of the first term to that of the last term. (1 : 16777216)

Example 4.9

Find the coefficient of x^3 in the expansion of $\left(2-\frac{1}{3}x\right)^5(1-x)^6$.

Solution

$$\left(2-\frac{1}{3}x\right)^5 = C_0^5(2)^5 + C_1^5(2)^4\left(-\frac{1}{3}x\right) + C_2^5(2)^3\left(-\frac{1}{3}x\right)^2 + C_3^5(2)^2\left(-\frac{1}{3}x\right)^3 + \dots$$

$$= 32 - \frac{80}{3}x + \frac{80}{9}x^2 - \frac{40}{27}x^3 + \dots$$

$$(1-x)^6 = C_0^6(1)^6 + C_1^6(1)^5(-x) + C_2^6(1)^4(-x)^2 + C_3^6(1)^3(-x)^3 + \dots$$

$$= 1 - 6x + 15x^2 - 20x^3 + \dots$$

$$\left(2-\frac{1}{3}x\right)^5(1-x)^6 = \left(32 - \frac{80}{3}x + \frac{80}{9}x^2 - \frac{40}{27}x^3 + \dots\right)(1 - 6x + 15x^2 - 20x^3 + \dots)$$

$\therefore$ The coefficient of $x^3 = 32(-20) - \frac{80}{3}(15) + \frac{80}{9}(-6) - \frac{40}{27}(1)$

$$= -1094\frac{22}{27}$$

Checkpoint 4.9

Find the coefficient of x^3 in the expansion of $(x-1)^5(2x+3)^6$.

Example 4.10

In the expansion of $(1-ax)(1-bx)^6$, the coefficients of x and x^2 are 0 and $-\dfrac{21}{4}$ respectively.

Find the values of a and b.

Solution

$$\begin{aligned}(1-ax)(1-bx)^6 &= (1-ax)\left[1+C_1^6(-bx)+C_2^6(-bx)^2+\ldots\right]\\ &= (1-ax)(1-6bx+15b^2x^2+\ldots)\\ &= 1-6bx+15b^2x^2-ax+6abx^2+\ldots\\ &= 1-(6b+a)x+(15b^2+6ab)x^2+\ldots\end{aligned}$$

The coefficient of $x = 0$

$$6b+a=0$$

$$a=-6b\text{..........(1)}$$

The coefficient of $x^2 = -\dfrac{21}{4}$

$$15b^2+6ab=-\frac{21}{4}\text{.........(2)}$$

$$\therefore \quad 15b^2+6(-6b)b=-\frac{21}{4}$$

$$-21b^2=-\frac{21}{4}$$

$$b^2=\frac{1}{4}$$

$$b=\pm\frac{1}{2}$$

By (1), when $b=\dfrac{1}{2}$, $\quad a=-6\left(\dfrac{1}{2}\right)=-3$;

when $b=-\dfrac{1}{2}$, $\quad a=-6\left(-\dfrac{1}{2}\right)=3$.

$\therefore \quad a=-3,\ b=\dfrac{1}{2}$ or $a=3,\ b=-\dfrac{1}{2}$.

Checkpoint 4.10

In the expansion of $\left(x+\dfrac{a}{x^3}\right)^n$ in descending powers of x, where n is a positive integer, the 4th term is the constant term and is equal to $\dfrac{55}{2}$. Find the values of n and a.

4.4 Trinomial Expansion

Problems of trinomial expansion can be solved by repeated binomial expansions and is illustrated in the following examples.

Example 4.11

Expand $(1+x-x^2)^4$ in ascending powers of x as far as the term in x^3.

Solution

$$
\begin{aligned}
(1+x-x^2)^4 &= [1+(x-x^2)]^4 \\
&= [1+x(1-x)]^4 \\
&= 1+4x(1-x)+6[x(1-x)]^2+4[x(1-x)]^3+[x(1-x)]^4 \\
&= 1+4x-4x^2+6x^2(1-2x+\ldots)+4x^3(1+\ldots)+\ldots \\
&= 1+4x-4x^2+6x^2-12x^3+4x^3+\ldots \\
&= 1+4x+2x^2-8x^3+\ldots
\end{aligned}
$$

Example 4.12

Given $(1-x+px^2)^3 = 1+qx+9x^2+$ terms involving higher powers of x. Find p and q.

Solution

$$
\begin{aligned}
(1-x+px^2)^3 &= [1+x(px-1)]^3 \\
&= 1+3x(px-1)+3[x(px-1)]^2+\ldots \\
&= 1+3px^2-3x+3x^2(p^2x^2-2px+1)+\ldots \\
&= 1-3x+(3p+3)x^2+\ldots
\end{aligned}
$$

$\therefore \quad (1-x+px^2)^3 = 1+qx+9x^2+\ldots$

i.e. $\quad 1-3x+(3p+3)x^2+\ldots = 1+qx+9x^2+\ldots$

$\therefore \quad q=-3$

$3p+3=9$

$p=2$

Example 4.13

Given $(1+ax+x^2)^n = 1-8x+28x^2 +$ terms involving higher powers of x, where n is a positive integer. Find a and n.

Solution

$$
\begin{aligned}
(1+ax+x^2)^n &= [1+x(a+x)]^n \\
&= 1 + C_1^n x(a+x) + C_2^n [x(a+x)]^2 + \ldots \\
&= 1 + nx(a+x) + \frac{n(n-1)}{2} x^2 (a^2 + \ldots) + \ldots \\
&= 1 + nax + nx^2 + \frac{n(n-1)}{2} a^2 x^2 + \ldots \\
&= 1 + nax + \frac{2n + n(n-1)a^2}{2} x^2 + \ldots
\end{aligned}
$$

$\therefore$ $(1+ax+x^2)^n = 1-8x+28x^2+\ldots$

i.e. $1 + nax + \dfrac{2n+n(n-1)a^2}{2}x^2 + \ldots = 1-8x+28x^2+\ldots$

$$\therefore \quad \begin{cases} na = -8 \ldots\ldots(1) \\ \dfrac{2n+n(n-1)a^2}{2} = 28 \ldots\ldots(2) \end{cases}$$

From (1), $a = -\dfrac{8}{n} \ldots\ldots(3)$

Substituting (3) into (2), we have

$n^2 + 4n - 32 = 0$

$(n-4)(n+8) = 0$

$\therefore$ $n = 4 \quad \text{or} \quad -8$ (rejected)

$a = -\dfrac{8}{4} = -2$

$\therefore$ $n = 4$ and $a = -2$.

Checkpoint 4.11

Expand $(1+x+2x^2)^3$ in ascending powers of x as far as the term in x^3.

Checkpoint 4.12

If the coefficient of x^3 is 1180 in the expansion of $(1+2x+kx^2)^{10}$, find the value of k.

Exercise 4 Binomial Theorem

4.2

1. Evaluate the following in terms of *n*.
   (a) $\frac{n!}{(n-2)!}$ (b) $\frac{1}{(n+2)!}-\frac{1}{(n+1)!}$
   (c) C_{n-1}^{n} (d) C_{3}^{n-1}

2. Find *r* if $C_{3r}^{12}=C_{10-r}^{12}$.

3. Show that $nC_{r}^{n-1}=(r+1)C_{r+1}^{n}$.

4.3

4. Expand each of the following:
   (a) $(2+a)^5$
   (b) $\left(3x^3-\frac{1}{x}\right)^6$

5. Find the coefficients of the terms xy^6 and x^3y^4 in each of the following expansions.

6. Find the coefficient of $\frac{a^2}{b^2}$ in the expansion of $\left(\frac{a}{b}-\sqrt{\frac{b}{a}}\right)^8$.

7. Find, in terms of *n*, the coefficient of x^2 in the expansion of $(3-x)(1+2x)^n$ where *n* is a positive integer.

8. In the expansion of $(1+3x)^6+(1+4x)^5+(1-x)^{10}$, find the coefficients of x^8 and x^4.

9. In the expansion of $(3x+b)^{17}$, where *b* is a positive integer, in descending powers of *x*, the ratio of the coefficients of 8th term and the 9th term is 8 : 45. Find the value of *b*.

10. In the expansion of $(2+3x)^n$, where *n* is a positive integer, the coefficient of x^3 and x^4 are in the ratio 8 : 15. Find the value of *n*.

11. Given $(1+px)^n = 1+20x+180x^2+kx^3+$ other terms involving higher powers of *x*, where *n* is a positive integer. Find *n*, *p* and *k*.

12. In the expansion of $\left(a+\frac{x}{b}\right)^6$, the coefficients of *x* and x^2 are 48 and 15 respectively. Find the possible values of *a* and *b*.

13. (a) Given that $a=b-1$. Show that $a^{2n}+2nb-1$, where *n* is a positive integer, is divisible by b^2.
    (b) Using the result of (a), show that $3^{38}+151$ is divisible by 16.

4.4

14. Expand the following expressions in ascending powers of *x* as far as the term containing x^3.
    (a) $[1+x(3x-1)]^4$
    (b) $(1-2x+3x^2)^3$

15. Expand $(1+2x^2-4x^3)^5$ in ascending powers of *x* up to and including the term containing x^4.

16. Find the constant term in the expansion of $\left(1+x^3-\frac{1}{x^2}\right)^5$.

17. Find, in terms of *n*, the coefficient of x^2 in the expansion of $(1+2x+x^2)^n$ where *n* is a positive integer.

18. Find the value of the constant *a* if the coefficient of x^2 in the expansion of $(1-x+ax^2)^4$ is zero. Hence obtain the coefficient of x^3.

19. (a) Expand $(1+ax+bx^2)^4$ in ascending powers of *x*.
    (b) If the coefficients of *x* and x^2 are –8 and 12 respectively, find the values of *a* and *b*.