# Chapter 2 Linear and Quadratic Inequalities

## 2.1 Linear Inequalities in One Variable

For $ax + b \geq 0$,

$$x \geq -\frac{b}{a} \quad \text{if } a > 0$$
$$x \leq -\frac{b}{a} \quad \text{if } a < 0$$

### Example 2.1

Solve the following inequalities and represent the solutions graphically.

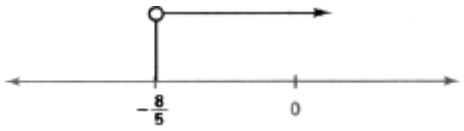
(a) $3(3x - 2) > 2(2x - 7)$

(b) $\frac{1}{2} + 5(x + 2) \leq \frac{3}{2}(2x - 1)$

### *Solution*

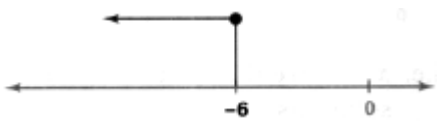
(a)
$$\begin{aligned} 3(3x - 2) &> 2(2x - 7) \\ 9x - 6 &> 4x - 14 \\ 9x - 6 - 4x &> 4x - 14 - 4x \\ 5x - 6 + 6 &> -14 + 6 \\ 5x &> -8 \\ \therefore \quad x &> -\frac{8}{5} \end{aligned}$$

The graphical representation of the solutions is




(b)
$$\begin{aligned} \frac{1}{2} + 5(x + 2) &\leq \frac{3}{2}(2x - 1) \\ 1 + 10(x + 2) &\leq 3(2x - 1) \\ 10x + 21 &\leq 6x - 3 \\ 10x + 21 - 6x - 21 &\leq 6x - 3 - 6x - 21 \\ 4x &\leq -24 \\ \therefore \quad x &\leq -6 \end{aligned}$$

The graphical representation of the solutions is



**Example 2.2**

Solve the following inequalities and represent the solutions graphically if possible:

(a) $3(5x-1) < 2(2x-7)$ and $\frac{3}{4}(4x-1) < \frac{1}{2}(3x+2)$

(b) $6 - \frac{5x}{4} \leq \frac{1-3x}{2} \leq \frac{4x-7}{3}$

***Solution***

(a)

$$3(5x-1) < 2(2x-7) \quad \text{and} \quad \frac{3}{4}(4x-1) < \frac{1}{2}(3x+2)$$

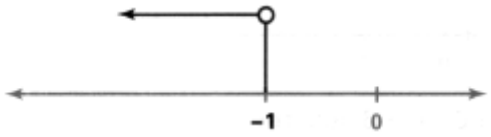
$$15x - 3 < 4x - 14 \quad \text{and} \quad 12x - 3 < 6x + 4$$

$$11x < -11 \quad \text{and} \quad 6x < 7$$

$$x < -1 \quad \text{and} \quad x < \frac{7}{6}$$

$\therefore \quad x < -1$

The graphical representation of the solutions is




(b) $6 - \frac{5x}{4} \leq \frac{1-3x}{2} \leq \frac{4x-7}{3}$

i.e. $6 - \frac{5x}{4} \leq \frac{1-3x}{2}$ and $\frac{1-3x}{2} \leq \frac{4x-7}{3}$

$24 - 5x \leq 2 - 6x$ and $3 - 9x \leq -8x + 14$

$x \leq -22$ and $x \geq -11$

$\therefore$ There is no solution for $x$.

**Example 2.3**

Solve the following inequalities and represent the solutions graphically if possible:

(a) $\dfrac{3x-1}{4} > \dfrac{2x+3}{3}$ or $-\dfrac{2x+1}{9} > -\dfrac{5}{3}$

(b) $3-\dfrac{5}{4}(2x-5) \le 4\left(1-\dfrac{3}{2}x\right)$ or $6(10-4x) > 5-12(3x-2)$

***Solution***

(a)
$$\frac{3x-1}{4} > \frac{2x+3}{3} \quad \text{or} \quad -\frac{2x+1}{9} > -\frac{5}{3}$$
$$9x-3 > 9x+12 \quad \text{or} \quad -2x-1 > -15$$
$$x > 15 \quad \text{or} \quad x < 7$$

$\therefore$ $x > 15$ or $x < 7$.

The graphical representation of the solutions is


(b)
$$3-\frac{5}{4}(2x-5) \le 4\left(1-\frac{3}{2}x\right) \quad \text{or} \quad 6(10-4x) > 5-12(3x-2)$$
$$12-5(2x-5) \le 16\left(1-\frac{3}{2}x\right) \quad \text{or} \quad 60-24x > 5-36x+24$$
$$12-10x+25 \le 16-24x \quad \text{or} \quad 12x > -31$$
$$14x \le -21 \quad \text{or} \quad x > -\frac{31}{12}$$
$$x \le -\frac{3}{2} \quad \text{or} \quad x > -2\frac{7}{12}$$

$\therefore$ All real numbers are solutions.

**Checkpoint 2.1**

Solve $\dfrac{4x}{5} \le 3+2x$ and represent the solutions graphically.

**Checkpoint 2.2**

Solve the following compound inequalities and represent the solutions graphically.

(a) $4x - 1 \leq 3 - 2x$ and $8x + 2 \geq 5 - 4x$

(b) $\begin{cases} 10x - 3 < 6(x + 1) \\ 5x - 1 \geq 3x + 4 \end{cases}$

**Checkpoint 2.3**

Solve the following compound inequalities and represent the solutions graphically.

(a) $2(x+1) \geq x+7$ or $3(1+2x) < 6(1-x)$

(b) $2(x-2) < 3(x-3)$ or $3(x-2) > 2(x-3)$

## 2.2 Quadratic Inequalities in One Variable

### 2.2.1 Standard Forms

(1) $ax^2 + bx + c > 0$

(2) $ax^2 + bx + c < 0$

(3) $ax^2 + bx + c \geq 0$

(4) $ax^2 + bx + c \leq 0$

### 2.2.2 Methods of Solutions

*A. Algebraic Method*

(5) If $ab > 0$, then $\begin{cases} a > 0 \\ b > 0 \end{cases}$ or $\begin{cases} a < 0 \\ b < 0 \end{cases}$.

If $(x-\alpha)(x-\beta) > 0$, where $\alpha > \beta$, then $x < \beta$ and $x > \alpha$.

(6) If $ab < 0$, then $\begin{cases} a > 0 \\ b < 0 \end{cases}$ or $\begin{cases} a < 0 \\ b > 0 \end{cases}$.

If $(x-\alpha)(x-\beta) < 0$, where $\alpha < \beta$, then $\beta < x < \alpha$.

**Example 2.4**

Solve the following inequalities:

(a) $x^2 - 4x - 5 > 0$

(b) $(x+3)(2x-5) \leq 0$

***Solution***

(a) $x^2 - 4x - 5 > 0$

$(x-5)(x+1) > 0$

$\therefore \quad x > 5 \quad \text{or} \quad x < 1$

(b) $(x+3)(2x-5) \leq 0$

$2(x+3)\left(x - \frac{5}{2}\right) \leq 0$

$(x+3)\left(x - \frac{5}{2}\right) \leq 0$

$\therefore \quad -3 \leq x \leq \frac{5}{2}$

**Checkpoint 2.4**

Solve the following inequalities.

(a) $x^2 - 4x - 12 < 0$

(b) $2x^2 + 3x - 2 \geq 0$

**Example 2.5**

Solve the following inequalities.

(a) $(2x-3)(x+5) > 3(x+5)$

(b) $(2x+3)(x+2) > 5$

***Solution***

(a)
$$(2x-3)(x+5) - 3(x+5) > 0$$
$$(x+5)(2x-3-3) > 0$$
$$(x+5)(2x-6) > 0$$
$$2(x+5)(x-3) > 0$$
$$\therefore \quad x < -5 \quad \text{or} \quad x > 3$$

(b)
$$(2x+3)(x+2) > 5$$
$$2x^2 + 7x + 6 > 5$$
$$2x^2 + 7x + 1 > 0$$
$$\left(x + \frac{-7+\sqrt{7^2-4(2)(1)}}{2(2)}\right)\left(x - \frac{-7-\sqrt{7^2-4(2)(1)}}{2(2)}\right) > 0$$
$$\left(x + \frac{-7+\sqrt{41}}{4}\right)\left(x - \frac{-7-\sqrt{41}}{4}\right) > 0$$
$$\therefore \quad x < \frac{-7-\sqrt{41}}{4} \quad \text{or} \quad x > \frac{-7+\sqrt{41}}{4}$$

*B. Tabulation*

A table is constructed to determine the signs of $(x-\alpha)(x-\beta)$ for $x < \alpha$, $\alpha < x < \beta$ and $x > \beta$ respectively.

**Example 2.6**

Solve the inequality $6+x-x^2<0$.

***Solution***

$$6+x-x^2<0$$
$$x^2-x-6>0$$
$$(x+2)(x-3)>0$$

| | $x < -2$ | $-2 < x < 3$ | $x > 3$ |
|---|---|---|---|
| $x + 2$ | – | + | + |
| $x - 3$ | – | – | + |
| $(x + 2)(x - 3)$ | + | – | + |

[From the table, $(x + 2)(x – 3)$ is positive only when $x < –2$ or $x > 3$.]

$\therefore$ The solutions are $x < –2$ or $x > 3$.

**Checkpoint 2.5**

Solve the inequality $2x^2+3x-5\le 0$ by tabulation method.

| | | | | | |
|---|---|---|---|---|---|
| | | | | | |
| | | | | | |
| | | | | | |

*C. Graphical Method*

A graph is sketched for the given quadratic function and the signs of the function in different regions along the $x$-axis is determined.

**Example 2.7**

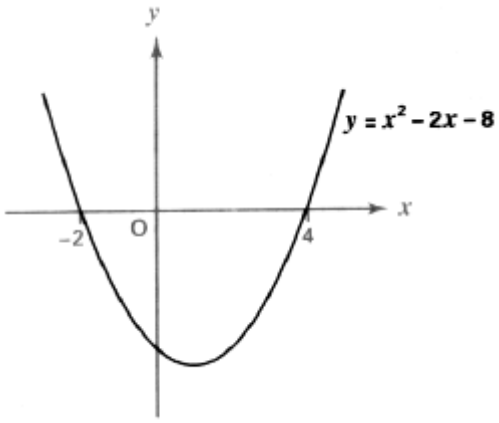
Solve the inequality $x^2 - 2x - 8 < 0$ graphically.

***Solution***

Let $y = x^2 - 2x - 8$

$= (x + 2)(x - 4)$

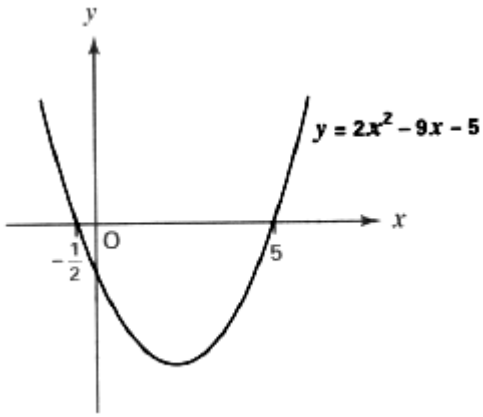
The $x$-intercepts of $y = x^2 - 2x - 8$ are –2 and 4, and its $y$-intercept is –8; it opens upwards as the coefficient of $x^2 = 1 > 0$.




For $x^2 - 2x - 8 < 0$, (i.e. $y$ <0), we have $-2 < x < 4$.

**Checkpoint 2.6**

The graph of $y = 2x^2 - 9x - 5$ is shown below. By using the graph, solve the inequality $2x^2 - 9x - 5 > 0$.



### 2.2.3 Special Cases

(1) If the roots of the corresponding quadratic equation $ax^2 + bx + c = 0$ are both $\alpha$ (i.e. repeated root), then , for the inequality, either

(a) all real numbers are solutions, or

(b) all real numbers except $\alpha$ are solutions, or

(c) there is no solution, or

(d) the solution is $x = \alpha$.

(2) If the corresponding quadratic equation $ax^2 + bx + c = 0$ has no real roots, then for the inequality, either

(a) all real numbers are solutions, or

(b) there is no real solution.

**Example 2.8**

Solve the following inequalities.

(a) $2x^2 - 8x + 8 \geq 0$

(b) $18x - 3x^2 < 27$

***Solution***

(a)
$$\begin{aligned} 2x^2 - 8x + 8 &\geq 0 \\ 2(x-2)^2 &\geq 0 \\ (x-2)^2 &\geq 0 \end{aligned}$$

$\therefore$ All real numbers are solutions.

(b)
$$\begin{aligned} 18x - 3x^2 &< 27 \\ -3x^2 + 18x - 27 &< 0 \\ -3(x-3)^2 &< 0 \\ (x-3)^2 &> 0 \end{aligned}$$

$\therefore$ The solutions are all real numbers except 3.

**Checkpoint 2.7**

Solve the following inequalities:

(a) $x^2 + 4x + 4 \leq 0$

(b) $4x^2 - 20x + 25 < 0$

(c) $x^2 - 2x + 2 > 0$

## 2.3 Applications of Quadratic Inequalities

Many problems concerning quadratic equations and quadratic functions require the solutions of inequalities.

**Example 2.9**

Find the range of values of $p$ for which the quadratic equation $2x^2-(3p-2)x+3p+4=0$ has

(a) two distinct real roots;

(b) two distinct positive roots.

***Solution***

(a) For the equation to have two distinct real roots, $\Delta > 0$.

i.e. $[-(3p-2)]^2-4(2)(3p+4)>0$

$$9p^2-12p+4-24p-32>0$$

$$9p^2-36p-28>0$$

$$(3p-14)(3p+2)>0$$

$\therefore$ $p<-\frac{2}{3}$ or $p>\frac{14}{3}$

(b) For the equation to have two distinct positive roots,

(i) $\Delta > 0$;

(ii) sum of roots > 0;

(iii) product of roots > 0.

For (i): $p<-\frac{2}{3}$ or $p>\frac{14}{3}$

For (ii): $\frac{3p-2}{2}>0$

$\therefore$ $p>\frac{2}{3}$

For (iii): $\frac{3p+4}{2}>0$

$\therefore$ $p>-\frac{4}{3}$

$\therefore$ Combining (i), (ii) and (iii), the required solutions are $p>\frac{14}{3}$.

**Example 2.10**

Find the range of values of $k$ so that the quadratic expression $x^2 - kx + x > -3$ is always greater than –3 for all real values of $x$.

***Solution***

$$x^2 - kx + k > -3 \quad \text{for all } x$$

i.e. $x^2 - kx + (k+3) > 0 \quad \text{for all } x$

Since the coefficient of $x^2 = 1 > 0$, the discriminant of $x^2 - kx + (k+3)$ must be negative.

i.e. $(-k)^2 - 4(1)(k+3) < 0$

$$k^2 - 4k - 12 < 0$$

$$(k-6)(k+2) < 0$$

$$-2 < k < 6$$

**Example 2.11**

Given that $y = \dfrac{2x^2+1}{4x+1}$.

(a) Show that $2x^2 - 4xy + (1-y) = 0$.

(b) Find the range of values of $y$ when $x$ is real.

***Solution***

(a)
$$y = \frac{2x^2+1}{4x+1}$$

$$4xy + y = 2x^2 + 1$$

$\therefore \quad 2x^2 - 4xy + (1-y) = 0$

(b) When $x$ is real, the discriminant of the equation must be greater than or equal to 0.

i.e. $(-4y)^2 - 4(2)(1-y) \geq 0$

$$16y^2 - 8 + 8y \geq 0$$

$$2y^2 + y - 1 \geq 0$$

$$(2y-1)(y+1) \geq 0$$

$$y \leq -1 \quad \text{or} \quad y \geq \frac{1}{2}$$

**Checkpoint 2.8**

Given the quadratic equation $2kx^2 + 3kx + 9 = 0$. Find the range of values of $k$ if $x$ is real.

**Checkpoint 2.9**

Find the range of values of $k$ such that $kx^2 + 4x - k > 4$ for all real values of $x$.

# Exercise 2 Linear and Quadratic Inequalities

**2.1**

1. Solve the following inequalities and represent the solutions graphically if possible.

   (a) $\dfrac{4x-3}{2}-\dfrac{3x-4}{5}<\dfrac{7x-9}{10}$ and $\dfrac{4-2x}{2}<\dfrac{7+3x}{3}$

   (b) $\begin{cases}\dfrac{1}{4}(x+3)-\dfrac{7}{10}x<\dfrac{3}{5}(9-2x)-3\\ 2(x+2)\le 7-x\end{cases}$

   (c) $\dfrac{2x-11}{4}+\dfrac{19-2x}{2}<2x$ or $\dfrac{2x+15}{9}\ge\dfrac{1}{5}(x-1)+\dfrac{x}{3}$

   (d) $\dfrac{13x-3}{4}+3<\dfrac{5x}{2}\le\dfrac{1}{5}(3x-2)+\dfrac{7}{10}$

2. Find all the integers that satisfy $\dfrac{x-1}{6}+\dfrac{10+2x}{5}<\dfrac{2x+7}{3}\le\dfrac{3x-1}{3}-\dfrac{x-5}{2}$.

3. Solve the compound inequality

   $$\frac{5x}{3}-8\le\frac{1}{7}(3x-4) \text{ or } \frac{3x}{2}-\frac{2x-1}{3}>3x-4 \text{ or } \frac{x+18}{4}<3-\frac{x}{4}.$$

**2.2**

4. Solve the following inequalities by algebraic method:

   (a) $x^2+6x-7<0$

   (b) $2x^2-5x-3<0$

   (c) $4x^2\ge 9$

5. Solve the following inequalities by tabulation method:

   (a) $-x^2+x+20>0$

   (b) $8x-5x^2>0$

   (c) $7-3x^2<-x^2+x-8$

6. Solve the following inequalities by graphical method:

   (a) $3x^2-8x+5>0$

   (b) $4x(x-2)<3(x+1)$

7. Use any method to solve the following inequalities:

(a) $(x-4)(x-2)<-1$

(b) $2x(2-x)<x^2+(x-2)^2$

(c) $(1-4x)(2x-5)<4+(x-4)^2$

(d) $\dfrac{(5x+4)(x+3)}{6}>x+1$

(e) $\dfrac{(x+2)(2x+5)}{2}-\dfrac{(3x+7)(x+1)}{3}>\dfrac{7x+16}{6}$

(f) $\dfrac{(x-3)(2x-3)}{4}\le\dfrac{(x-4)(x+5)}{3}+\dfrac{3-x}{4}$

8. Solve the following inequalities:

(a) $\begin{cases}15-x-6x^2\le 0\\ 6x^2-25x-9>0\end{cases}$

(b) $(6-x)(3-2x)-2\ge x(x-5)$

(c) $x^2+4x-(2x-5)(x+4)\ge 0$ or $(x+3)(x-3)<8x$

9. Find the greatest and smallest values that satisfy $2x^2-x-3\le 0$.

**2.3**

10. Find the range of values of $k$ for which the quadratic expression $-5x^2+(k-3)x+1$ is always less than 6 for all real values of $x$.

11. Find the range of values of $k$ for which the quadratic expression $kx^2+(3k+2)x+2$ is not less than $\dfrac{7}{4}$ for all real values of $x$.

12. Find the range of values of $\lambda$ such that $x^2+(3\lambda-2)x+2\lambda^2+\lambda-6=0$ has

(a) real and distinct roots;

(b) one positive root and one negative root.

13. Let $y=\dfrac{3x-2}{4x^2+1}$ ……(*).

(a) Express (*) in the form $ax^2+bx+c=0$.

(b) If the solution of (*) is real, find

(i) the range of values of $y$;

(ii) the maximum value of $|y|$.