A Bibliography for n-sided Surfaces

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Abstract

A bibliography of work on n-sided surfaces ($n > 4$), primarily since 1990. Results are grouped from the point of view of interpolation problem to be solved and type of solution. The forms of interpolation examined are: bounding curves in general, and the special case of vertex blends. The techniques examined are: subdivison, multple patch solutions, and single patch solutions.

1 Introduction

This paper is intended to update [48] and [13] from a different organizational perspective. A need for n-sided, or more precisely, non-4-sided surfaces has arisen from a variety of situations in CAGD and a number of different solutions have been proposed.

I will look primarily at the problem where $n > 4$, although there is a large literature on triangular surfaces, and at least one paper by Sabin [41] on 2-sided surfaces.

Techniques will be categorized according to the interpolation problem being solved and the construction of the solution.

2 Interpolation problem

For a curve $f : [a, b] \to R^3$ a *cross-tangency constraint* is a vector field defined on [a,b] , $T : [a, b] \to R^3$. For example, $T = df/dt \times d^2f/dt^2$, the cross product of the tangent and the normal provides a *default* cross-tangency constraint. If $F : [a, b] \times [c, d] \to R^3$ is a surface, and $f(t) = F(a, t)$ (i.e. hold u fixed at a) then $T = \partial F/\partial u$ is another example.

I will then distinguish the interpolation problems as either *filling in curves* or *filling in faces*. In fact, some techniques start from a set of points which need interpolation, but these techniques take as a first step the creation of a requisite set of curves.

2.1 Filling in curves

The input for this case is a set of curves which form a closed loop.

The lack of any (explicit) cross-tangency information may make finding a solution simpler. However complexities where the curves meet may result in poorly behaved surfaces with some methods. This was in fact the issue that Gregory [10] first solved. Hall and Mullineux [14] discuss how to tweak the shape of surfaces developed by Gregory and Yuen [12].

This is the case which started my investigation of the subject. It arises in the context of the *cover* functionality provided by the ACIS$^{(r)}$ [1] solid modeling kernel. ACIS$^{(r)}$ provides functions for filling in curves, edges and vertex blends, using the approach described by Charrot and Plowman in [4].

2.2 Filling in faces

One of the classical problems is filling in a hole bounded by a set of surfaces. In the general setting (e.g. non-manifold) the edges may not bound a hole but a new internal face. There may or may not be cross-tangency data supplied.

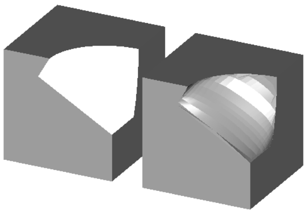

FIGURE 1. Filling in a face.

The other classical problem is *vertex blends*.

In this case the maximum amount of cross-tangency data is available, and the surface is forced to be well-behaved at corners. This may make finding a solution difficult, but the solution is usually a *fair* surface. Varady and Hoffmann [43] provides a survey of methods for this specific problem.

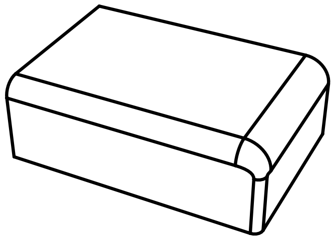

FIGURE 2. A Vertex Blend.

3 Constructions

Three approaches will be discussed: subdivision, multiple patch and single patch. Dyn's survey of John Gregory's work [8] provides examples of all three approaches.

3.1 Subdivision

A *recursively generated surface* is defined initially by a polygon P_0. A new polygon P_1 is generated from P_0 by cutting its corners with planes. The surface is then described as

$S = lim P_i$.

Subdivision is a technique which applies to a much wider range of problems than n-sided surfaces. Some general references [2], [7], [42], [26], [28] are provided in the bibliography. Peters [32], [34], [35] introduces a *surface* spline to solve the problem of irregrular meshes globally. Other authors look at the problem on a patch-by patch basis, and this is the relevance to n-sided surfaces.

I distinguish two types. The first which I call *internal* arises from a general subdivision problem producing internal non-rectangular regions.

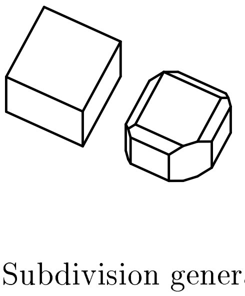

FIGURE 3. Internal: Subdivision generates a pentagonal face.

The second which I call *external* arises from using subdivision techniques specifically to solve an n-sided surface problem. Hermann and Renner [15] mention this approach. It is also described by Nasri in [29]: "...to force a recursively generated surface over arbitrary topology to interpolate pre-defined quadratic B-spline curves". In [30] he extends the initial result for closed curves to open curves, using tehcniques from [27] to deal with corners. In fact, Nasri generates an internal network to solve the external problem.

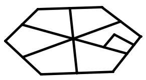

FIGURE 4. External:One subdivision and start of second.

Wang *et. al.* [51] and [50] use Varady's overlap patches [48], [47] to solve the internal problem of filling n-sided holes arising from the subdivision process.

3.2 Multiple patches

The method here is to subdivide the region to be filled, typically only once, and then fill in the new smaller regions with surfaces. The main difficulty with this approach is maintaining a given level of continuity between adjacent patches, since besides the cross-tangency conditions, the new internal edges must also satisfy some continuity constraints. Besides aesthetic considerations, G^1 continuity may be required by some solid modeling systems to allow further operations (booleans, offsets) to proceed.

Dahmen [6] provides the lowest degree approach: using quadric patches. The input data in this case is a polygon. Renner [37] fills in a hole bounded by a cubic with a quartic polynomial patch. Gregory and Zhou [56] use bicubic patches to fill in an n-sided hole.

The relationships between adjacent surface patches become cyclic when the last patch meets the first one. In [31] Peters examines the constraint systems required to obtain C^k continuity around a vertex. Varady and Hermann [45] examine the control of surface curvature for the adjoining patches.

Hsu and Tsay [18] use a blend between two edges to fill in a triangular subset of the n-sided hole, then continue to subdivide the remaining pieces so that the entire hole is filled in with multiple blend surfaces. This approach is different from the usual multiple patch approach as it may require multiple subdivisions to arrive at regions suitable for blending.

Zhao and Rockwood [54] use 4-sided biquintic Bezier patches to obtain G^1 cotninuity, Peters [33] uses 4-sided and triangular, bicubic biquadratic patches for C^1 continuity. Hermann [16] subdivides the n-sided hole into rectangles and then uses biquintic Gregory patches, assuming the boundary curves are quintic. Zhou and Lau in [57] use 4-sided biquintic patches, and examine a bicubic approximation which gives up to GC^2 continuity, and in [58] use 4 sided high degree $(2k^2 + 3k + 1)$ patches to give GC^k continuity.

Loop in [23] and [24] uses triangular quartic Bezier patches to approximate an irregular mesh. For regular (4 sided) portions of the mesh, the 4 triangular patches may be replaced by a single bi-quadratic.

Varady and Rockwood in [46], [44] use a multiple patch approach for vertex blends with setbacks.

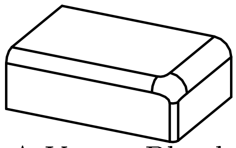

FIGURE 5. A Vertex Blend with setback.

3.3 Single patches

The objective of this approach is to create a single surface which satisfies the boundary conditions.This is the most convenient approach for a solid modeling environment (at least for vertex blends): the single patch can be easily connected as a single face.

Sabin in [40] presents a theoretical approach, primarily looking at only inerpolating the curves. [41] provides a specific example in the formulation of a 2-sided patch.

Daehlen and Skytt [5] use box splines to generate vertex blends. They point out that this method has some difficulties matching B-spline surfaces adjacnt to the vertex.

A rational function $f(u, v) = \frac{p(u,v)}{q(u,v)}$ has a *base point* where p and q are simultaneously zero. Warren in [52] and [53] has pointed out that when the surface is rational, results from algebraic geometry guarantee that these methods are specific cases of rational maps with base points. He then classifies single surface solutions as either having base points on the boundaries or external

3.3.1 Base point surfaces:boundary base points

Warren's method is limited to four, five or six sided patches. Since the conditions Sabin proposes in [39] uses basis functions which are zero along the edges, a rational surface defined with such a symmetric domain would provide another example of this type.

3.3.2 S-patches: external base points

Instead of using a polygonal domain, the S-patch approach uses a polyhedral domain.

Loop [22] has a good description of S-patches: Starting with an n-gon P a map $L : P \to \Delta$ maps P into an n-simplex Δ. The map L uses a generalized barycentric coordinate system for points in P. A Bezier simplex map $B : \Delta \to M$ provides the contol net for the S-patch which is defined as $B \circ L$. Loop and DeRose [21] apply S-patches specifically to filling an n-sided hole.

Karciauskas [19] modifies the S-patch approach by using a different map L to obtain a more tractable control net.

3.3.3 Polygonal domains:external base points

In [4]Charrot and Plowman extend the method used for a pentagon in [3] to an n-gon in the vertex blend setting. The polygon is mapped directly to a surface using a blend of patches defined for each edge. The final shape is controlled by a parameter choice, as opposed to the control net available in the S-patch approach. Both approaches provde a C^1 solution. Gregory and Hahn in [11] provide a C^2 extension. All of these approaches involve functions of n variables.

Gregory and Yuen [12] start with a larger problem: interpolating an arbitrary mesh. Their solution is to interpolate the "edges" by rational splines to create polygonal curved regions, then extend the splines into strips, and blend the strips into the interior using the same technique as [3]. Hall and Mullineux [14] discuss shape adjustment for this technique.

Kato [20] uses a similar approach to generate surface patches with holes. He notes that "..in many cases, modification of the inner boundaries [holes] in u, v space is needed...in the case of a surface patch with more than one hole, manual modification is needed."

Varady [47] uses parametrizations based initially on the vertices rather than the edges. This approach requires $2n$ variables to get the individual components to match.

3.3.4 Non-rational approaches

These solutions use Bezier forms, so the mathematics is non-rational. In the following three papers, the idea is to create an n-sided patch whose boundary edges can be connected to existing 4-sided patches

Sabin [38] uses quadratic functions to fill in three-sided and five-sided patches, with the property that the edges "look like" the edges of bi-quadratic tensor product Bezier patches. Hosaka and Kimura [17] do the same thing using quadratics and cubics for three-sided, five-sided and six-sided patches. Their solution is described in the general n-sided setting. Zheng and Ball [55] extend those two approaches by using higher degrees for higher numbers of sides.

3.3.5 Closed objects

All of the examples discussed so far have similar topologies in both parameter space and model space. I close this section with two papers which address the problem which arises in a more global setting from a classical result in topology.

Theorem 3.1 *[25]: If M is a compact connected 2-manifold, M is a 2-sphere with h handles and m cross caps.*

This theorem implies M can be represented by a polygon with edges identified: e.g. a torus is $ABA^{-1}B^{-1}$ Ferguson and Rockwood [9] apply this result to model objects with a single surface. Wallner and Pottman [49] apply the same idea to *orbifolds*: surface defined as images of group actions.

These techniques fall into the scope of this bibliography insofar as the marked polygon defining the surface is an n-sided object in parameter space.

4 Conclusion

Three techniques have been examined for generating n-sided surfaces, where $n > 4$. *Subdivision* starts from a mesh of points to generate the surface as a limit of polygons. *Multiple surface* approaches fill in an n-sided region with triangular and quadrilateral patches. Generating a smoother complex forces higher degrees for the individual pieces. *Single surface* methods provide a single surface solution. When rational mathematics is used, these solutions are seen to all be cases of surfaces with base points.

The opinions expressed in this paper are the author's and do not represent the views of Spatial Technology, Inc. The figures were created using ACIS$^{(r)}$ 4.0.

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