Chapter 3 Mathematical Induction

3.1 The Principle of Mathematical Induction

The principle of mathematical induction is stated as follows:

Suppose $P(n)$ is a statement in n for each positive integer n. If

(1) $P(1)$ is true, and

(2) for all positive integers $k \geq 1$, the assumption that $P(k)$ is true implies that $P(k + 1)$ is true,

then $P(n)$ is true for all positive integers n.

Note: $P(n)$ is also known as proposition. It is a statement but **not** a function in n.

Note: (1) and (2) are independent to each other, i.e. "$P(1)$ is true" does not imply the other and vice versa.

Example 3.1

Consider the statement $P(n)$

$$‘1+2+3+...+n=1+3+5+...+(2n-1)’,$$

where n is a positive integer.

(a) Verify $P(1)$ is true.

(b) Write down the statements $P(k)$ and $P(k + 1)$ for a positive integer k. Does "$P(k)$ is true" imply "$P(k + 1)$ is true"?

(c) Is $P(2)$ true? Can you conclude that the statement is true for all positive integers n?

Solution

(a) For $n = 1$,

L.H.S. of $P(1) = 1$

R.H.S. of $P(1) = 1$

L.H.S. = R.H.S.

$\therefore$ $P(1)$ is true.

(b) For a positive integer k,

$P(k)$ is the statement:

$$1+2+3+...+k=1+3+5+...+(2k-1)$$

$P(k + 1)$ is the statement:

$$1+2+3+...+k+(k+1)=1+3+5+...+(2k-1)+[2(k+1)-1]$$

Assume that $P(k)$ is true.

i.e. $1+2+3+...+k=1+3+5+...+(2k-1)$

For $n = k + 1$,

$$\text{L.H.S. of } P(k+1) = 1 + 2 + 3 + \ldots + k + (k+1)$$
$$= 1 + 3 + 5 + \ldots + (2k-1) + (k+1)$$
$$\text{R.H.S. of } P(k+1) = 1 + 3 + 5 + \ldots + (2k-1) + [2(k+1) - 1]$$
$$= 1 + 3 + 5 + \ldots + (2k-1) + (2k+1)$$

$\because$ $k + 1 < 2k + 1$

$\therefore$ L.H.S. $\neq$ R.H.S.

$\therefore$ "$P(k)$ is true" does not imply "$P(k + 1)$ is true".

(c) $P(2)$ is the statement '$1 + 2 = 1 + 3$' which is obviously false.

$\therefore$ It is wrong to conclude that the statement is true for all positive integers n.

Example 3.2

Given the statement $P(n)$

$$\text{'} 2 + 6 + 10 + \ldots + 2(2n-1) = 2n^2 + 2 \text{'},$$

where n is a positive integer.

(a) Write down the statements $P(k)$ and $P(k + 1)$ for a positive integer k. Prove that "$P(k)$ is true" imply "$P(k + 1)$ is true".

(b) Is $P(1)$ true?

(c) Can you conclude that the statement is true for all positive integers n?

Solution

(a) For a positive integer k,

$P(k)$ is the statement:

$$2 + 6 + 10 + \ldots + 2(2k-1) = 2k^2 + 2$$

$P(k + 1)$ is the statement:

$$2 + 6 + 10 + \ldots + 2(2k-1) + 2[2(k+1) - 1] = 2(k+1)^2 + 2$$

Assume that $P(k)$ is true.

i.e. $2 + 6 + 10 + \ldots + 2(2k-1) = 2k^2 + 2$

For $n = k + 1$,

$$\text{L.H.S. of } P(k+1) = 2 + 6 + 10 + \ldots + 2(2k-1) + 2[2(k+1) - 1]$$
$$= 2k^2 + 2 + 2[2(k+1) - 1]$$
$$= 2k^2 + 2 + 4k + 2$$
$$= 2(k^2 + 2k + 1) + 2$$
$$= 2(k+1)^2 + 2$$
$$= \text{R.H.S. of } P(k+1)$$

$\therefore$ "$P(k)$ is true" implies "$P(k + 1)$ is true".

(b) For $n = 1$,

L.H.S. of $P(1) = 2$

R.H.S. of $P(1) = 2(1)^2 + 2 = 4$

L.H.S. $\neq$ R.H.S.

$\therefore$ $P(1)$ is false.

(c) $\because$ $P(1)$ is false.

$\therefore$ It is wrong to conclude that the statement is true for all positive integers n.

Checkpoint 3.1

Let $P(n)$ be the statement

$$\text{‘}1\times 2+2\times 3+3\times 4+...+n\times(n+1)=n^2+1\text{’},$$

where n is a positive integer.

(a) Verify that $P(1)$ is true.

(b) If $P(k)$ is true, does it imply that $P(k + 1)$ is true?

(c) Can you conclude that the statement is true for all positive integers n?

Checkpoint 3.2

Let $P(n)$ be the statement

$$‘1+3+5+7+...+(2n-1)=n^2-1’,$$

where n is a positive integer.

(a) Prove that If $P(k)$ is true, then $P(k+1)$ is true.

(b) Is $P(1)$ true?

(c) Can you conclude that $P(n)$ is true for all positive integers n?

Example 3.3

Prove, by mathematical induction, that $1+2+3+...+n=\dfrac{n(n+1)}{2}$ for all positive integers n.

Solution

Let $P(n)$ be the statement '$1+2+3+...+n=\dfrac{n(n+1)}{2}$', where n is a positive integer.

[Step 1: Prove that P(1) is true.]

For $n = 1$,

$$\text{L.H.S. of } P(1)=1$$

$$\text{R.H.S. of } P(1)=\frac{1(1+1)}{2}=1$$

$$\text{L.H.S.}=\text{R.H.S.}$$

$\therefore$ $P(1)$ is true.

[Step 2: Assume that $P(k)$ is true, prove that $P(k + 1)$ is also true.

$P(k)$ states that $1+2+3+...+k=\dfrac{k(k+1)}{2}$

$P(k+1)$ states that $1+2+3+...+k+(k+1)=\dfrac{(k+1)[(k+1)+1]}{2}$.]

Assume that $P(k)$ is true for some positive integer k.

i.e. $1+2+3+...+k=\dfrac{k(k+1)}{2}$

For $n = k + 1$,

$$\begin{aligned}\text{L.H.S. of } P(k+1)&=1+2+3+...+k+(k+1)\\&=\frac{k(k+1)}{2}+(k+1)\\&=(k+1)\left(\frac{k+2}{2}\right)\\&=\frac{(k+1)[(k+1)+1]}{2}\\&=\text{R.H.S. of } P(k+1)\end{aligned}$$

$\therefore$ $P(k + 1)$ is true.

By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

Checkpoint 3.3

Prove, by mathematical induction, that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{2^n}=1-\frac{1}{2^n}$ is true for all positive integers n.

3.2 Application of Mathematical Induction

3.2.1 Application to Proofs of Summation Formulae for Series

A series is an expression for the sum of a list of numbers following a particular pattern. For example, 1 + 4 + 7 + 10 + 13 + 16 is a series of 6 terms; 1 + 2 + 3 + …+ n is a series of n terms.

Example 3.4

Prove, by mathematical induction, that

$$1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \frac{1}{3}n(4n^2 - 1)$$

for all positive integers n.

Solution

Let $P(n)$ be the statement '$1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \frac{1}{3}n(4n^2 - 1)$', where n is a positive integer.

For $n = 1$,

$$\text{L.H.S. of } P(1) = 1^2 = 1$$

$$\text{R.H.S. of } P(1) = \frac{1}{3}(1)[4(1)^2 - 1] = 1$$

$$\text{L.H.S.} = \text{R.H.S.}$$

$\therefore$

$\therefore$ $P(1)$ is true.

Assume that $P(k)$ is true for some positive integer k.

i.e. $1^2 + 3^2 + 5^2 + ... + (2k-1)^2 = \frac{1}{3}k(4k^2 - 1)$

For $n = k + 1$,

$$\begin{aligned}
\text{L.H.S.} &= 1^2 + 3^2 + 5^2 + ... + (2k-1)^2 + [2(k+1)-1]^2 \\
&= \frac{1}{3}k(4k^2 - 1) + (2k+1)^2 \\
&= \frac{1}{3}k(2k+1)(2k-1) + (2k+1)^2 \\
&= \frac{1}{3}(2k+1)[k(2k-1) + 3(2k+1)] \\
&= \frac{1}{3}(2k+1)(2k^2 + 5k + 3) \\
&= \frac{1}{3}(2k+1)(2k+3)(k+1)
\end{aligned}$$

$$\text{R.H.S.} = \frac{1}{3}(k+1)[4(k+1)^2 - 1]$$
$$= \frac{1}{3}(k+1)[2(k+1)+1][2(k+1)-1]$$
$$= \frac{1}{3}(k+1)(2k+3)(2k+1)$$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ $P(k+1)$ is true.

By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

Example 3.5

(a) Prove, by mathematical induction, that

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for all positive integers n.

(b) Hence find the value of $\frac{1}{50\times 51} + \frac{1}{51\times 52} + \frac{1}{52\times 53} + \ldots + \frac{1}{99\times 100}$.

Solution

(a) Let $P(n)$ be the statement '$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$', where n is a positive integer.

For $n = 1$,

$$\text{L.H.S.} = \frac{1}{1\times 2} = \frac{1}{2}$$
$$\text{R.H.S.} = \frac{1}{1+1} = \frac{1}{2}$$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ $P(1)$ is true.

Assume that $P(k)$ is true for some positive integer k.

i.e. $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \ldots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

For $n = k + 1$,

$$\text{L.H.S.} = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + ... + \frac{1}{k(k+1)} + \frac{1}{(k+1)[(k+1)+1]}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+(k+1)}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

$$\text{R.H.S.} = \frac{k+1}{(k+1)+1}$$

$$= \frac{k+1}{k+2}$$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ $P(k + 1)$ is true.

By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

(b) Putting $n = 99$ in (a), we have

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + ... + \frac{1}{99\times 100} = \frac{99}{100} \quad \ldots\ldots(1)$$

Putting $n = 49$ in (a), we have

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + ... + \frac{1}{49\times 50} = \frac{49}{50} \quad \ldots\ldots(2)$$

(1) – (2): $$\frac{1}{50\times 51} + \frac{1}{51\times 52} + \frac{1}{52\times 53} + ... + \frac{1}{99\times 100} = \frac{99}{100} - \frac{49}{50}$$

$$= \frac{1}{100}$$

Checkpoint 3.4

Prove, by mathematical induction, that $4 + 14 + 30 + ... + (3n^2 + n) = n(n+1)^2$ for all positive integers *n*.

Checkpoint 3.5

Prove, by mathematical induction, that $\frac{1^2}{1\cdot 3}+\frac{2^2}{3\cdot 5}+\frac{3^2}{5\cdot 7}+...+\frac{n^2}{(2n-1)(2n+1)}=\frac{n(n+1)}{2(2n+1)}$ for all positive integers n.

Sometimes a series is represented using the symbol 'Σ', meaning 'the sum of'. For example,

$$\sum_{r=1}^{4} 2r = 2(1) + 2(2) + 2(3) + 2(4)$$

$$\sum_{r=2}^{6} r^2 = 2^2 + 3^2 + 4^2 + 5^2 + 6^2$$

$$\sum_{r=1}^{n} r3^r = 1(3^1) + 2(3^2) + 3(3^3) + ... + n(3^n)$$

Note that we may use other letters instead of r. For example, $\sum_{k=1}^{4} 2k = \sum_{r=1}^{4} 2r$.

Example 3.6

Prove, by mathematical induction, that $\sum_{r=1}^{n} 2r(r-1) = \frac{n(n+1)(4n-1)}{3}$ for all positive integers n.

Solution

Let $P(n)$ be the statement '$\sum_{r=1}^{n} 2r(r-1) = \frac{n(n+1)(4n-1)}{3}$', where n is a positive integer.

When $n = 1$,

$$\begin{aligned}
\text{L.H.S.} &= \sum_{r=1}^{n} 2r(r-1) \\
&= 2(1)[2(1)-1] \\
&= 2 \\
\text{R.H.S.} &= \frac{1(1+1)[4(1)-1]}{3} \\
&= 2
\end{aligned}$$

$\therefore$ L.H.S. = R.H.S.

$\therefore$ $P(1)$ is true.

Assume that $P(k)$ is true for some positive integer k,

i.e. $\sum_{r=1}^{k} 2r(r-1) = \frac{k(k+1)(4k-1)}{3}$

When $n = k + 1$,

$$\text{L.H.S.} = \sum_{r=1}^{k+1} 2r(r-1)$$

$$= \sum_{r=1}^{k} 2r(r-1) + 2(k+1)[2(k+1)-1]$$

$$= \frac{k(k+1)(4k-1)}{3} + 2(k+1)(2k+1)$$

$$= (k+1)\left[\frac{k(4k-1)}{3} + 2(2k+1)\right]$$

$$= (k+1)\left[\frac{4k^2 - k + 12k + 6}{3}\right]$$

$$= \frac{(k+1)(4k^2 + 11k + 6)}{3}$$

$$= \frac{(k+1)(k+2)(4k+3)}{3}$$

$$= \frac{(k+1)[(k+1)+1][4(k+1)-1]}{3}$$

$$\text{R.H.S.} = \frac{(k+1)[(k+1)+1][4(k+1)-1]}{3}$$

L.H.S. = R.H.S.

∴ $P(k + 1)$ is true.

By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

Checkpoint 3.6

Prove, by mathematical induction, that $\sum_{r=1}^{n} (1+4r) = n(2n+3)$ for all positive integers n.

3.2.2 Application to Proofs of Divisibility

Useful fact:

For two integers a and b, a is divisible by b if $a = bm$ for some integer m.

Example 3.7

Prove, by mathematical induction, that $23^n - 1$ is divisible by 11 for all positive integers n.

Solution

Let $P(n)$ be the statement that '$23^n - 1$ is divisible by 11', where n is a positive integer.

When $n = 1$,

$$\begin{aligned} 23^n - 1 &= 23^1 - 1 \\ &= 22 \\ &= 11(2) \end{aligned}$$

$\therefore$ $P(1)$ is true.

Assume that $P(k)$ is true for some positive integer k,

i.e. $23^k - 1$ is divisible by 11, and $23^k - 1 = 11m$ for some integer m.

When $n = k + 1$,

$$\begin{aligned} 23^n - 1 &= 23^{k+1} - 1 \\ &= 23(23^k) - 1 \\ &= 23(11m + 1) - 1 \\ &= 23(11m) + 22 \\ &= 11(23m + 2) \end{aligned}$$

Since $11(23m + 2)$ is divisible by 11,

$\therefore$ $P(k + 1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all positive integers n.

Example 3.8

Prove, by mathematical induction, that $7^n - 6^n$ is divisible by 13 for all positive **even** integers n.

Solution

Let $P(n)$ be the statement that '$7^n - 6^n$ is divisible by 13', where n is a positive even integer.

When $n = 2$,

$$\begin{aligned} 7^n - 6^n &= 7^2 - 6^2 \\ &= 13 \end{aligned}$$

$\therefore$ $P(1)$ is true.

Assume that $P(k)$ is true for some positive integer k,

i.e. $7^k - 6^k$ is divisible by 13, and $7^k - 6^k = 13m$ for some integer m.

When $n = k + 2$,

$$\begin{aligned} 7^n - 6^n &= 7^{k+2} - 6^{k+2} \\ &= 7^2(7^k) - 6^{k+2} \\ &= 7^2(13m + 6^k) - 6^{k+2} \\ &= 7^2(13m) + 7^2 \cdot 6^k - 6^2 \cdot 6^k \\ &= 13(49m) + 6^k(49 - 36) \\ &= 13(49m + 6^k) \end{aligned}$$

Since $13(49m + 6^k)$ is divisible by 13,

$\therefore$ $P(k + 2)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all positive even integers n.

Checkpoint 3.7

Prove, by mathematical induction, that $8n^3 - 2n$ is divisible by 6 for all positive integers n.

Checkpoint 3.8

Prove, by mathematical induction, that $5^n - 4^n$ is divisible by 9 for all positive even integers n.

Example 3.9

Show, by mathematical induction, that $a^{2n+1}+b^{2n+1}$ is divisible by $a + b$ for all positive integers n.

Solution

Let $P(n)$ be the statement that '$a^{2n+1}+b^{2n+1}$ is divisible by $a + b$', where n is a positive integer.

When $n = 1$,

$$\begin{aligned} a^{2n+1}+b^{2n+1} &= a^3+b^3 \\ &= (a+b)(a^2-ab+b^2) \end{aligned}$$

$\therefore$ $P(1)$ is true.

Assume that $P(k)$ is true for some positive integer k,

i.e. $a^{2k+1}+b^{2k+1}$ is divisible by $a + b$, and $a^{2k+1}+b^{2k+1}=(a+b)Q$, where Q is a polynomial in a and b.

When $n = k + 1$,

$$\begin{aligned} a^{2n+1}+b^{2n+1} &= a^{2(k+1)+1}+b^{2(k+1)+1} \\ &= a^{2k+3}+b^{2k+3} \\ &= a^{2k+1}\cdot a^2+b^{2k+1}\cdot a^2-b^{2k+1}\cdot a^2+b^{2k+1}\cdot b^2 \\ &= (a^{2k+1}+b^{2k+1})a^2-b^{2k+1}(a^2-b^2) \\ &= (a+b)Q\cdot a^2-b^{2k+1}(a+b)(a-b) \\ &= (a+b)[Qa^2-b^{2k+1}(a-b)] \end{aligned}$$

Since $Qa^2-b^{2k+1}(a-b)$ is a polynomial in a and b, $(a+b)[Qa^2-b^{2k+1}(a-b)]$ is divisible by $a + b$.

$\therefore$ $P(k + 1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all positive integers n.

Checkpoint 3.9

Show, by mathematical induction, that $a^{2n-1} - b^{2n-1}$ is divisible by $a - b$ for all positive integers n.

Exercise 3 Mathematical Induction

3.2

1. Prove, by mathematical induction, that

$$1\times 5+3\times 7+5\times 9+\ldots+(2n-1)(2n+3)=\frac{n}{3}(4n^2+12n-1)$$

for all positive integers n.

2. Prove, by mathematical induction, that

$$1\times 3\times 5+2\times 4\times 6+3\times 5\times 7+\ldots+n(n+2)(n+4)=\frac{1}{4}n(n+1)(n+4)(n+5)$$

for all positive integers n.

3. Prove, by mathematical induction, that

$$1^2\times 2+2^2\times 3+3^2\times 4+\ldots+n^2(n+1)=\frac{n(n+1)(n+2)(3n+1)}{12}$$

for all positive integers n.

4. Prove, by mathematical induction, that

$$\frac{1}{1\cdot 3\cdot 5}+\frac{2}{3\cdot 5\cdot 7}+\frac{3}{5\cdot 7\cdot 9}+\ldots+\frac{n}{(2n-1)(2n+1)(2n+3)}=\frac{n(n+1)}{2(2n+1)(2n+3)}$$

for all positive integers n.

5. Prove, by mathematical induction, that

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$

for all integers $n\geq 2$.

6. Prove, by mathematical induction, that $\sum_{r=1}^{n} r2^{r-1}=1+(n-1)2^n$ for all positive integers n.

7. (a) Prove, by mathematical induction, that

$$\frac{1}{1\times 3}+\frac{1}{2\times 4}+\frac{1}{3\times 5}+\ldots+\frac{1}{n(n+2)}=\frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)}$$

for all positive integers n.

(b) Hence, find the value of $\frac{1}{8\times 10}+\frac{1}{9\times 11}+\frac{1}{10\times 12}+\ldots+\frac{1}{34\times 36}$.

8. (a) Prove, by mathematical induction, that

$$1^3 + 2^3 + 3^3 + ... + n^3 = \frac{1}{4}n^2(n+1)^2$$

for all positive integers n.

(b) Using the result of (a), find $2^3 + 4^3 + 6^3 + ... + (2n)^3$.

(c) Using the results of (a) and (b), find $1^3 + 3^3 + 5^3 + ... + (2n+1)^3$.

9. (a) Prove, by mathematical induction, that

$$1\times 7 + 2\times 8 + 3\times 9 + ... + n(n+6) = \frac{1}{6}n(n+1)(2n+19)$$

for all positive integers n.

(b) Using the result of (a), and the fact $1+2+3+...+n = \frac{n(n+1)}{2}$, show that

$$1\times 2 + 2\times 3 + 3\times 4 + ... + n(n+1) = \frac{1}{3}n(n+1)(n+2).$$

10. Prove, by mathematical induction, that $45^n - 1$ is divisible by 11 for all positive integers n.

11. Prove, by mathematical induction, that $n(n^2 - 1)$ is divisible by 3 for all positive integers n.

12. Prove, by mathematical induction, that $9^n - 2^n$ is divisible by 7 for all positive integers n.

13. Prove, by mathematical induction, that $5^{2n-1} + 6^{2n-1}$ is divisible by 11 for all positive integers n.

14. Prove, by mathematical induction, that $a^n + b^n$ is divisible by $a + b$ for all positive odd integers n.

15. Prove, by mathematical induction, that $x^{2n} - y^{2n}$ is divisible by $x^2 - y^2$, where x and y are integers, for all positive integers n.

16. (a) Prove, by mathematical induction, that $n(n + 1)$ is divisible by 2 for all positive integers n.

(b) Using the result of (a), prove, by mathematical induction, that $n^3 + 5n$ is divisible by 6.