Chapter 1 Quadratic Equations and Quadratic Functions

1.1 Solutions of Quadratic Equations

1.1.1 The method of Completing the Square

A quadratic equation can be solved by first transforming it into the form $(x + p)^2 = q$ using the method of completing the square.

Example 1.1

Solve the equation $x^2 + 6x + 5 = 0$.

Solution

$$
\begin{aligned}
x^2 + 6x + 5 &= 0 \\
x^2 + 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 5 &= 0 \\
(x+3)^2 &= 3^2 - 5 \\
(x+3)^2 &= 4 \\
x + 3 &= \pm\sqrt{4} \\
x + 3 &= 2 \quad \text{or} \quad -2 \\
x &= -1 \quad \text{or} \quad -5
\end{aligned}
$$

Example 1.2

Solve the equation $2x^2 - 5x - 9 = 0$.

Solution

$$
\begin{aligned}
2x^2 - 5x - 9 &= 0 \\
x^2 - \left(\frac{5}{2}\right)x - \left(\frac{9}{2}\right) &= 0 \\
x^2 - \left(\frac{5}{2}\right)x + \left(\frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2 - \left(\frac{9}{2}\right) &= 0 \\
\left(x - \frac{5}{4}\right)^2 &= \frac{97}{16} \\
x - \frac{5}{4} &= \pm\sqrt{\frac{97}{16}} \\
x &= \frac{5 \pm \sqrt{97}}{4}
\end{aligned}
$$

Checkpoint 1.1

Use the method of completing the square, solve the quadratic equation $3x^2 - 2x - 3 = 0$.

1.1.2 Quadratic Formula

The two roots of the quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$) are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example 1.3

Solve the equation $3x^2 - 6x - 2 = 0$.

Solution

$$3x^2 - 6x - 2 = 0$$

$$\begin{aligned} x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(-2)}}{2(3)} \\ &= \frac{6 \pm \sqrt{60}}{6} \\ &= \frac{6 \pm 2\sqrt{15}}{6} \\ &= \frac{3 \pm \sqrt{15}}{3} \end{aligned}$$

Example 1.4

Solve the equation $x^2 = x - 4$.

Solution

$$
\begin{aligned}
x^2 &= x - 4 \\
x^2 - x + 4 &= 0 \\
x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(4)}}{2(1)} \\
&= \frac{1 \pm \sqrt{-15}}{2} \qquad \text{(rejected)}
\end{aligned}
$$

$\therefore$ The equation $x^2 = x - 4$ has no real roots.

Checkpoint 1.2

Solve the equation $3x^2 - 8x + 1 = 0$.

Checkpoint 1.3

Solve the equation $mx^2 - 2x + m = 0$, where $m > 1$.

Example 1.5

Solve the equation $(x^2 - 4x)^2 - 5(x^2 - 4x) + 4 = 0$.

Solution

Let $y = x^2 - 4x$. Then the equation becomes

$$y^2 - 5y + 4 = 0$$
$$(y-1)(y-4) = 0$$
$$y = 1 \quad \text{or} \quad 4$$

Consider $y = 1$.

$$x^2 - 4x = 1$$
$$x^2 - 4x - 1 = 0$$
$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-1)}}{2(1)}$$
$$= \frac{4 \pm \sqrt{20}}{2}$$
$$= 2 \pm \sqrt{5}$$

Consider $y = 4$.

$$x^2 - 4x = 4$$
$$x^2 - 4x - 4 = 0$$
$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-4)}}{2(1)}$$
$$= \frac{4 \pm \sqrt{32}}{2}$$
$$= 2 \pm 2\sqrt{2}$$

$\therefore$ The solutions are $x = 2 \pm \sqrt{5}$ or $2 \pm 2\sqrt{2}$.

Example 1.6

Solve the simultaneous equations

$$\begin{cases} x - y = 3 \\ y = -x^2 + 2x \end{cases}.$$

Solution

$$\begin{cases} x - y = 3 & \quad \ldots\ldots\ldots(1) \\ y = -x^2 + 2x & \quad \ldots\ldots\ldots(2) \end{cases}$$

(1) + (2): $\quad x = 3 - x^2 + 2x$

i.e. $\quad x^2 - x - 3 = 0$

$$\therefore \quad x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{13}}{2}$$

When $x = \dfrac{1+\sqrt{13}}{2}$, $y = x - 3 = \dfrac{1+\sqrt{13}}{2} - 3 = \dfrac{-5+\sqrt{13}}{2}$.

When $x = \dfrac{1-\sqrt{13}}{2}$, $y = x - 3 = \dfrac{1-\sqrt{13}}{2} - 3 = \dfrac{-5-\sqrt{13}}{2}$.

$\therefore$ The solutions are $\left(\dfrac{1+\sqrt{13}}{2}, \dfrac{-5+\sqrt{13}}{2}\right)$ or $\left(\dfrac{1-\sqrt{13}}{2}, \dfrac{-5-\sqrt{13}}{2}\right)$.

Checkpoint 1.4

Solve the equation $(x^2 - 2x)^2 + (x^2 - 2x) - 2 = 0$.

Checkpoint 1.5

Solve the simultaneous equations $\begin{cases} x + y = 5 \\ y = x^2 + x + 2 \end{cases}$.

1.2 Nature of Roots of the Quadratic Equation

For the quadratic equation $ax^2 + bx + c = 0$, where a, b, c are real. Let $\Delta = b^2 - 4ac$.

Case 1:

If $\Delta = b^2 - 4ac > 0$, then the equation has two distinct (unequal) real roots.

If Δ is a perfect square, then the roots are rational; if Δ is NOT a perfect square, then the roots are irrational.

Case 2:

If $\Delta = b^2 - 4ac = 0$, then the equation has two equal real roots.

The roots are rational.

Case 3:

If $\Delta = b^2 - 4ac < 0$, then the equation has no real roots.

Example 1.7

Without solving each of the given quadratic equations, determine whether the equation has real roots. If the equation has real roots, state whether the roots are equal and rational.

(a) $4x^2 + x - 3 = 0$

(b) $3x^2 - 5x + 1 = 0$

(c) $9x^2 - 30x + 25 = 0$

(d) $x^2 - ax + a^2 = 0$, where a is a non-zero real number.

Solution

(a) $4x^2 + x - 3 = 0$

$$\begin{aligned}\Delta &= 1^2 - 4(4)(-3)\\ &= 49\\ &> 0\end{aligned}$$

Also, 49 is a perfect square.

$\therefore$ The equation has two distinct roots which are rational.

(b) $3x^2 - 5x + 1 = 0$

$$\begin{aligned}\Delta &= (-5)^2 - 4(3)(1)\\ &= 13\\ &> 0\end{aligned}$$

Also, 13 is not a perfect square.

$\therefore$ The equation has two distinct roots which are irrational.

(c) $x^2 - ax + a^2 = 0$

$\Delta = (-30)^2 - 4(9)(25)$

$= 0$

$\therefore$ The equation has two equal roots which are rational.

(d) $4x^2 + x - 3 = 0$

$\Delta = (-a)^2 - 4(1)(a^2)$

$= -3a^2$

< 0

$\therefore$ The equation has no real roots.

Checkpoint 1.6

Determine if each of the following equations has real roots. State also whether the roots are equal and rational.

(a) $10x^2 + 14x + 5 = 0$

(b) $4x^2 - 12x + 9 = 0$

Example 1.8

Determine the possible value(s) of k so that the equation $4x^2 + 6(k+3)x - 9k = 0$ has equal roots.

Solution

For the equation to have equal roots, $\Delta = 0$.

$\therefore \quad [6(k+3)]^2 - 4(4)(-9k) = 0$

$k^2 + 10k + 9 = 0$

$(k+9)(k+1) = 0$

$\therefore \quad k = -9 \quad \text{or} \quad -1$

Example 1.9

Find the range of values of k so that the equation $3x^2 - 6x - (5-k) = 0$ has real roots.

Solution

For the equation to have real roots, $\Delta \geq 0$.

$\therefore \quad (-6)^2 - 4(3)[-(5-k)] \geq 0$

$36 - 12k + 60 \geq 0$

$12k \leq 96$

$\therefore \quad k \leq 8$

Example 1.10

Determine the range of values of k so that the simultaneous equations $\begin{cases} y = x + k \\ y = 2x^2 - x - 3 \end{cases}$ will have real solutions.

Solution

Substituting $y = x + k$ and $y = 2x^2 - x - 3$, we have

$$x + k = 2x^2 - x - 3$$

$$2x^2 - 2x - (k+3) = 0 \,..........\, (1)$$

[If (1) has real solutions, then the simultaneous equations will have real solutions.]

$\therefore \quad \Delta \geq 0$

i.e. $\quad (-2)^2 - 4(2)[-(k+3)] \geq 0$

$4 + 8k + 24 \geq 0$

$\therefore \quad k \geq -\dfrac{7}{2}$

Checkpoint 1.7

Find the possible values of *k* so that the equation $x^2 + 2kx + 4 = 0$ has equal roots.

Checkpoint 1.8

Find the range of values of *k* so that the equation $x^2 - 3x + 2k = 0$

(a) has distinct real roots;

(b) has no real roots.

Example 1.11

Prove that the quadratic equation $(a-1)x^2 + ax + 1 = 0$ has real roots for all values of a.

Solution

$$\begin{aligned}\Delta &= a^2 - 4(a-1)(1) \\ &= a^2 - 4a + 4 \\ &= (a-2)^2 \\ &\geq 0\end{aligned}$$

$\therefore$ The equation has real roots for all values of a.

Checkpoint 1.9

Prove that the quadratic equation

$$2(a^2 + b^2)x^2 + 6(a-b)x + 9 = 0\text{, where } a + b \neq 0,$$

has no real roots for x.

1.3 Relations Between Roots and Coefficients

(1) If α and β are the roots of $ax^2 + bx + c = 0$, then

sum of roots = $\alpha + \beta = -\dfrac{b}{a}$ and product of roots = $\alpha\beta = \dfrac{c}{a}$.

(2) Any quadratic equation in x can be written in the form

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0.$$

Example 1.12

If α and β are the roots of the quadratic equation $4x^2 + 8x - 1 = 0$, find, without solving the equation, the values of

(a) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ (b) $\alpha^2 + \beta^2$

(c) $\alpha^3 + \beta^3$ (d) $\alpha - \beta$, where $\alpha > \beta$.

Solution

We have $\alpha + \beta = -\dfrac{8}{4} = -2$ and $\alpha\beta = \dfrac{-1}{4} = -\dfrac{1}{4}$.

(a)
$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{-2}{-\frac{1}{4}} = 8$$

(b)
$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (-2)^2 - 2\left(-\frac{1}{4}\right) = \frac{9}{2}$$

(c)
$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha^2\beta - 3\alpha\beta^2 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = (-2)^3 - 3\left(-\frac{1}{4}\right)(-2) = -\frac{19}{2}$$

(d) $$\begin{aligned}(\alpha-\beta)^2 &= \alpha^2+\beta^2-2\alpha\beta \\ &= \alpha^2+2\alpha\beta+\beta^2-4\alpha\beta \\ &= (\alpha+\beta)^2-4\alpha\beta \\ &= (-2)^2-4\left(-\frac{1}{4}\right) \\ &= 5\end{aligned}$$

$\because \quad \alpha > \beta.$

$\therefore \quad \alpha-\beta=\sqrt{5}$

Checkpoint 1.10

If α and β are the roots of the quadratic equation $x^2-4x+2=0$, find, the values of

(a) $\dfrac{\beta}{\alpha}+\dfrac{\alpha}{\beta}$ (b) $(\alpha-\beta)^2$

Example 1.13

If α and β are the roots of the quadratic equation $2x^2 - 5x - 1 = 0$, in each of the following, form a quadratic equation in x whose roots are

(a) $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$ (b) $\alpha + \frac{1}{\beta}, \beta + \frac{1}{\alpha}$

Solution

We have $\alpha + \beta = \frac{5}{2}$ and $\alpha\beta = -\frac{1}{2}$.

(a) Sum of new roots $= \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2 + \alpha^2}{(\alpha\beta)^2}$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

$$= \frac{\left(\frac{5}{2}\right)^2 - 2\left(-\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)^2}$$

$$= 29$$

Product of new roots $= \frac{1}{\alpha^2} \cdot \frac{1}{\beta^2}$

$$= \frac{1}{(\alpha\beta)^2}$$

$$= \frac{1}{\left(-\frac{1}{2}\right)^2}$$

$$= 4$$

$\therefore$ The required equation is $x^2 - 29x + 4 = 0$.

(b) Sum of new roots $= \left(\alpha + \frac{1}{\beta}\right) + \left(\beta + \frac{1}{\alpha}\right)$

$$= \alpha + \beta + \frac{\alpha+\beta}{\alpha\beta}$$

$$= \frac{5}{2} + \frac{\frac{5}{2}}{-\frac{1}{2}}$$

$$= -\frac{5}{2}$$

$$\text{Product of new roots} = \left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right)$$

$$= \alpha\beta + 1 + 1 + \frac{1}{\alpha\beta}$$

$$= -\frac{1}{2} + 2 + \frac{1}{-\frac{1}{2}}$$

$$= -\frac{1}{2}$$

$\therefore$ The required equation is $x^2 + \frac{5}{2}x - \frac{1}{2} = 0$, i.e. $2x^2 + 5x - 1 = 0$.

Checkpoint 1.11

If α and β are the roots of the quadratic equation $3x^2 - x - 1 = 0$, form a quadratic equation in x whose roots are $\frac{1+\alpha}{\beta}$ and $\frac{1+\beta}{\alpha}$.

Example 1.14

If one root of the equation $ax^2 + bx + c = 0$ is k times the other root, show that $kb^2 = (1+k)^2 ac$.

Solution

Let α and $k\alpha$ be the roots of the equation $ax^2 + bx + c = 0$, then

$$\begin{cases} \alpha + k\alpha = -\dfrac{b}{a} \text{..........(1)} \\ \alpha(k\alpha) = \dfrac{c}{a} \text{.............(2)} \end{cases}$$

From (1), $\alpha = -\dfrac{b}{a(1+k)}$ ....(3)

Substitute (3) into (2), we have

$$k\left[-\frac{b}{a(1+k)}\right]^2 = \frac{c}{a}$$

$$\frac{kb^2}{a^2(1+k)^2} = \frac{c}{a}$$

$$kb^2 = (1+k)^2 ac$$

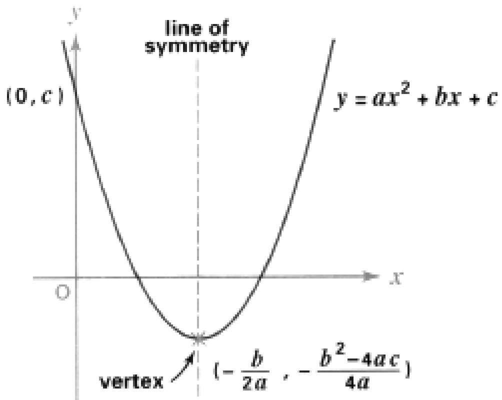
Checkpoint 1.12

Given the equation $x^2 - (k-4)x + 2k - 12 = 0$, find the possible values of k such that the two roots of the equation differ by 4.

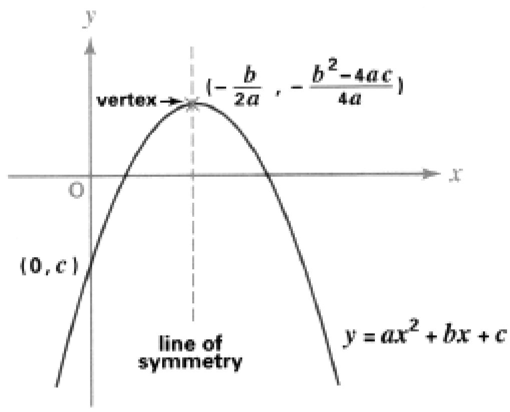
1.4 Quadratic Functions

For the graph of $y = ax^2 + bx + c$, $a \neq 0$, we have:

		$a > 0$	$a < 0$
Direction of opening		Upwards	Downwards
Vertex		Minimum value when $x = -\dfrac{b}{2a}$	Maximum value when $x = -\dfrac{b}{2a}$
Line of symmetry		$x = -\dfrac{b}{2a}$	
y-intercept		c	
Intersection with the x-axis	$\Delta > 0$	$y > 0$ for all x	$y < 0$ for all x
		At no points	
	$\Delta < 0$	At two distinct points	
	$\Delta = 0$	At only one point	



(a) $a > 0$



(b) $a < 0$

Example 1.15

By expressing the quadratic function $f(x) = 2x^2 + 3x + 4$ in the form $a(x+p)^2 + q$, determine the minimum value of the function and the corresponding value of x.

Solution

$$
\begin{aligned}
f(x) &= 2x^2 + 3x + 4 \\
&= 2\left[x^2 + \frac{3}{2}x + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2\right] + 4 \\
&= 2\left(x + \frac{3}{4}\right)^2 - 2\left(\frac{3}{4}\right)^2 + 4 \\
&= 2\left(x + \frac{3}{4}\right)^2 + \frac{23}{8}
\end{aligned}
$$

Hence, the minimum value of $f(x)$ is $\frac{23}{8}$, and the corresponding value of x is $-\frac{3}{4}$.

Explanation

Since $\left(x + \frac{3}{4}\right)^2 \geq 0$ for any x, $2\left(x + \frac{3}{4}\right)^2 \geq 0$, we have

$f(x) = 2\left(x + \frac{3}{4}\right)^2 + \frac{23}{8} \geq 0 + \frac{23}{8} = \frac{23}{8}$ and $f(x) = \frac{23}{8}$ when $x = -\frac{3}{4}$.

Example 1.16

Find the maximum value of the function $f(x) = -x^2 + x + 6$ and its corresponding value of x.

Solution

$$
\begin{aligned}
f(x) &= -x^2 + x + 6 \\
&= -\left[x^2 - x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right] + 6 \\
&= -\left(x - \frac{1}{2}\right)^2 + \frac{25}{4}
\end{aligned}
$$

Hence, the maximum value of $f(x)$ is $\frac{25}{4}$ and the corresponding value of x is $\frac{1}{2}$.

Checkpoint 1.13

Find the minimum value of the function $f(x) = 3x^2 + 6x + 5$ and the corresponding value of x.

Checkpoint 1.14

Find the maximum/minimum value of the function $f(x) = 1 + 4x - 2x^2$ and the corresponding value of x.

Example 1.17

Find the range of values of k so that the function $y = 5x^2 + 4x - k$ is always positive.

Solution

$y = 5x^2 + 4x - k$ attains a minimum value since $a = 5 > 0$.

$y = 5x^2 + 4x - k$ being always positive means that the curve does not meet the x-axis,

$\therefore \quad \Delta < 0$

i.e. $4^2 - 4(5)(-k) < 0$

$$20k < -16$$

$$k < -\frac{4}{5}$$

Alternative Solution

$$\begin{aligned} y &= 5x^2 + 4x - k \\ &= 5\left[x^2 + \frac{4}{5}x + \left(\frac{2}{5}\right)^2 - \left(\frac{2}{5}\right)^2\right] - k \\ &= 5\left(x + \frac{2}{5}\right)^2 - \frac{4}{5} - k \end{aligned}$$

Since $y = 5x^2 + 4x - k$ is always positive and $\left(x + \frac{2}{5}\right)^2 \geq 0$, $-\frac{4}{5} - k > 0$, i.e. $k < -\frac{4}{5}$.

Checkpoint 1.15

Find the range of values of k so that the function so that the function $y = -x^2 - 5x + k$ < 0 for all values of x.

1.5 Graphical Solutions of Simultaneous Equations: One Linear and One Quadratic

The coordinates of the point(s) of intersection are the roots of the simultaneous equations.

Example 1.18

Solve the following simultaneous equations graphically:

$$\begin{cases} y = 2x^2 - x \\ y = -2x + 2 \end{cases}$$

Solution

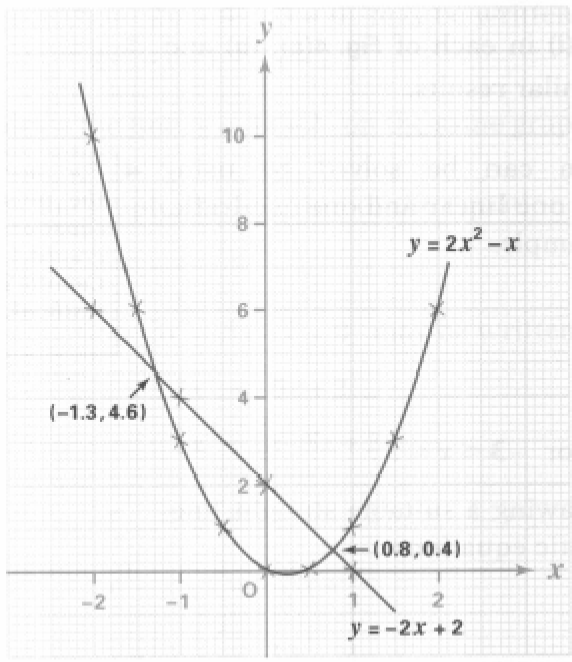
$y = 2x^2 - x$

x	–2	–1.5	–1	–0.5	0	0.5	1	1.5	2
y	10	6	3	1	0	0	1	3	6

$y = -2x + 2$

x	–2	–1	0	1
y	6	4	2	0

The graphs of the two equations are shown below:



From the graph, the solution of (x, y) is (–1.3, 4.6) or (0.8, 0.4).

Checkpoint 1.16

Solve the system of equations graphically:

$$\begin{cases} y = -x^2 + 4x \\ 2x + y - 9 = 0 \end{cases}$$

$y = -x^2 + 4x$

x	–2	–1	0	1	2	3	4	5	6
y									

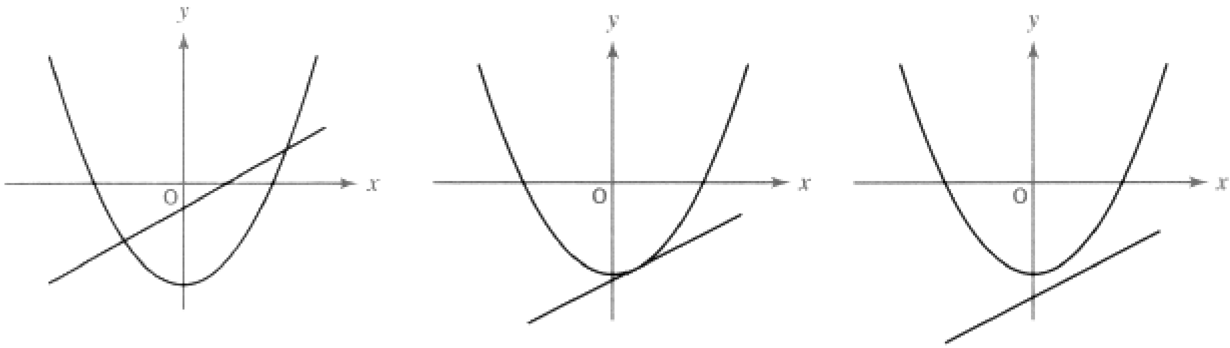
$2x + y - 9 = 0$

x	–2	–1	0	1
y				

In general, the simultaneous equations $\begin{cases} y = ax^2 + bx + c \\ y = mx + k \end{cases}$ can be reduced to

$$ax^2 + (b - m)x + (c - k) = 0 \ \ldots\ldots\ldots (*)$$

Then (*) has 2 distinct real roots, one real root or no real roots when $\Delta > 0$, $\Delta = 0$ and $\Delta < 0$ respectively.



$\Delta > 0$, TWO points of intersection $\quad$ $\Delta = 0$, ONE point of intersection $\quad$ $\Delta < 0$, NO points of intersection

In **Example 1.18**, the simultaneous equations have 2 distinct real roots. Thus $\Delta > 0$.
In **Checkpoint 1.16**, the simultaneous equations have 1 real root. Thus $\Delta = 0$.

1.6 Absolute Values

1.5.1 Definition

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

1.5.2 Properties

(1) $|x| \geq 0$

(2) $|x| = |-x|$

(3) $|xy| = |x||y|$

(4) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, if $y \neq 0$

(5) $|x|^2 = x^2$

(6) If $|y| = a$, where $a \geq 0$, then $y = a$ or $y = -a$.

The concept, definition and properties we learnt about absolute values are used to solve equations when they involve absolute values.

Example 1.19

Solve the following equations.

(a) $\left|\frac{4-5x}{2}\right| = 1$

(b) $|2x-3| = |7-4x|$

(c) $|2x+5| - x = 2$

Solution

(a) $\left|\frac{4-5x}{2}\right| = 1$

$\frac{4-5x}{2} = 1$ or $\frac{4-5x}{2} = -1$

$4-5x = 2$ or $4-5x = -2$

$x = \frac{2}{5}$ or $x = \frac{6}{5}$

(b) $|2x-3| = |7-4x|$

$2x-3 = 7-4x$ or $2x-3 = -(7-4x)$

$6x = 10$ or $2x = 4$

$x = \frac{5}{3}$ or $x = 2$

(c)

$|2x+5| - x = 2$

$|2x+5| = 2 + x \geq 0$

$\therefore \quad x \geq -2$

$2x+5 = 2+x$ or $2x+5 = -(2+x)$

$x = -3$ (rejected) or $x = -\frac{7}{3}$ (rejected)

Example 1.20

Solve $(x-3)^2 - 2|x-3| - 3 = 0$.

Solution

$$(x-3)^2 - 2|x-3| - 3 = 0$$

$$|x-3|^2 - 2|x-3| - 3 = 0$$

$$(|x-3|)-3(|x-3|+1) = 0$$

$\therefore$ $|x-3| = 3$ or $|x-3| = -1$ (rejected)

For $|x-3| = 3$, we have

$x-3=3$ or $x-3=-3$

$x=6$ or $x=0$

Checkpoint 1.17

Solve the following equations.

(a) $|4x-3| = 1$

(b) $|2-3x| = |4x+1|$

(c) $|-6x| = -2$

(d) $|4x+3| = x$

Example 1.21

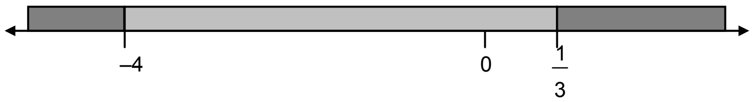
Solve $|x+4|+|3x-1|=6$.

Solution

Notice that

$$|x+4|=\begin{cases} x+4, & \text{if } x+4\geq 0 \\ -(x+4), & \text{if } x+4<0 \end{cases} \quad \text{and} \quad |3x-1|=\begin{cases} 3x-1, & \text{if } 3x-1\geq 0 \\ -(3x-1), & \text{if } 3x-1<0 \end{cases},$$

$$\text{i.e.} \quad |x+4|=\begin{cases} x+4, & \text{if } x\geq -4 \\ -(x+4), & \text{if } x<-4 \end{cases} \quad \text{and} \quad |3x-1|=\begin{cases} 3x-1, & \text{if } x\geq \frac{1}{3} \\ -(3x-1), & \text{if } x<\frac{1}{3} \end{cases}$$



So consider 3 cases: $x<-4$, $-4\leq x<\frac{1}{3}$ and $x\geq\frac{1}{3}$.

Case 1: $x<-4$.

$$\begin{aligned} |x+4|+|3x-1| &= 6 \\ -(x+4)-(3x-1) &= 6 \\ -4x-3 &= 6 \\ x &= -\frac{9}{4} \quad \text{(rejected)} \end{aligned}$$

← $\because x<4$

Case 2: $-4\leq x<\frac{1}{3}$.

$$\begin{aligned} |x+4|+|3x-1| &= 6 \\ x+4-(3x-1) &= 6 \\ -2x+5 &= 6 \\ x &= -\frac{1}{2} \end{aligned}$$

Case 3: $x\geq\frac{1}{3}$.

$$\begin{aligned} |x+4|+|3x-1| &= 6 \\ x+4+(3x-1) &= 6 \\ 4x+3 &= 6 \\ x &= \frac{3}{4} \end{aligned}$$

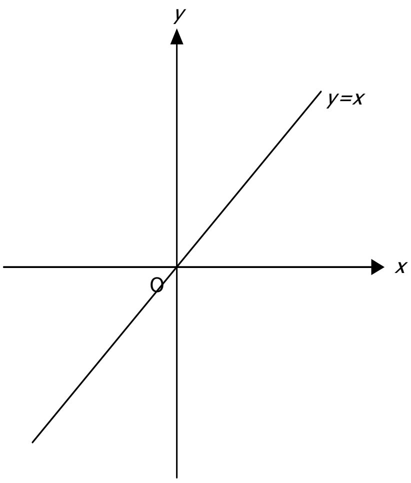
$\therefore$ The solutions are $x=-\frac{1}{2}$ or $x=\frac{3}{4}$.

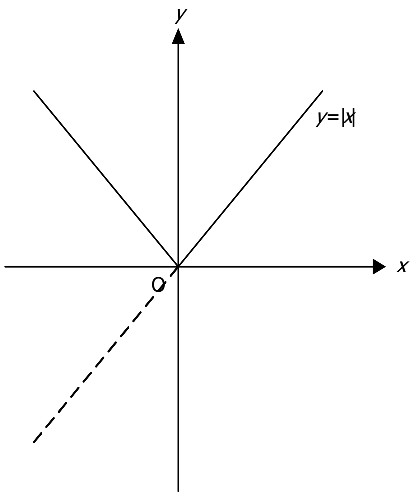
Checkpoint 1.18

Solve $|2x-3|+|x+5|=4$.

1.7 Graphical Solutions Involving Absolute Values

To draw the graph of the function in the form of $y = |f(x)|$, we can first draw the graph of $y = f(x)$, then take the portion of the graph below the x-axis and reflect it about the x-axis.





Example 1.22

(a) Draw the graph of $y = |2x-1|$.

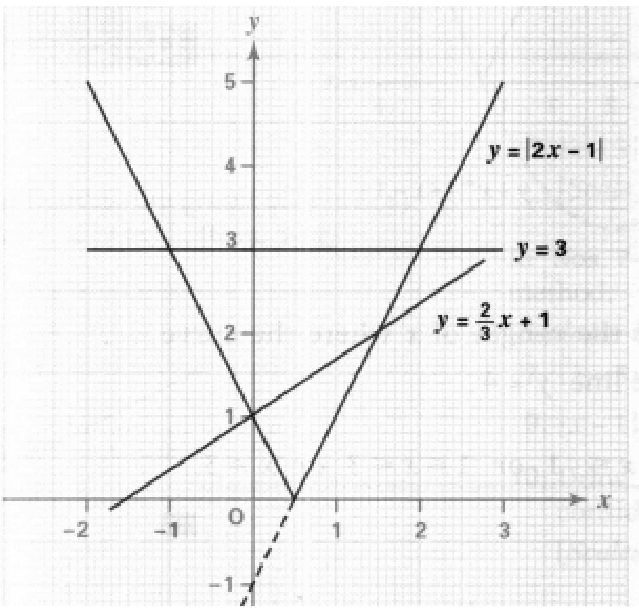
(b) Hence solve the equations

(i) $|2x-1| = 3$

(ii) $3|2x-1| = 2x+3$

Solution

(a)



(b) (i) $y = |2x-1| = 3$, so the line $y = 3$ is added to the graph.

From the graph, $x = -1$ or 2.

(ii) $3|2x-1| = 2x+3$

$$y = |2x-1| = \frac{2}{3}x+1$$

So the line $y = \frac{2}{3}x+1$ is added to the graph.

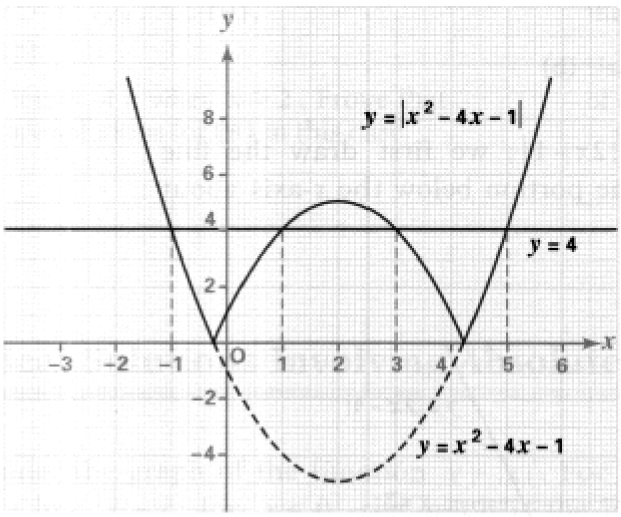
From the graph, $x = 0$ or $\frac{3}{2}$.

Example 1.23

Solve $|x^2-4x-1| = 4$ graphically.

Solution

The graph of $y = |x^2-4x-1|$ is shown below:



$y = |x^2-4x-1| = 4$, so the line $y = 4$ is added to the graph.

From the graph, $x = -1$, 1, 3 or 5.

Checkpoint 1.19

(a) Draw the graph of $y = |x^2 - 6x + 5|$ from $x = -1$ to $x = 7$.

(b) Solve $|x^2 - 6x + 5| = 3$ graphically.

(c) Find the value(s) of a in the equation $|x^2 - 6x + 5| = a$ if the equation has

(i) 4 distinct roots;

(ii) 3 distinct roots;

(iii) 2 distinct roots.

(a) $y = |x^2 - 6x + 5|$

x	–1	0	1	2	3	4	5	6	7
y									

Exercise 1 Quadratic Equations and Quadratic Functions

1.1

1. Solve the following equations.
   (a) $2x(3-x)=-1$
   (b) $4x^2-2\sqrt{3}x-1=0$

2. Solve the equation $16y+\dfrac{1}{y}=8$.

3. Solve the equation $10(2x-1)^2-(2x-1)-2=0$.

4. Solve the equation $3x^{\frac{4}{3}}-10x^{\frac{2}{3}}-8=0$.

5. Solve the equation $\dfrac{1}{x}+\dfrac{1}{x+1}=\dfrac{1}{3}$.

6. Solve the equation $9^x-2(3^x)-8=0$.
   (*Give the answers correct to 3 significant figures.*)

7. Solve the equation $\log(x-1)+\log(x-3)=1$.
   (*Give the answers correct to 3 significant figures.*)

8. Solve the system of equations $\begin{cases} x+y=2 \\ x^2+y^2-4x-2y+3=0 \end{cases}$.

9. Solve $2\left(x^2+\dfrac{1}{x^2}\right)-5\left(x+\dfrac{1}{x}\right)+6=0$

1.2

10. Determine the nature of roots of the following quadratic equations.
    (a) $2x^2-x-2=0$
    (b) $(x-1)(x+3)=4$
    (c) $3x^2-\sqrt{8}x-1=0$
    (d) $-3x^2+7x-2=0$

11. Find the possible values of k so that the quadratic equation $x^2+2(k-3)x+k^2=0$ has real roots.

12. Find the range of values of k so that the simultaneous equations $\begin{cases} y = kx-1 \\ y^2 = 4x \end{cases}$ has real roots.

13. Show that the equation $kx^2+2x-(k-2)=0$ has real roots for all values of k.

14. Given the equation $(2x^2+4x+1)h=3x^2-2x+2$. Find the value(s) of h such that the equation has equal roots.

15. Prove that, if a, b, c are rational, the roots of the equation $x^2-6ax+9a^2-4b^2+12bc-9c^2=0$ are rational.

16. Given the quadratic equation

$$2x^2\log(k-2)-4x\log(k-2)+\log k=0, \text{ where } k>2 \text{ and } k\neq 3.$$

Find the value of k such that the equation has equal roots.

1.3

17. The equation $3x^2-4x+1=0$ has roots α and β ($\alpha>\beta$). Find the values of
    (a) $\alpha^2+\beta^2$
    (b) $\alpha^3\beta+\alpha\beta^3$
    (c) $\alpha-\beta$
    (d) $\left(\alpha+\frac{1}{\alpha}\right)+\left(\beta+\frac{1}{\beta}\right)$

18. If α and β are the roots of the equation $2x^2-x-5=0$, find quadratic equations with integral coefficients such that the roots of the equations are as follows:
    (a) $\alpha-3, \beta-3$
    (b) $\frac{1}{\alpha+1}, \frac{1}{\beta+1}$
    (c) $2\alpha-\beta, 2\beta-\alpha$
    (d) $\alpha^2+\frac{1}{\beta}, \beta^2+\frac{1}{\alpha}$

19. If one root of the equation $3x^2 - (k-2)x + 4 = 0$ is $\frac{2}{3}$, find the other root and hence find the value of k.

20. State the condition that the sum of reciprocals of the roots of the equation $ax^2 + bx + c = 0$ is 1.
[The reciprocal of a is $\frac{1}{a}$.]

21. α and β are the roots of the equation $2x^2 - (m-1)x + (m+1) = 0$ such that $\alpha^2 + \beta^2 = \alpha\beta + 1$. Find the value of m where $m < 0$ and hence solve the equation.

22. α and β are the roots of the equation $x^2 + px + q = 0$ such that $\alpha^3 + \beta^3 = 1$. Without solving the equation, show that $p^3 + 1 = 3pq$.

23. α and β are the real roots of the equation $2x^2 + 4x + m = 0$ while $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$ are the roots of the equation $9x^2 - nx + 4 = 0$. Find the values of m and n.

24. Let α and β be the roots of the equation $x^2 + px + q = 0$.
   (a) Express $(\alpha^3 - \beta)(\beta^3 - \alpha)$ in terms of p and q.
   (b) If one root of the equation is the cube of the other, using the result of (a), show that $p^2(p^2 - 4q) = q(q-1)^2$.

1.4

25. Given that the vertex of a quadratic curve $y = ax^2 + bx + c$ is (3, 2). If the curve passes through (–1, 4), find the values of a, b and c.

26. Determine the maximum value of the expression $\frac{11}{3x^2 - 4x + 5}$ and the corresponding value of x.

27. Sketch the graph of $y = ax^2 + bx + c$ if $a > 0$, $b < 0$ and $c < 0$.

28. Find the range of values of k if $(k^2 + 1)x^2 + 2kx + 1 \geq 0$ for all values of x.

29. (a) Prove that the quadratic function $y = 4x^2 + 4hkx + h^4 + k^4 - h^2k^2$ can never be negative if h and k are real constants.

(b) If the minimum value of y is 0, express h in terms of k.

30. Given that the maximum value of the function $f(x) = 2 + \frac{9}{k}x - x^2$, where $k \neq 0$, is 18.

Find the values of k.

1.5

31. Plot the graph of $y = 2x - x^2$ for $-3 \leq x \leq 3$. Use this graph, and by drawing a suitable straight line, solve each of the following quadratic equations.

(a) $3x^2 - 4x + 5 = 0$

(b) $4x^2 + 12x + 9 = 0$

(c) $2x^2 - 5x - 3 = 0$

[*Scales: x-axis, 5 divisions (1 cm) = 1 unit; y-axis,5 divisions (1 cm) = 5 units.*]

1.6

32. Solve $3|2x - 1| - 4 = 5$.

33. Solve $|4 - x - x^2| = 2$.

34. Solve $|3x - 1| = \left|\frac{x - 7}{2}\right|$.

35. Solve $|x^2 - 3x + 1| = 5$.

36. Solve $\left|\frac{x^2 - 2x + 3}{x - 1}\right| = 3$.

37. Solve $6(1 - 4x)^2 + |1 - 4x| - 12 = 0$.

38. Solve $|x + 7| + |5x - 6| = 10$.

39. Solve $|6x-4|-|2x+9|=-5$.

40. Solve $||3-2x|-1|=2|x|$.

1.7

41. Let $f(x)=|x|+|2x-3|$.

(a) Show that

$$f(x)=\begin{cases} 3-3x & \text{if } x<0 \\ 3-x & \text{if } 0\le x<\dfrac{3}{2} \\ 3x-3 & \text{if } x\ge\dfrac{3}{2} \end{cases}$$

Hence draw the graph of $y=f(x)$ for $-2\le x\le 4$

[*Scale for both axes: 5 divisions (1 cm) = 1 unit.*]

(b) Using the graph drawn in (a), solve the equation $|x|+|2x-3|=6$.