Credit risk analysis of cashflow CDO structures

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Abstract

We develop a method that offers consistent and computationally efficient credit risk analysis of cashflow CDO structures. The proposal makes use of simple portfolio models that admit semi-analytic representations of the loss distribution, combined with detailed and fast calculations of realistic interest and principal cashflow waterfalls. We define in this context and study credit tranche risk measures such as the probability of loss and expected loss-given-default and the variance of the latter. We benchmark our approach against the stress-scenario based analysis favored by cashflow CDO market practitioners.

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[†] Disclaimer: The results, opinions and conclusions presented in this article reflect the personal opinion of the authors and not that of ABN AMRO Bank.

1 Introduction

The Collateralized Debt Obligation (CDO) market continues to grow on both the synthetic (6.5 trillion USD outstanding in June 2006 compared to 3.5 trillion USD outstanding December 2005, see [3]) and cash side (414 billion USD in 2006 compared to 205 billion USD in 2007, see [2]). Despite some similarities between the two market segments, there are significant structural differences in the respective contracts, with the synthetic instruments being significantly more transparent to describe and model. In contrast to the standard synthetic tranche contract (implemented with a credit default swap) a typical cash-flow CDO structure is a complex true sale securitization, whereby a diversified pool of credit assets is sold into a Special Purpose Vehicle which in turn issues multiple classes of notes. The CDO liabilities are serviced exclusively by the cash flows generated by the collateral assets. Cash flow CDOs are characterized by a large number of structural elements and covenants which govern very precisely the distribution of income and principal to the issued notes (see [8]).

In common with other credit risky bonds, CDO investors are interested in the credit risk profile of a given tranche. Commonly used risk measures for credit risky bonds are the probability of default and recovery (or loss given default). Modelling default rates and recoveries for corporate bonds and loans has received enormous attention in recent years (see [4]). However, CDO cash flows and their risk characteristics differ significantly from the cash flows of a corporate bond. In widely used market practise, the credit risk analysis of cashflow CDOs is based on the following strategy: first, estimate a default rate distribution of the collateral portfolio using (possibly) a portfolio model (see [9]). Next, using this distribution, establish representative stress scenarios (each associated with a scenario realization probability). Then, calculate cash flow patterns for each stress scenario. Finally, the severest stress scenarios "sustained" by a given tranche without monetary loss determines the rating of that tranche. For instance, if under the assumptions of a default scenario corresponding to a AAA stress (and expected to occur with AAA frequency), the tranche still receives all the scheduled cashflows, the tranche is rated AAA. Note, that traditional cash flow calculations transform a single default scenario into a single cash flow scenario.

While reasonable as a starting point, the above procedure suffers from a key shortcoming: The tranche risk profile is probed only partially (on a single cashflow scenario basis) instead of considering the entire stochastic distribution of cashflows. We investigate a more comprehensive approach where we calculate and weight cash flow scenarios with the probabilities assigned by the underlying collateral model, thereby creating the full probability distribution of cash flow scenarios. Using this distribution we define risk measures for the various tranches: e.g., probability of tranche loss, expected loss or LGD volatility.

Combining a portfolio credit risk model with detailed cashflow calculations is eminently feasible in a Monte Carlo context, but not without substantial technical overhead. We illustrate here that a simple semi-analytical framework based on Vasicek's large pool model permits fast and insightful analysis of structures while still capturing realistic levels of complexity. We focus on a generic CLO structure (CDOs where the underlying pool is a pool of leveraged loans) to avoid the additional complexity of a CDO of ABS structure. Since most actual CLO transactions are managed CDO's, any actual cashflows are complicated by the uncertain (and very path dependent) portfolio profile. For concreteness, in this implementation we adopt a static portfolio view. Consequently, we incorporate a key subset of structural CLO features (the OC/IC triggers), but leave out elements that are linked to portfolio composition. As with any analysis of a financial instrument with the amount of complexity of a cashflow CDO, our results cover only a subset of the relevant risk factors.

The contribution of this paper is twofold: First, we illustrate that the modelling of complex cashflow CDO waterfalls can be achieved with simple (low dimensional) portfolio models that can be computed semi-analytically (in the simplest case reducing the analysis to one-dimensional integration). This result follows because, within the assumptions of the current study, interest and principal cashflows are deterministic functions of the portfolio loss rate. Secondly, we propose that consistent measures of tranche risk can be defined using the entire distribution of cashflows, instead of a partial, scenario based, approach. The latter method can run into difficulties in the presence of cashflow mechanisms that violate standard seniority prioritization.[1]

The structure of the remaining paper is as follows: Section 2 outlines a simple multi-period extension of the analytic large pool model (Vasicek model), which is a well known approximate description of the credit risk in a pool of homogeneous assets. We illustrate how this model can be used to derive an aggregate description of collateral performance for a static pool of assets. Next, in Section 3 we discuss a mathematical specification of a cashflow CDO structure. The detailed waterfall mechanics is presented in an appendix B.2. In Section 4 we introduce various risk measures for capturing the performance of CDO tranches and in Section 5 we apply our method to a typical structure and compare the results with a stylized scenario based analysis.

2 Using the Large Pool Model as the cashflow CDO Collateral model

We adopt the large pool (Vasicek) credit risk model as the basis for modeling collateral performance. This implies a homogeneous portfolio with infinitely many assets, all with the identical default curves q_t and identical expected recoveries g per asset and uniform asset correlation ρ^2. More details on the large pool model are given in Vasicek's paper (2002, see [7]). This model has been widely used to provide an approximate description of portfolio loss, not least for estimating regulatory capital for financial institutions. We note that this model can be (and has been) improved in many ways. Besides removing the homogeneity (as in Lucas, Klaassen, Spreij and Straetmans (2000), see [6]) and large-pool assumption (as in

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[1] This is likely to be the case each and every time the following two statements are not strictly identical: "The tranche will experience no loss in a fraction p of all possible cashflow realisations" versus "The tranche will experience no loss in a fraction p of all possible portfolio loss realisations". Clearly, we only really care about the former measure.

Anderson, Basu and Sidenius (2003), see [1], and as in Laurent and Gregory (2003), see [5]). Any of those advances in semi-analytic techniques (which are primarily spurred by studies of pricing in the synthetic CDO market) can be substituted here for a collateral model.

The simplicity of the chosen collateral model means that we need to abstract from several features of the actual cash structure in the following areas: the portfolio is assumed fully ramped-up at closing time. There is no scheduled amortization (all exposures are bullets with same maturity). There are no substitutions or reinvestments (static structure) of principal proceeds, which in turn requires that principal recoveries from workouts are paid down to the various tranches before maturity. Modeling the reinvestment of principal repayments, recoveries and excess spread would require a more dynamic portfolio model, where new assets can be introduced at forward time points, contingent on current market events (rating migration of the portfolio, spread moves), together with some predefined *portfolio management strategy.* At present, those additional complexities would only hinder our aim to understand and define the basic cashflow CDO risk properties.

We introduce a time grid (discrete time approximation), with the time of closing denoted as $T_0 = 0$ and N subsequent periods at times T_t and stated maturity at $T_N = T$. If a variable X is indexed by t, we mean that the variable is evaluated at T_t.

$$X_t = X\left(T_t\right).$$

To simplify accounting the cashflows we assume that defaults happen at the start of each period, hence coupon from defaulted assets is lost for that entire period (and all subsequent ones). We assume recoveries are delayed by one period. Coupon payments to the tranche investors, equity distributions, any principal adjustments and any allocations to auxiliary accounts are done instantly after the current period collections.

Recalling the basic features of the Large Homogeneous Pool model, the credit standing of each asset j is driven by an associated normal variable $W^j = \rho Z + \sqrt{1-\rho^2}\epsilon^j$, where both Z and ϵ^j have a standard normal distribution. Default before time t happens with frequency $P(W^j < \alpha_t) = q_t$ which implies that $\alpha_t = N^{-1}(q_t)$. Calibration is to the historical default performance of similarly rated pools. Conditioning on a realization $Z = z$, the cumulative default rate by period t is denoted by $q_t(z)$, and the conditional default rate[2] is $\lambda_t(Z)$ (conditional on both the realization of Z and survival till the previous period).

$$\begin{aligned} q_t(z) &= N(\frac{\alpha_t - \rho z}{\sqrt{1-\rho^2}}) && (1) \\ \lambda_t(z) &= \frac{q_t(z) - q_{t-1}(z)}{1 - q_{t-1}(z)} && (2) \end{aligned}$$

In our analysis we calibrate to historical default performance (we use widely available S&P data). The cumulative default probabilities for the ratings we have used are presented in

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[2] We note that using the Vasicek model in a multi-period context implies a very concrete inter-temporal distribution of default rates which may or may not be provide an adequate fit to actual data.

Appendix A.1. A summary of this table (excluding rating modifiers "-" and "+") is

PD	1 year	2 years	3 years	4 years	5 years	6 years	7 years
AAA	0.02%	0.06%	0.12%	0.19%	0.28%	0.39%	0.52%
AA	0.11%	0.24%	0.39%	0.57%	0.76%	0.97%	1.20%
A	0.14%	0.32%	0.54%	0.81%	1.11%	1.45%	1.81%
BBB	0.22%	0.64%	1.18%	1.81%	2.50%	3.21%	3.94%
BB	2.77%	5.26%	7.50%	9.49%	11.25%	12.82%	14.20%
B	8.59%	14.51%	18.59%	21.45%	23.49%	25.00%	26.15%
CCC	19.82%	30.18%	35.83%	39.09%	41.08%	42.39%	43.32%
D	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%

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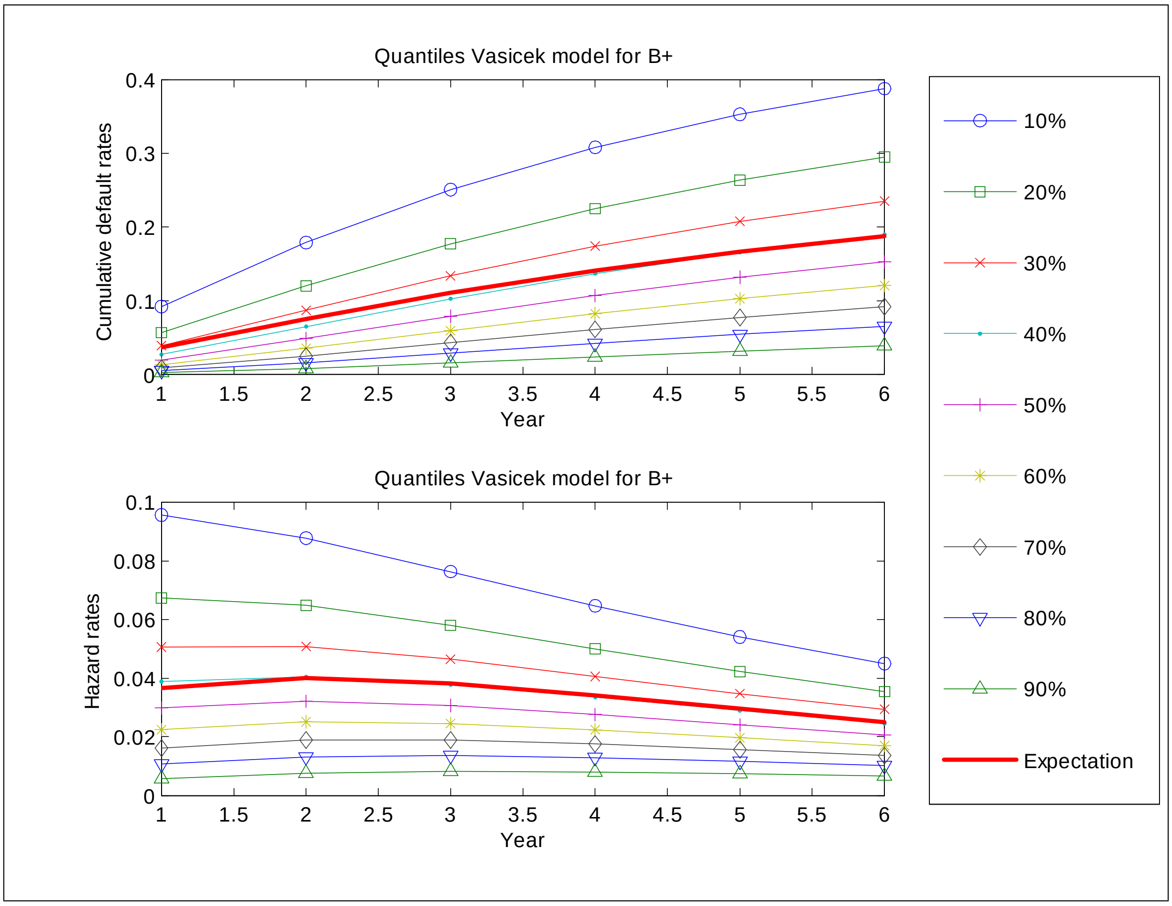
As an example, for a B+ rated portfolio we show the quantiles and cumulative default rates q_t and hazard rates λ_t in figure 1. The average default rates depend on the choice of a B+ rated portfolio (and the corresponding historical figures). For some context, we also present

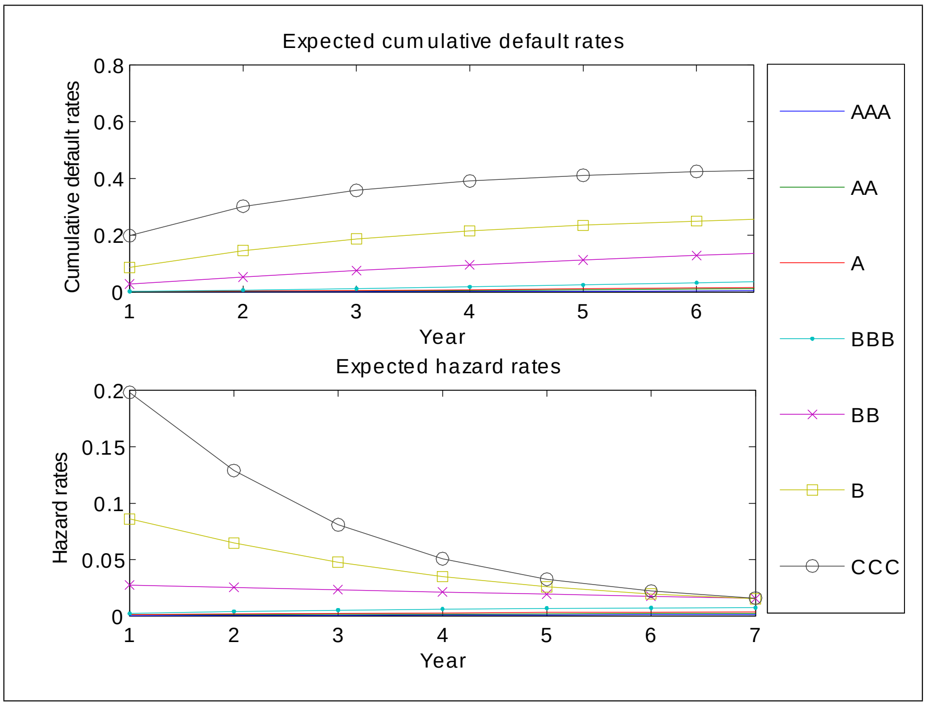
Figure 1: Quantiles of the Vasicek distribution for B+ rated portfolio

Quantiles (e.g. 10%) of the cumulative default rates and hazard rates of a B+ rated portfolio in a Vasicek model. Note that the hazard rate curve is downward sloping. We also show the average (cumulative) default rates and the corresponding hazard rates.



q_t and λ_t for the other ratings in figure 2. Any other functional specification $q_t(z)$ could also be used instead of the Vasicek distribution. E.g. the $q_t(z)$ could be based on a t-distribution instead of a Gaussian distribution, in order to introduce fatter tails.

Figure 2: Cumulative default rates and hazard rates for different rating classes
For each rating class (e.g. AAA), the cumulative default rates and hazard rates. Note that the hazard rate curve is downward sloping for the lower rating classes and upward sloping for the higher rating classes.



We denote by N_t the outstanding notional at any period. By convention this is initially scaled to unity. The reduction of performing collateral due to defaults is governed by the equation

$$N_t = (1 - \lambda_t)N_{t-1}. \tag{3}$$

In equation (3) and in the sequel we drop the z dependence on all equations in order to simplify the notation. All relationships are assumed to hold conditionally on the factor realization, unless explicitly noted.

We assume all assets and liabilities are floating rate notes. We define as c the weighted average interest payment (coupon rate) of the performing collateral. The floating rate collateral is assumed to have an average coupon of $c = r + \bar{s}$, where $\bar{s}$ is the average spread over some reference rate r with the same maturity T.

It follows from the homogeneous and static portfolio assumption that the coupon rate c remains the same for the lifetime of the structure. We define for each period the interest cashflow based on the coupon and performing capital

$$w_t = c\ N_t. \tag{4}$$

The recovery cashflow equals the default rate times the recovery rate (g)

$$r_t = g\ \lambda_t\ N_{t-1}. \tag{5}$$

When handling recovery cashflow, in principle we need to distinguish between recovered principal and recovered coupons, as those may be processed differently in the cashflow waterfall. We will treat recovered amounts as principal recoveries.

3 The cashflow CDO structure

In this section we describe in mathematical terms the cashflow CDO structure. The structure involves M tranches (alternatively notes or bonds), indexed by $i = 1, \ldots, M$ in order of decreasing seniority, plus an equity position. We denote T_0^i the initial size for the i-th tranche. The tranche size may be reduced by contractual repayments or be increased by the addition of deferred interest. T_t^i denotes the running notional for the i-th tranche during the t-th period. Similarly, E_0 is the initial equity size while E_t denotes the current "book" equity,

$$E_t = N_t - \sum_{j=1}^{M} T_t^j. \tag{6}$$

The *subordination* available to the i-th tranche at each period is defined as:

$$\Sigma_t^i = E_t + \sum_{j=i+1}^{M} T_t^j = N_t - \sigma_t^i, \tag{7}$$

where in the above equation it is useful to define the complementary notion of the *supported debt level* for a tranche, i.e., the sum of total tranche sizes down to and including the i-th tranche:

$$\sigma_t^i = \sum_{j=1}^{i} T_t^j. \tag{8}$$

In order to keep track of the collateral cashflows we will introduce the following accounts: The *interest proceeds* account i_t and the *principal proceeds* account p_t. The initialization and update of those accounts is given in the appendix.

Moving on to the liability side, for each tranche, the paid coupon c^i is a sum of a risk-free rate and a tranche spread ($c^i = r + s_i$), which at any given period leads to *scheduled* interest payments, $S_t^i = c^i T_t^i$. Subject to the payment waterfall each note will receive an *actual* tranche payment b_t^i. The residual cashflow, after all debt servicing and costs, is paid out as dividend d_t to the equity interest. Scheduled interest that is *not* paid during a payment period (i.e., when $b_t^i < S_t^i$) is added to the outstanding notional of the tranche as deferred interest [3]:

$$T_t^i := T_t^i + S_t^i - b_t^i. \tag{9}$$

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[3] The implication of this treatment of missed coupons is that in subsequent periods deferred interest is accruing at the tranche coupon rate. For simplicity the above structure applies to all notes. In actual transactions it may be the case that some senior notes do not defer interest but default immediately.

In equation (9) we introduced the ":=" symbol to denote the "update" of a variable during the distributions occurring on a CDO payment date. This notation implies that the actual final values of any assigned variable depend on the ordering of those distributions and adjustments, as per the specific waterfall which we will define later in this section. An important note here is that due to deferred interest and possible unscheduled amortization (in order to comply with debt covenants), the scheduled payments are path dependent. E.g., we only know of the required final repayments one period before maturity.

In contrast to standard synthetic CDO structures, the notional of a cashflow CDO tranche is not contractually reduced due to defaults during the life of the transaction (except for the adjustments mentioned above). Such reduction is obviously implicit in the accumulating notional losses in the collateral, but is normally only revealed as an explicit notional write-down (loss) when, as per the terms of the indenture, the tranche must be redeemed and there are no available funds.

Ultimately in analyzing the credit risk of the *i*-th tranche, we are concerned with the probability distribution of the realized returns b_t^i. Since missed coupons accrue at the tranche rate, with little loss of generality we can focus on the final period payments b_T^i. If the realization of those variables is lower than the scheduled final redemptions S_T^i there has been some loss. More on defining suitable risk measures in Section 4.

We define running measures of the available *overcollateralization* (OC) for the various tranches:

$$L_t^i = \frac{\tilde{N}_t}{\sigma_t^i}. \tag{10}$$

In analogy with current practice, we used in the numerator an *adjusted outstanding notional*

$$\tilde{N}_t = h\, N_t + r_t, \tag{11}$$

in order to properly reflect the current leverage. Hence any risk-free cash (available in the form of current recoveries r_t) is added to the adjusted notional. We introduced h as a haircut applied to the outstanding notional. The purpose of this haircut is to provide protection against a market value decline of deteriorating credits. For simplicity in the calculations presented here we will assume that $h = 1$.[4]

In addition to the leverage considerations above, we define interest coverage ratios as running measures of the debt service payment ability. This is defined as the ratio of the *interest coverage amount* (IC), $\tilde{w}_t = w_t - f_s$ (defined as the current interest proceeds minus senior fees), over the cumulative scheduled interest payments down to the tranche under consideration:

$$l_t^i = \frac{\tilde{w}_t}{\kappa_t^i}, \tag{12}$$

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[4] In practice this haircut is applied only to the most deteriorated credits in collateral pool, i.e., the part that would be expected to sell at a discount to par value due to downward migration. In the current aggregate portfolio model haircuts cannot be applied consistently. In addition, it is the *current* collateral rating that determines the haircut, which requires a dynamic portfolio model.

where we have defined the *supported debt service level*, κ_t^i

$$\kappa_t^i = \sum_{j=1}^{i} c^j T_t^j = \sum_{j=1}^{i} S_t^j. \tag{13}$$

The cashflow structure is distinguished by the fact that the running leverage and debt service ratios (L_t^i, l_t^i) are checked against pre-specified (and constant) triggers or barriers (L_B^i, l_B^i). Depending on the outcome of those comparisons, the indenture may modify the manner in which interest and principal payments are distributed to the various notes and equity. For the sake of simplicity we assume here that each tranche has an associated OC/IC pair of tests, whereas in practice several tranches may share a pair of tests. We also assume, as is usual in practice, that the two types of tests are always examined in pairs and only the joint outcome affects the waterfall. At each period, each tranche's OC test status is given by the indicator variable

$$O_t^i = 1_{\{L_t^i > L_B^i\}}. \tag{14}$$

Similarly, each tranche's IC test status is given by the indicator variable

$$I_t^i = 1_{\{l_t^i > l_B^i\}}. \tag{15}$$

Finally, we are interested in the joint indicator

$$J_t^i = 1_{\{L_t^i > L_B^i, l_t^i > l_B^i\}}. \tag{16}$$

If the i-th OC test is failing during a period, the payment waterfall may require that at that point principal is (re)paid sequentially to all notes senior to or equal the seniority of the test level, until the failing i-th test is met.[5] For each OC test we define the "supportable" notional size Q_t^i:

$$Q_t^i = \frac{\tilde{N}_t}{L_B^i}. \tag{17}$$

This is the (reduced) size of the various tranches that will restore the OC indicator for this period). The *scheduled notional reduction* of the k-th tranche for the curing of the i-th OC test will be given (recursively, starting from the senior-most tranche) by the expression

$$\delta T_t^{ki} = max(min(\sigma_t^i - \sum_{m=1}^{k-1} \delta T_t^{mi} - Q_t^i, T_t^k), 0). \tag{18}$$

The meaning of the above expression is that the currently supported debt level σ_t^i must be reduced iteratively by pursuing all possible tranche reductions, starting from the senior-most $(i = 1)$ tranche.

Similarly, if the i-th IC test is failing, the indenture requires that principal is repaid sequentially to all senior notes until the test is met. We define the supportable tranche notional for each IC test:

$$R_t^i = \frac{\tilde{w}_t}{l_B^i}. \tag{19}$$

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[5] E.g., if the second (Class B) OC/IC test fails, we need to repay notional to the class A and Class B notes

The *scheduled debt service reduction* of the k-th tranche, for the curing of the i-th IC test will be given (recursively) by the expression

$$\Delta P_t^{ki} = max(min(\kappa_t^i - \sum_{m=1}^{k-1} \Delta P_t^{mi} - R_t^i, S_t^k), 0). \tag{20}$$

Reducing the debt service can only be achieved via a corresponding notional reduction of the notes. Using the tranche coupon, we calculate the required notional reduction as of the k-th tranche, for the curing of the i-th IC test:

$$\Delta T_t^{ki} = \frac{\Delta P_t^{ki}}{c^k}. \tag{21}$$

Due to the customary joint application of the OC/IC tests, we combine the reduction requirements δT_n^{ki} and ΔT_t^{ki} into:

$$\partial T_t^{ki} = max(\delta T_t^{ki}, \Delta T_t^{ki}). \tag{22}$$

Obviously the joint required reduction ∂T_t^{ki} can only be achieved up to the available funds as per the cashflow waterfall. It will be useful here to introduce the quantity

$$M_t^i = \sum_{m=1}^{i} \partial T_t^{mi} \tag{23}$$

which denotes the total required debt reduction per test. As proceeds are applied sequentially to the various tranches to achieve the reduction, the updated variables M_t^i control whether the curing of the tests was successful or not.[6]. Formally this is achieved with the indicator function C_t^i

$$C_t^i = 1_{\{M_t^i = 0\}}. \tag{24}$$

With the definitions given in the previous subsection we are ready to provide a precise, even if high level definition of the cashflow waterfall. For conciseness we do use the indicator functions we defined above. A more detailed description is given in Appendix B.2.

For all payment periods before maturity:

- Collect current period payments
- Pay senior fees
- Pay senior tranche interest
- Loop over mezzanine tranches ($i < M$)
  - If J_t^i is true, pay scheduled interest to the $i + 1$ tranche, proceed to the next tranche
  - If J_t^i is false, attempt to amortize senior tranches to the required level
    - If C_t^i is true, pay scheduled interest, proceed to next tranche

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[6] In the evaluation of the required debt reduction for junior tests we always incorporate previous debt reductions for more senior tests

    - $*$ If C_t^i is false, defer interest on junior tranches, proceed to next period

- If J_t^M is true, pay mezzanine fees, pay equity dividend, proceed to next period.
- If J_t^M is false, attempt to amortize senior tranches to the required level
  - If C_t^M is true, pay mezzanine fees, pay equity dividend, proceed to next period.
  - If C_t^M is false, proceed to next period

4 Consistent CDO Risk Measures

In credit risk modelling of corporate bonds or loans, it is market practise to estimate default probability (PD) and loss given default (LGD). For a credit risk analysis of cash CDOs we are interested primarily in the probability of experiencing a loss, rather than the probability of default. The probability of loss for each tranche can be derived once all final cash distributions have been computed (intermediate losses may be remedied). This is given by

$$PD^i = \int_{-\infty}^{+\infty} 1_{\{b_T^i(z) < S_T^i(z)\}} n(z) dz \tag{25}$$

where $b_T^i(z)$ is the final coupon and principal payment. Once we know the final scheduled payment $S_T^i(z)$ (which, depending on the default path, includes notional and any deferred interest) we simply check whether $b_T^i(z)$ is enough to make this payment, else the tranche experiences a loss event.

Apart from the probability of not receiving promised cash flows, we are interested in the expected amount of loss (i.e., loss given loss). We discount cash flows received at different times with risky discount factors DF_t including the credit spread on the tranche. Before defining the LGD, the loss rate $LR^i(z)$ is defined as:

$$LR^i(z) = 1 - \frac{\sum_{t=1}^{T} b_t^i(z) DF_t}{\sum_{t=1}^{T} S_t^i(z) DF_t}, \tag{26}$$

where

$$DF_t = \frac{1}{(1+s_i)^t}.$$

The LGD^i can then be defined as

$$LGD^i = \frac{\int_{-\infty}^{+\infty} LR^i(z) n(z) dz}{PD^i},$$

whereas the Tranche Expected Loss is defined as

$$\begin{aligned} EL^i &= E[LR^i] \\ &= \int_{-\infty}^{+\infty} LR^i(z) n(z) dz \\ &= LGD^i \cdot PD^i \end{aligned}$$

$Var[LR^i]$ is then defined to be the Tranche Loss Volatility and $Var\left[LR^i|LR^i > 0\right]$ the LGD Volatility.

$$Var[LR^i] = \int_{-\infty}^{+\infty} LR^i(z)^2 n(z)dz - E[LR^i]^2 \tag{27}$$

$$Var\left[LR^i|LR^i > 0\right] = \int_{-\infty}^{+\infty} LR^i(z)^2 \frac{n(z)}{PD^i}dz - \left(LGD^i\right)^2. \tag{28}$$

When we refer to the LGD distribution, we mean the conditional distribution LR^i conditioned on $LR^i > 0$.

5 Analysis of representative cashflow structure

We illustrate here the proposed approach on a realistic example.

5.1 Description structure

The asset portfolio consists of B+ rated assets with a spread of 300 bps and a recovery rate of 40%. The senior fees are 20 bps. The mezzanine fees are 45 bps. There are four tranches. The class A note is considered to be a senior tranche in the waterfall structure. The structure looks as follows

	Lower Bound	Upper Bound	Spread (bps)	OC Triggers	IC Triggers
Class A	22.08%	100%	10	120%	120%
Class B	18.05%	22.08%	25	111.4%	110%
Class C	12.2%	18.05%	100	104%	105%
Class D	7.42%	12.2%	318	103.4%	100%

The transaction maturity is 7 years.

5.2 Credit risk measures

We illustrate here the main credit risk measures derived for this structure:

	Rating	PD	LGD	LGD Volatility	EL
Class A	AAA	0.39%	4.17%	3.52%	0.02%
Class B	AA	1.01%	63.67%	36.92%	0.64%
Class C	BBB	4.00%	53.89%	36.72%	2.16%
Class D	BB	12.71%	61.21%	32.92%	7.78%

As expected, the most senior tranche, has a high rating, low PD and a low LGD. We observe in addition that the LGD volatility of the most senior tranche is much lower than the LGD volatility of lower tranches. We observe also that the LGD Volatility of all mezzanine tranches is of similar order of magnitude.[7]

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[7] This is an interesting contrast with the corporate bond market, where the recovery amount is an independent risk factor. For a tranche, all the moments of the LGD distribution can (in theory) be modeled rather directly

5.3 Comparison with the stress scenario approach

The advertised difference in our analysis compared to standard stress scenarios is that we input the full default distribution into the cash flow model. This results in a full cash flow distribution (instead of a set of stressed cash flow scenarios per tranche). While a judicious choice of waterfall mechanics can maximize the observed differences, we benchmark our results using a typical structure. In order to perform a meaningful comparison we re-interpret the stress scenario methodology within the quantitative framework established here. This is done as follows: Using the Vasicek default model we determine a set of stressed default scenarios. More specifically, for each period we use the quantiles of the Vasicek default model which correspond to the default probabilities of the various rating classes. Dependent on the initial rating of the portfolio this leads to stressed default scenarios. We input these scenarios in our cash flow model. The severest stressed default scenario a tranche can handle without losing scheduled cash flows determines the rating of this tranche.

The stressed default scenarios (for the complete table including rating modifiers "+" and "-", we refer to Appendix A.2) correspond to the following default probabilities:

Quantiles	1 year	2 years	3 years	4 years	5 years	6 years	7 years
AAA	48.25%	58.09%	63.41%	66.54%	68.41%	69.50%	70.08%
AA	38.12%	48.68%	54.89%	58.76%	61.21%	62.75%	63.67%
A	36.79%	46.68%	52.38%	55.87%	58.02%	59.32%	60.06%
BBB	33.45%	41.23%	45.82%	48.71%	50.56%	51.72%	52.43%
BB	16.80%	23.45%	28.09%	31.42%	33.83%	35.58%	36.87%

Note that the AAA quantile scenario is the severest stress scenario. The stress scenarios could be compared with the expected default scenario[8], which is:

Expected	1 year	2 years	3 years	4 years	5 years	6 years	7 years
B+	3.67%	7.53%	11.08%	14.12%	16.66%	18.74%	20.44%

For concreteness, we compare the derived probability of loss (PD^i) with the historical default probabilities and infer a "rating". E.g., if the probability of loss of the Class A notes matches the corresponding cumulative default probability of the AAA rating class, we assign a AAA rating. We obtain the following results:

CDO notes	Full cashflow analysis	Stress scenario analysis
Class A	AAA	AA+
Class B	AA	AA-
Class C	BBB	BBB-
Class D	BB	BB

We observe that the full cashflow and stress scenario results are broadly similar for a typical structure. This reflects the underlying seniority based prioritization of cashflows in such structures. Nevertheless, there are measurable differences, with the full cashflows framework producing slightly higher ratings (one notch difference) compared to the traditional approach.

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[8] The stress scenarios are quantile scenarios corresponding to the cumulative default rates of the different ratings. The quantiles are substantially affected by the skew of the distribution compared to the mean.

6 Conclusions

The continuous expansion of the structured credit market requires ever more incisive analytical tools. Much of the motivation for further modelling development comes from the credit derivatives area, where contracts are usually restricted to relatively simple cashflow patterns. The advent of CDS on non-corporate securities and the need to better understand the credit risk of cash CDO motivates a closer look at the quantitative analysis of cash structures. In this direction, we introduced an approach that significantly facilitates this analysis. At the computational level, the method illustrates that one can combine the widely used "conditional independence" (or factor models) for portfolio credit risk with very detailed cashflow calculations (in particular cashflow redirections triggered by breaching leverage thresholds). In essence we showed by construction that the path-dependency of the basic CLO structure can be captured in a few state variables (the various indicators) linked to the cumulative loss rate. The entire calculation is then reduced to simple one-dimensional integration, which is executed almost instantaneously in current computer hardware.

We defined and computed key risk measures applying to CDO tranches utilizing the full relevant cash flow distribution. Those measures are the probability of loss, the LGD but also the LGD volatility. For a typical structure, the first measure is in broad agreement with a scenario based approach, but there are nevertheless measurable (one notch level) differences. The latter two measures cannot be computed consistently in a scenario framework.

We presented results using a widely adopted single factor model, and a stylized cash flow scheme representative of a static CLO transaction. In the current paper we do not pursue an exhaustive analysis of the impact and interaction of structural elements with the portfolio model but leave this for future work.

A key advantage of a scenario based approach that is not yet incorporated here is that it allows flexibility in specifying the timing (inter-temporal distribution) of a given number of defaults. In future research we aim to incorporate this aspect in a consistent manner. Relaxing the other simplifying assumptions of this study (e.g., the static portfolio assumption) may require portfolio models that are too complex to be captured in semi-analytic form.

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A Appendix: Vasicek model

A.1 Cumulative default probabilities

Cum. PD	1 year	2 years	3 years	4 years	5 years	6 years	7 years
AAA	0.02%	0.06%	0.12%	0.19%	0.28%	0.39%	0.52%
AA+	0.02%	0.07%	0.14%	0.24%	0.36%	0.50%	0.66%
AA	0.11%	0.24%	0.39%	0.57%	0.76%	0.97%	1.20%
AA-	0.14%	0.29%	0.46%	0.66%	0.88%	1.11%	1.37%
A+	0.14%	0.30%	0.50%	0.73%	0.98%	1.26%	1.57%
A	0.14%	0.32%	0.54%	0.81%	1.11%	1.45%	1.81%
A-	0.14%	0.36%	0.63%	0.96%	1.33%	1.74%	2.17%
BBB+	0.22%	0.53%	0.91%	1.35%	1.84%	2.37%	2.92%
BBB	0.22%	0.64%	1.18%	1.81%	2.50%	3.21%	3.94%
BBB-	0.54%	1.36%	2.32%	3.34%	4.39%	5.42%	6.41%
BB+	1.67%	3.32%	4.92%	6.44%	7.87%	9.19%	10.41%
BB	2.77%	5.26%	7.50%	9.49%	11.25%	12.82%	14.20%
BB-	2.79%	5.67%	8.38%	10.83%	12.97%	14.83%	16.44%
B+	3.67%	7.53%	11.08%	14.12%	16.66%	18.74%	20.44%
B	8.59%	14.51%	18.59%	21.45%	23.49%	25.00%	26.15%
B-	9.56%	16.63%	21.56%	24.96%	27.32%	28.99%	30.21%
CCC+	14.69%	23.40%	28.70%	32.02%	34.20%	35.69%	36.76%
CCC	19.82%	30.18%	35.83%	39.09%	41.08%	42.39%	43.32%
CCC-	46.55%	53.45%	57.22%	59.39%	60.72%	61.60%	62.21%
D	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%

A.2 Stressed default scenarios corresponding to B+ portfolio

Stressed Quantile Cum. PDs	1 year	2 years	3 years	4 years	5 years	6 years	7 years
AAA	48.25%	58.09%	63.41%	66.54%	68.41%	69.50%	70.08%
AA+	48.22%	57.25%	62.17%	65.09%	66.84%	67.85%	68.37%
AA	38.12%	48.68%	54.89%	58.76%	61.21%	62.75%	63.67%
AA-	36.80%	47.35%	53.61%	57.54%	60.03%	61.59%	62.53%
A+	36.79%	47.01%	53.00%	56.72%	59.06%	60.50%	61.36%
A	36.79%	46.68%	52.38%	55.87%	58.02%	59.32%	60.06%
A-	36.38%	45.75%	51.13%	54.42%	56.46%	57.69%	58.39%
BBB+	33.46%	42.68%	48.07%	51.41%	53.50%	54.77%	55.52%
BBB	33.45%	41.23%	45.82%	48.71%	50.56%	51.72%	52.43%
BBB-	27.51%	35.06%	39.70%	42.73%	44.76%	46.10%	46.99%
BB+	20.07%	27.45%	32.38%	35.79%	38.17%	39.84%	41.02%
BB	16.80%	23.45%	28.09%	31.42%	33.83%	35.58%	36.87%

B Appendix: Generic Cashflow Payments

In this appendix we define with the necessary precision the payments waterfall. This waterfall reflects the main stylized features of contemporary cash CLO structures. Some adaptations/simplifications were necessary.

B.1 *Payment* Function

In order to specify the waterfall concisely, it is helpful to introduce a function *Payment* which takes three arguments:

- *Source*: The source (account) from which the scheduled amount is to be paid.
- SA: Scheduled payment amount.
- PA: Total paid amount.

The arguments are updated within the function as follows:

- If $Source \geqslant SA$ then the total scheduled amount is paid:
  - $PA := PA + SA$ (Total paid amount is updated.)
  - $Source := Source - SA$ (The source is updated for the payment and there might be still something available for other payments).
  - $SA := 0$ (This indicates that the scheduled amount has been successfully made).
- If $Source < SA$ then:
  - $PA := PA + Source$ (Only the total available source is paid.)
  - $SA := SA - Source$ (Part of the scheduled amount is paid, the remainder is still scheduled for other available sources).
  - $Source := 0$ (The source is fully exhausted. No other payments can be made anymore from this source).

B.2 Waterfall

The following is a description of a generic interest/principal proceeds waterfall. For specific transactions there can be substantial variations in structure.

1. *Initialization of accounts:* For each period before maturity, we track the flow of the current interest proceeds i_t and the current principal proceeds p_t. The size of the interest proceeds account at the end of period and before distributions is given by any leftover interest proceeds from the previous cycle (normally zero) accruing at the risk free rate r, plus any new interest proceeds w_t from the collateral. Similarly, the value of the principal proceeds account at the end of period and before distributions is given

by any leftover principal proceeds from the previous cycle (normally zero) accruing at the risk free rate, plus any new principal proceeds r_t from the collateral. I.e.,

$$\begin{aligned} i_t &= (1+r)i_{t-1} + w_t\,, \quad (29)\\ p_t &= (1+r)p_{t-1} + r_t\,. \quad (30)\end{aligned}$$

Next, the payment and reserve accounts are initialized at 0:

$$\begin{aligned} f_t^s &= 0\\ f_t^m &= 0\\ b_t^i &= 0\\ a_t &= 0. \end{aligned}$$

2. *Calculate scheduled payments* SF_t^s, SF_t^m, S_t^i:
    - $SF_t^s = f^s N_t$
    - $SF_t^m = f^m N_t$
    - $S_t^i = (r + s_i)\, T_t^i$.

3. *Payment of senior fees:* Use the available interest proceeds account i_t to pay the senior fees SF_t^s. If there is a shortfall in senior fees payment, use the principal proceeds account p_t to compensate

$$\begin{aligned} &Payment\left(SF_t^s, i_t, f_s^t\right) \quad (31)\\ &Payment\left(SF_t^s, p_t, f_s^t\right). \quad (32)\end{aligned}$$

4. *Payment of senior bonds:* Use any remaining interest proceeds i_t to pay scheduled interest to senior notes. At this point the interest waterfall receives funds from the principal waterfall to make up (if possible) for any shortfall in paying the senior notes coupon. The size of the actual payment affects the relevant tranche sizes and accounts as follows: Increment the senior tranche notional T_t^1 with any interest payment shortfall. This means that while we allow senior tranches to default only at maturity, we do keep track of missed coupons and their accrued interest.

$$\begin{aligned} &Payment\left(S_t^1, i_t, b_t^1\right) \quad (33)\\ &Payment\left(S_t^1, p_t, b_t^1\right) \quad (34)\end{aligned}$$

$$T_t^1 := T_t^1 + S_t^1 \quad (35)$$

5. *Mezzanine payments loop:* In this portion of the waterfall, a loop of identical operations is performed for every OC/IC test $i = 1, \ldots, M-1$. For each test, check the i-th joint indicator J_t^i. [9].

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[9] The last OC/IC test (i=M) will lead to equity instead of bond payments and is hence treated separately

(a) If $J_t^i = 1$, pay scheduled interest to the $i+1$ notes from i_t. (So the Class B notes will receive payment if the Class A OC/IC test is PASS).

$$Payment\left(S_t^{i+1}, i_t, b_t^{i+1}\right) \tag{36}$$

$$T_t^{i+1} := T_t^{i+1} + S_t^{i+1} \tag{37}$$

Once the above payments have been made, the loop proceeds to the next lower OC/IC test.

(b) If $J_t^i = 0$, we enter into OC/IC "cure mode". In the first instance we use interest proceeds i_t, to sequentially amortize principal on notes down to i-th note, and then attempt to meet any shortfall using notional reduction from principal proceeds p_t or the reserve account a_t. In sequence for each of the notes $j = 1, \ldots, i$ this leads to the updates. Introduce also a temporary variable $\widetilde{b}_t^j$:

$$\widetilde{b}_t^j = 0 \tag{38}$$

$$Payment\left(\partial T_t^{ji}, i_t, \widetilde{b}_t^j\right) \tag{39}$$

$$Payment\left(\partial T_t^{ji}, p_t, \widetilde{b}_t^j\right) \tag{40}$$

$$Payment\left(\partial T_t^{ji}, a_t, \widetilde{b}_t^j\right) \tag{41}$$

$$b_t^j := b_t^j + \widetilde{b}_t^j \tag{42}$$

$$T_t^j := T_t^j - \widetilde{b}_t^j \tag{43}$$

We next check the indicator C_t^i, which indicates whether there were enough funds to achieve the necessary reductions.

i. If $\sum_{j=1}^{i} \partial T_t^{ji} = 0$ and $C_t^i = 1$, the required notional reduction has been achieved. Now the waterfall reverts back to Point 5a of the Mezzanine portion above.

ii. If $\sum_{j=1}^{i} \partial T_t^{ji} = 0$ and $C_t^i = 0$, the required reduction was not successful. We now have exhausted all funds and simply defer interest on notes from $(i+1)$ onwards, while equity receives no dividend this period. Hence for $j = i{+}1, \ldots, M$

$$b_t^j = 0 \tag{44}$$

$$T_t^j = T_t^j + S_t^j \tag{45}$$

$$d_t = 0 \tag{46}$$

With those adjustments, we proceed to the next period (Point 1).

6. *Payment of mezzanine fees:* Use the available interest proceeds account i_t to pay the mezzanine fees f_m^t.

$$Payment\left(SF_t^m, i_t, f_m^t\right) \tag{47}$$

7. *Payment of Equity dividends:* In this portion of the waterfall, passing the junior most OC/IC test ($i = M$) will lead to equity payments.
   (a) If $J_t^M = 1$, pay any remaining interest proceeds i_t to equity and amortize any remaining principal proceeds to notes

   $$Payment\,(i_t, i_t, d_t)$$

   $$\begin{aligned} a_t &: = (1+r)\,a_{t-1} + p_t \\ p_t &: = 0. \end{aligned}$$

   The payment waterfalls is now complete and we revert to the next period (Point 1).

   (b) If $J_t^M = 0$, similarly to the mezzanine loop, we use any remaining i_t to sequentially amortize *all* the notes, until the test is cured. Hence for each of the notes $j = 1, \ldots, M$

   $$\widetilde{b}_t^j = 0 \tag{48}$$

   $$Payment\left(\partial T_t^{jM}, i_t, \widetilde{b}_t^j\right) \tag{49}$$

   $$b_t^j := b_t^j + \widetilde{b}_t^j \tag{50}$$

   $$T_t^j := T_t^j - \widetilde{b}_t^j. \tag{51}$$

   We now check whether the M-th (junior) OC/IC cure has been successful. We check the indicator C_t^M

   i. If $C_t^M = 1$, the required notional reduction has been achieved. The waterfall reverts back to Point 7a.

   ii. If $C_t^M = 0$, the required reduction was not successful. Equity receives no payment this period.

   $$d_t := 0 \tag{52}$$

   We now have exhausted all funds and simply move on to the next period (Point 1).

During the final period ($t = T$), all notes receive final scheduled coupon and principal (if possible).

- Accumulate the interest rate and principal proceeds into one account a_T

$$a_T := (1+r)\,a_{T-1} + N_T + i_T + p_T \tag{53}$$

- For all tranches $j = 0, \ldots, M$, attempt sequential repayment of principal and accrued coupons

$$Payment\left(S_T^j, a_T, b_T^j\right)$$

- Any remaining funds accumulate to the equity interest.

$$d_T = a_T. \tag{54}$$

Figure 3: Interest waterfall

In the figure we demonstrate how the interest waterfall flows through the cash flow CDO structure before maturity.

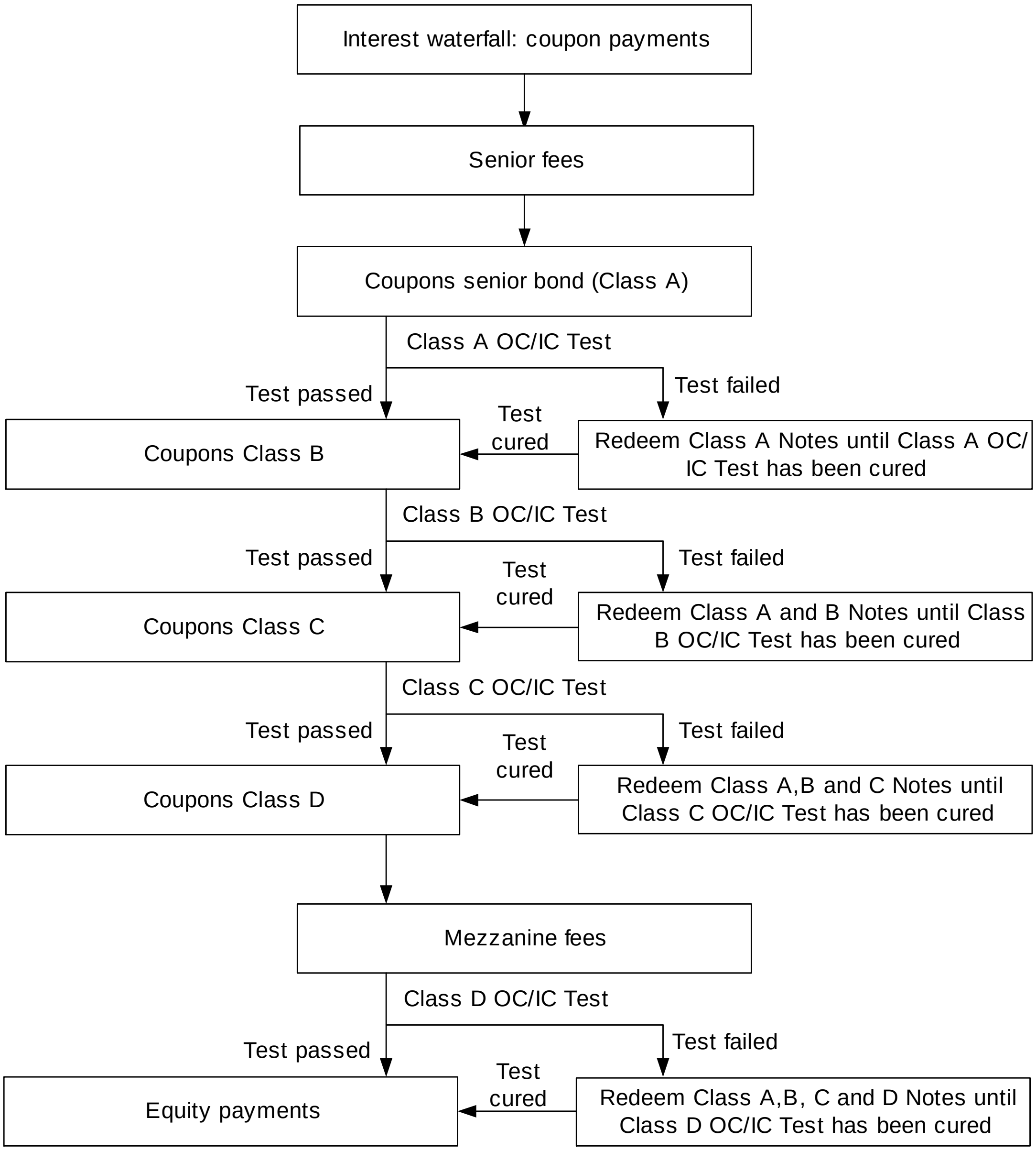
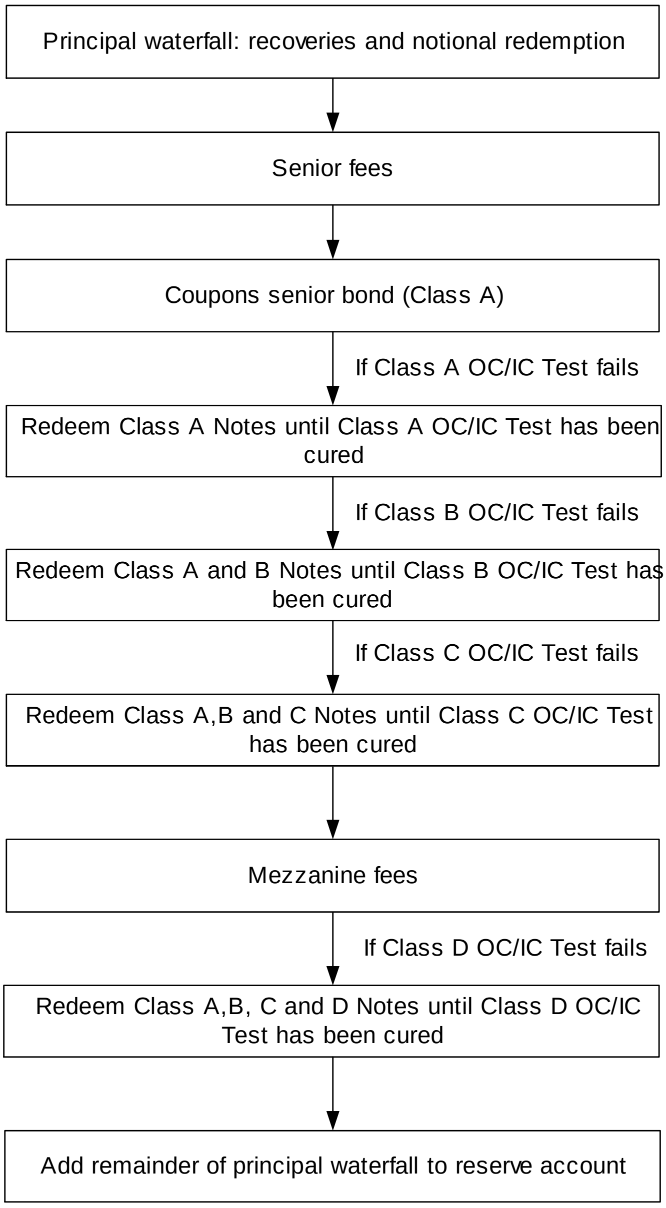


Figure 4: Principal waterfall before maturity

In the figure we demonstrate how the principal waterfall flows through the cash flow CDO structure before maturity.

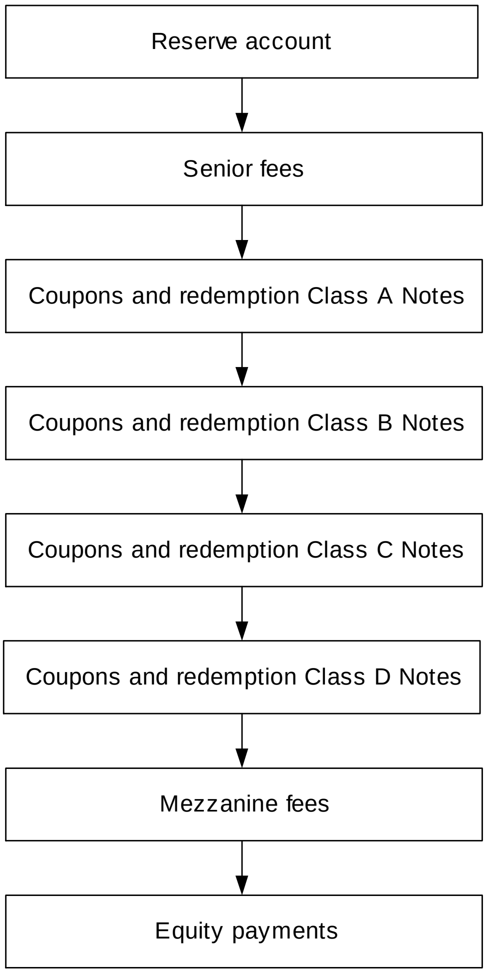


C Diagram of cash flow waterfalls

Before maturity the interest and principal waterfall is presented in figures 3 and 4. At maturity the interest and principal waterfall and the outstanding notional of the portfolio are all accumulated in the reserve account and distributed according to figure 5.

Figure 5: Waterfall at maturity

In the figure we demonstrate how the waterfall at maturity flows through the cash flow CDO structure.



D Notation

Symbol	Definition
q_t	Default curve
g	Recovery
ρ^2	Correlation
t_k	Evaluation time-points
W^j	Asset credit factors
Z	Systematic credit factor
α_t	Default thresholds
λ_t	Hazard rate
N_t	Outstanding notional
T	Maturity
r	Risk free rate
c	(Weighted) average asset coupon
$\bar{s}$	(Weighted) average asset spread
w_t	Interest proceeds
r_t	Principal proceeds
T_0^i	Initial tranche size
T_t^i	Running tranche size
c^i	Tranche coupon
s^i	Tranche spread
S_t^i	Scheduled tranche payment
D_t^i	Deferred tranche payment
b_t^i	Actual tranche payment
E_0	Initial equity size
E_t	Running equity size
i_t	Interest proceeds account
p_t	Principal proceeds account
L_t^i	Overcollateralization ratio (leverage)
$\tilde{N}_t$	Adjusted Notional
L_B^i	Overcollateralization ratio trigger
l_t^i	Interest cover ratio
l_B^i	Interest cover trigger
$\tilde{w}_t$	Interest cover amount
f^s	Senior fees and costs
f^m	Mezzanine fees and costs
SF_t^s	Scheduled senior fee payments
SF_t^m	Scheduled mezzanine fee payments
f_t^s	Actually realised senior fee payments
f_t^m	Actually realised mezzanine fee payments
d_t	Equity payment (dividend)

Symbol	Definition
a_t	Reserve account
I_t^i	Indicator of IC test
O_t^i	Indicator of OC test
J_t^i	Indicator of joint IC/OC test
C_t^i	Indicator of cured IC/OC test
δT_t^{ki}	Scheduled notional reduction of k-th note to cure i-th OC test
ΔT_t^{ki}	Scheduled notional reduction of k-th note to cure i-th IC test
∂T_t^{ki}	Scheduled notional reduction of k-th note to cure joint i-th IC/OC test