On perfect matchings of the 2-by-2-by-n grid graph

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1 Introduction

In considering perfect matchings of a 2-by-2-by-n grid graph, we came upon an interesting fact that each number of the sequence was either a perfect square or twice a perfect square. Namely, if n is an odd number, then the number of perfect matchings is twice a square, and if n is an even number, then the number of perfect matchings is a perfect square. The number of perfect matchings for a 2-by-2-by-n grid graph is given by the sequence $\{a\} = \{1, 2, 9, 32, 121, 450, 1681, 6272, \dots\}\ for\ n \geq 0$. This sequence, and some interesting properties of it, can be found on Sloane's Online Encyclopedia of Integer Sequences(A006253). The generating function for the sequence is given by

$$A(w) = \frac{2 + 3w + w^2}{1 - 3w - 3w^2 + w^3} = \sum_{i=0}^{\infty} k_i w^i \tag{1}$$

where k_i is the number of perfect matchings of the 2-*by*-2-*by*-i grid graph.

2 Bijection between perfect matchings of different combinatorial objects

Define $\{e\}$ to be the sequence given by $e_i = \sqrt{a_{2i}}$ for $i \geq 0$. Then $\{e\} = \{1, 3, 11, 41, 153, 571, \dots\}$. Analogously, define a sequence $\{o\}$ to be the sequence given by $o_i = \sqrt{\frac{a_{2i+1}}{2}}$ for $i \geq 0$. Then $\{o\} = \{1, 4, 15, 56, 209, 780, \dots\}$. What is interesting here is that the first terms of each of these sequences seem to define sequences of the matchings of different combinatorial objects. Namely, e_i coincides with the number of perfect matchings of a 3-by-$2i$ grid and o_i coincides with the number of perfect matchings of a 3-by-$(2i + 1)$ grid with a corner vertex removed (it also coincides with the number of spanning trees of the 2-by-i grid). These observances naturally led to the following conjecture.

Conjecture 1 *There exists a nice way to bijectively map that maps each perfect matching of a 2-by-2-by-$2n$ cube snake to an ordered pair of perfect matchings of the 3-by-$2n$ grid. Additionally, there exists a nice way to bijectively map that maps half of the perfect matchings of a 2-by-2-by-$(2n + 1)$ grid to an ordered pair of perfect matchings of the 3-by-$(2n + 1)$ grid for $n \geq 0$*

3 Introducing weighted edges to the cube snake

In attempting to prove conjecture (1), it was natural to try to extract more information from the combinatorics of perfect matchings of the 2-by-2-by-n grid.

Theorem 1 *Consider the cube snake given weights of x, y and z to edges of the cube snake that exist in the conventional axes x, y and z of R^3, respectively. Then the generating function of perfect matchings of the 2-by-2-by-n grid graph involving the formal variables x, y and z is*

$$A(w;x,y,z) = \frac{y^2+z^2+x^2(x^2+y^2+z^2)w - x^6w^2}{1-(x^2+y^2+z^2)w - x^2(x^2+y^2+z^2)w^2 + x^6w^3} \quad (2)$$

Proof: Let $\{a\}$ be the sequence where a_i is the sum of weighted perfect matchings of the 2-by-2-by-i grid graph. Let $\{b\}$ be the sequence where b_i is the sum of weighted perfect matchings of the 2-by-2-by-i grid graph with the two rightmost bottommost vertices removed. Let $\{c\}$ be the sequence where c_i is the sum of weighted perfect matchings of the 2-by-2-by-i grid graph with the two rightmost backmost vertices removed. Then choosing all possibilities of perfect matchings of $\{a\}$, we see that

$$a_n = (y^2+z^2)a_{n-1} + 2x^2zb_{n-1} + 2x^2yc_{n-1} + x^4a_{n-2}.$$

Proceeding analogously for $\{b\}$ and $\{c\}$, we see that

$$b_n = za_{n-1} + x^2b_{n-1}$$

and

$$c_n = ya_{n-1} + x^2c_{n-1}.$$

Now we can let A(w;x,y,z), B(w;x,y,z) and C(w;x,y,z) be the generating functions for $\{a\}$, $\{b\}$ and $\{c\}$, respectively. Now multiply the recurrences for a_n, b_n and c_n by x^n and sum from $n = 0$ to ∞. Solving these three equations simultaneously gives that the generating function for $\{a\}$ is

$$A(w;x,y,z) = \frac{y^2+z^2+x^2(x^2+y^2+z^2)w - x^6w^2}{1-(x^2+y^2+z^2)w - x^2(x^2+y^2+z^2)w^2 + x^6w^3},$$

as you can check.

Notice that the first few coefficients of the terms of the Taylor expansion of $A(w,x,y,z)$ are perfect squares in the polynomials of x, y and z for the even terms and the product of (y^2+z^2) and a perfect square of a polynomial in x, y and z for odd terms. Notice that setting $x = y = z = 1$ gives the generating function (1). This extended version of the generating function should make finding the nice bijective maps of conjecture (1) easier.