Design of Experiments: One Factor and Randomized Block Experiments

Prof. Daniel A. Menasce
Dept. of Computer Science
George Mason University

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Basic Notions in Design of Experiments

- Response: what you want to measure.
- Factor: what affects the response.
- Level: value of a factor.

	Factors			Response
	CPU Clock Frequency (MHz)	Number of CPUs	Main Memory (MB)	Benchmark Execution Time (sec)
	550	1	128	25
	750	1	128	32
	1000	1	128	48
	550	2	128	19
[redacted]	750	2	128	14
	1000	2	128	10
	550	1	256	23
	750	1	256	29
	1000	1	256	45

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Comparing Means of Various Groups

- ANOVA: Analysis of Variance.
- Consider c groups.
- Subdivide total variation in the response into variations attributable to differences among the c groups and differences within the c groups (experimental error).

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- A, B, C, and D are different page replacement algorithms.
- Factor: page replacement algorithm.
- Levels: A, B, C, and D.
- Number in each column: running times of programs under each replacement algorithm.

Page Replacement Algorithm

A	B	C	D
11	12	18	11
13	14	16	12
17	17	18	16
17	19	20	15
15	21	22	14
16	18	15	17
14	19	17	13
10	18	21	16
12	16	16	17
14	18	20	18

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How about the Influence of Uncontrolled and Unforeseen Factors?

- The running time of a program depends on many other factors. Its locality of reference plays a role in the effectiveness of a page replacement algorithm.
- Randomization: consider a large set of programs and randomly assign programs to each group.

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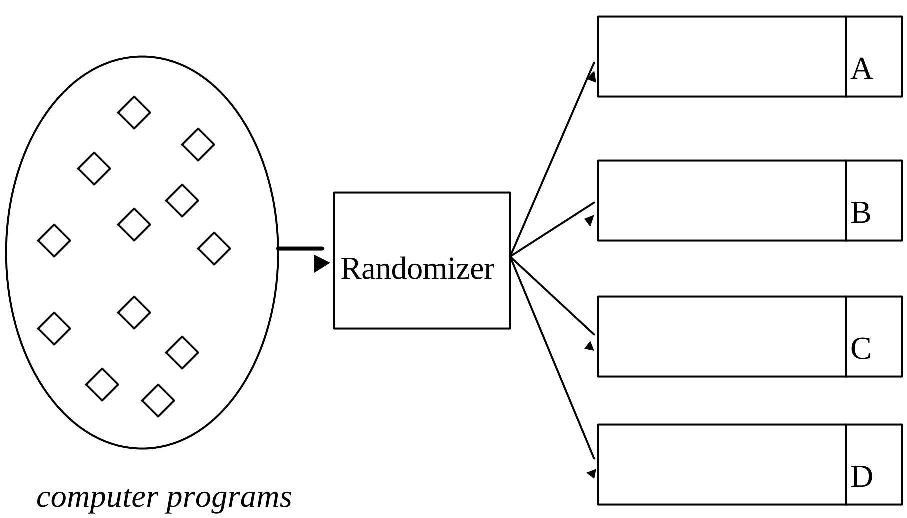
Randomization



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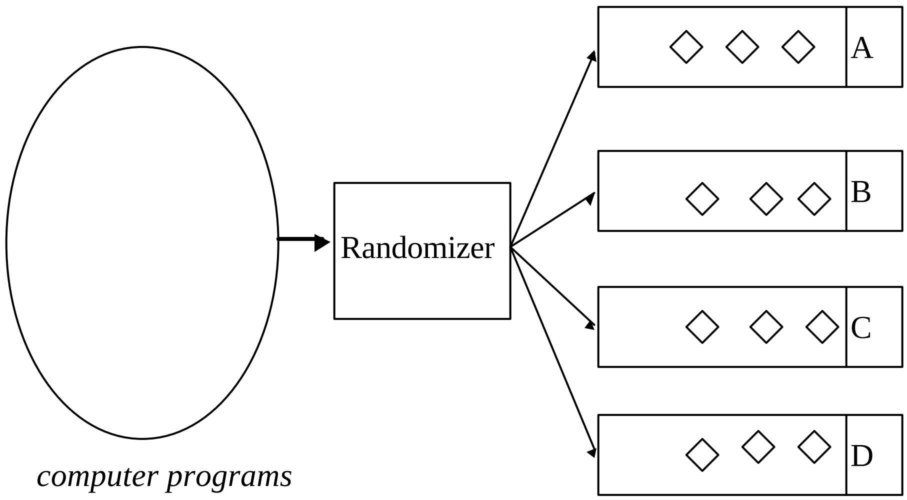
Randomization



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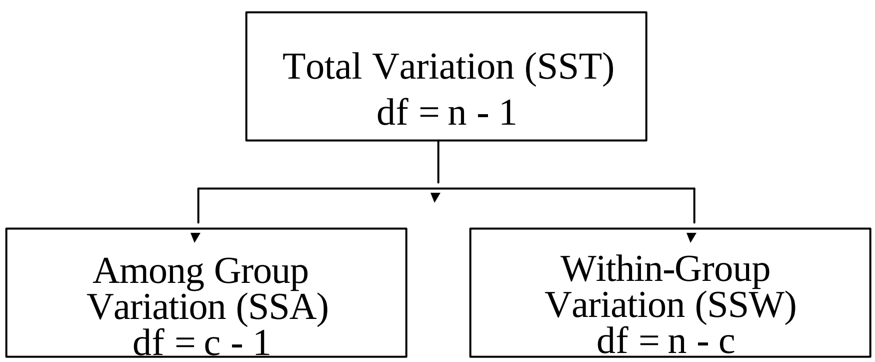
ANOVA Model



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ANOVA

- Assumptions:
  - c groups or levels of the factor being examined represent populations whose outcome measurements are randomly and independently drawn and follow a normal distribution and have equal variances.
- Hypotheses:

$$H_0 : m_1 = m_2 = ... = m_c$$
$$H_1 : \text{not all } m_j \text{ are equal (j} = 1,..., \text{c)}$$

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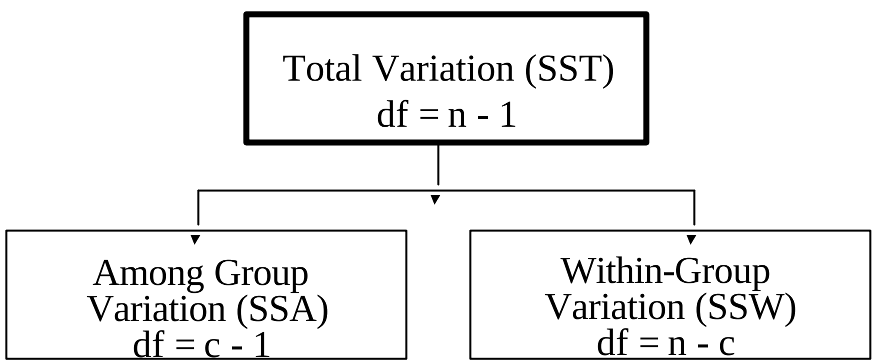
ANOVA Model



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SST (Sum of Squares Total)

$$SST = \sum_{j=1}^{c} \sum_{i=1}^{n_j} \left(X_{ij} - \overline{\overline{X}} \right)^2$$

where

$$\overline{\overline{X}} = \frac{\sum_{j=1}^{c} \sum_{i=1}^{n_j} X_{ij}}{n}$$: overall or grand mean.

X_{ij} : i-th observation in group or level j.

n_j : number of observations in group or level j.

n: total number of observations: $\sum_{j=1}^{c} n_j$

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SST Example

Page Replacement Algorithm

A	B	C	D
11	12	18	11
13	14	16	12
17	17	18	16
17	19	20	15
15	21	22	14
16	18	15	17
14	19	17	13
10	18	21	16
12	16	16	17
14	18	20	18

Grand Mean 16.075

SST = $(11-16.075)^2 + (13-16.075)^2 + \ldots + (18-16.075)^2 =$ 336.75

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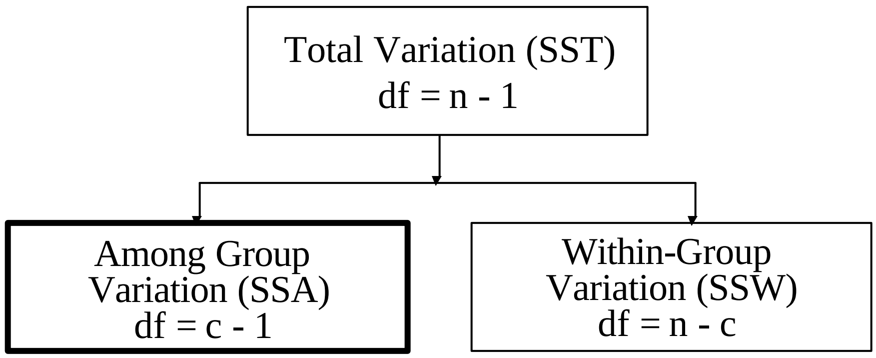
ANOVA Model



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SSA (Sum of Squares Among Groups)

$$SSA = \sum_{j=1}^{c} n_j \left(\overline{X}_j - \overline{\overline{X}} \right)^2$$

where

$$\overline{\overline{X}} = \frac{\sum_{j=1}^{c} \sum_{i=1}^{n_j} X_{ij}}{n}$$: overall or grand mean.

$\overline{X}_j$: sample mean corresponding to group or level j.

n_j : number of observations in group or level j.

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SSA Example

	Page Replacement Algorithm			
	A	B	C	D
	11	12	18	11
	13	14	16	12
	17	17	18	16
	17	19	20	15
	15	21	22	14
	16	18	15	17
	14	19	17	13
	10	18	21	16
	12	16	16	17
	14	18	20	18
Mean	13.9	17.2	18.3	14.9
Grand Mean	16.075			

$$SSA = 10\,(13.9-16.075)^2 + 10\,(17.2-16.075)^2 + 10\,(18.3-16.075)^2 + 10\,(14.9-16.075)^2 = 123.275$$

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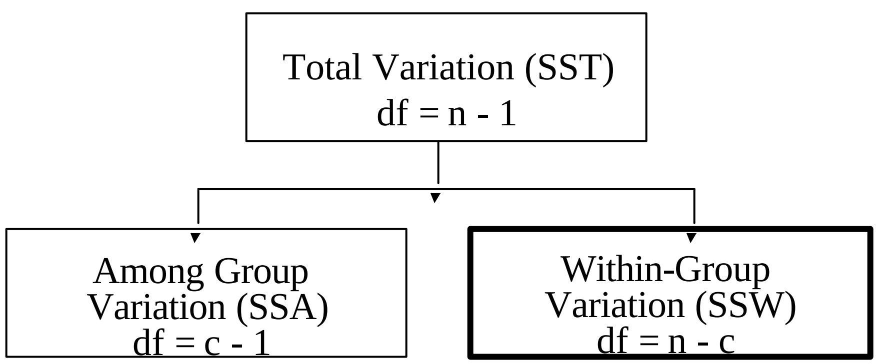
ANOVA Model



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SSW (Sum of Squares Within Groups)

$$SSW = \sum_{j=1}^{c} \sum_{i=1}^{n_j} \left(X_{ij} - \overline{X}_j \right)^2$$

where

X_{ij} : i-th observation in group or level j.

$\overline{X}_j$: sample mean corresponding to group or level j.

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SSW Example

	Page Replacement Algorithm			
	A	B	C	D
	11	12	18	11
	13	14	16	12
	17	17	18	16
	17	19	20	15
	15	21	22	14
	16	18	15	17
	14	19	17	13
	10	18	21	16
	12	16	16	17
	14	18	20	18
Mean	13.9	17.2	18.3	14.9

SSW = $(11-13.9)^2 + \ldots + (14-13.9)^2 +$
$(12-17.2)^2 + \ldots + (18-17.2)^2 +$
$(18-18.3)^2 + \ldots + (20-18.3)^2 +$
$(11-14.9)^2 + \ldots + (18-14.9)^2$
= 213.5

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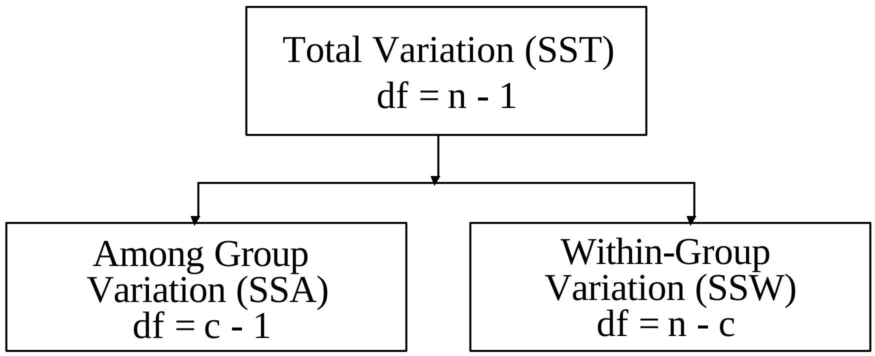
ANOVA Model



$$SST = SSA + SSW$$

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ANOVA Model: Mean Squares

$$MSA = \frac{SSA}{c-1}$$

$$MSW = \frac{SSW}{n-c}$$

$$MST = \frac{SST}{n-1}$$

The mean squares are variances!

If there are no real differences among the c groups, MSA, MSW, and MST provide estimates for the variance inherent in the data.

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The one-way ANOVA F Test Static

$$F = \frac{MSA}{MSW}$$

- The F-test statistic follows an F distribution with c-1 degrees of freedom in the numerator corresponding to MSA and n-c degrees of freedom in the denominator corresponding to MSW.

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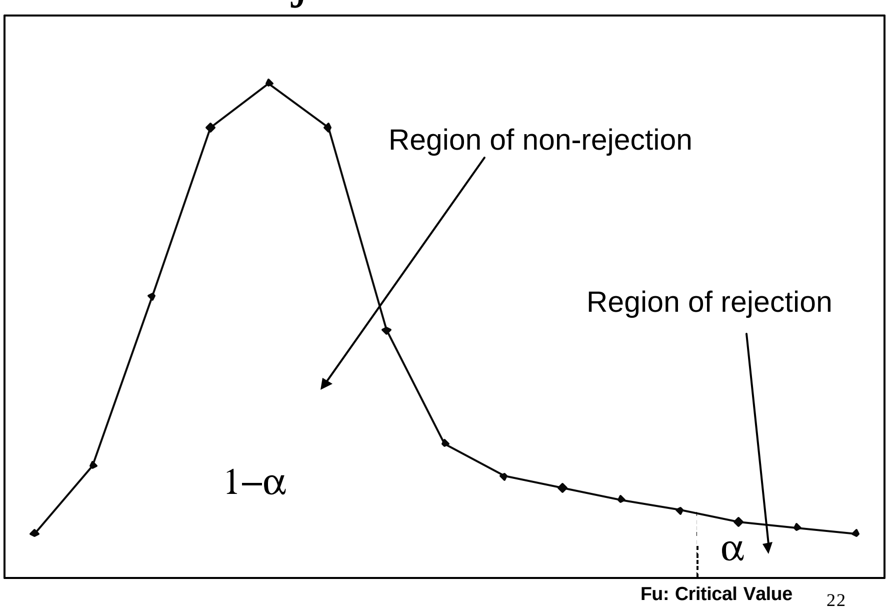
Reject Ho if F > Fu



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ANOVA Summary Table

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square (Variance)	F
Among groups	c-1	SSA	MSA= SSA/(c-1)	F=MSA/MSW
Within Groups	n-c	SSW	MSW=SSW/(n-c)	
Total	n-1	SST		

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ANOVA Example

Page Replacement Algorithm

	A	B	C	D	
	11	12	18	11	
	13	14	16	12	
	17	17	18	16	
	17	19	20	15	
	15	21	22	14	
	16	18	15	17	
	14	19	17	13	
	10	18	21	16	
	12	16	16	17	
	14	18	20	18	
Mean	13.9	17.2	18.3	14.9	
Grand Mean	16.075				
SSA	47.30625	12.65625	49.50625	13.80625	**123.275**
SSW	213.5				
SST	336.775				
MSA	41.091667				
MSW	5.9305556				
F	6.93				
df numer.	3				
df denom.	36				
Fu	2.87	(from table)			

F=MSA/MSW

c-1 = 4-1

n-c = 40-4

F > Fu ⇨ reject Ho. Algorithms A, B, C, and D have a significant difference at 0.05 level of significance.

α

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ANOVA With Excel

Anova: Single Factor

SUMMARY

Groups	*Count*	*Sum*	*Average*	*Variance*
Column 1	10	139	13.9	5.877778
Column 2	10	172	17.2	6.844444
Column 3	10	183	18.3	5.566667
Column 4	10	149	14.9	5.433333

ANOVA

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	123.275	3	41.09167	6.928806	0.000844	2.866265
Within Groups	213.5	36	5.930556			
Total	336.775	39				

Since the p-value is less than $\alpha = 0.05$, reject Ho.

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Multiple Comparisons: The Tukey-Kramer Procedure

- If Ho is rejected, then the question is "Which groups are different?"
- Use the Tukey-Kramer procedure to compare all pairs of groups simultaneously.
- Must compute the differences $\overline{X}_j - \overline{X}_{j'}$ for $j \neq j'$ among all c(c-1)/2 pairs of means.

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Multiple Comparisons: The Tukey-Kramer Procedure

- Obtain the critical range:

$$\text{critical range} = q_u \sqrt{\frac{MSW}{2}\left(\frac{1}{n_j} + \frac{1}{n_{j'}}\right)}$$

where q_u is the upper-tail critical value from a *Studentized range** distribution with c degrees of freedom in the numerator and (n-c) degrees of freedom in the denominator.

* See table A9.

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Multiple Comparisons: The Tukey-Kramer Procedure

- A pair is considered significantly different if the absolute difference between the sample means exceeds the critical range.

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Multiple Comparisons: The Tukey-Kramer Procedure

			critical range	
|XA-XB|	3.3	$>$	2.9341	A significantly different than B
|XA-XC|	4.4	$>$	2.9341	A significantly different than C
|XA-XD|	1	$<$	2.9341	A not significantly different than D
|XB-XC|	1.1	$<$	2.9341	B not significantly different than C
|XB-XD|	2.3	$<$	2.9341	B not significantly different than D
|XC-XD|	3.4	$>$	2.9341	C significantly different than D

qu	3.81	(from table)
MSW	5.930556	

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Multiple Comparisons: The Tukey-Kramer Procedure with PhStat

Tukey Kramer Multiple Comparisons

Group	**Sample Mean**	**Sample Size**
1	13.9	10
2	17.2	10
3	18.3	10
4	14.9	10

Comparison	Absolute Difference	Std. Error of Difference	Critical Range	**Result**
Group 1 to Group 2	3.3	0.770101	2.9341	**Means are different**
Group 1 to Group 3	4.4	0.770101	2.9341	**Means are different**
Group 1 to Group 4	1	0.770101	2.9341	**Means are not different**
Group 2 to Group 3	1.1	0.770101	2.9341	**Means are not different**
Group 2 to Group 4	2.3	0.770101	2.9341	**Means are not different**
Group 3 to Group 4	3.4	0.770101	2.9341	**Means are different**

Intermediate Calculations

MSW	5.930556
Q Statistic	3.81

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Reviewing ANOVA Assumptions

- Randomness and independence: must always be met.
- Normality: ANOVA F test is robust as long as distributions are not extremely different from a normal distribution particularly for large samples.
- Homogeneity of variance: $s_1^2 = s_2^2 = \ldots = s_c^2$
  - If unequal sample sizes between groups, different variances is a problem.
  - Should try to use same-size groups.

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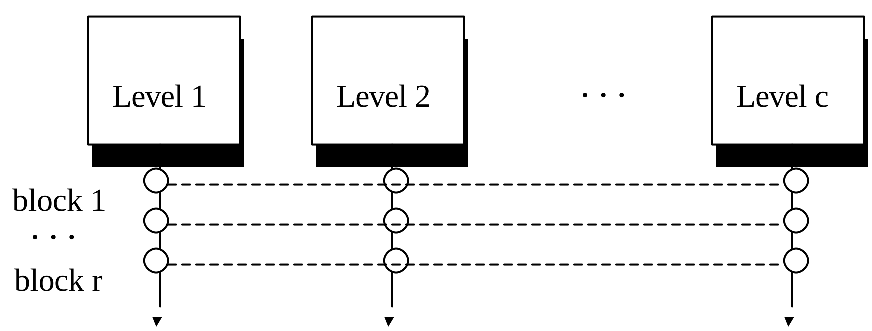
Randomized Block Model



- each block contains the response of the same item to the c levels of the factor being analyzed.
- Purpose: remove as much block or subject variability as possible by reducing experimental error.

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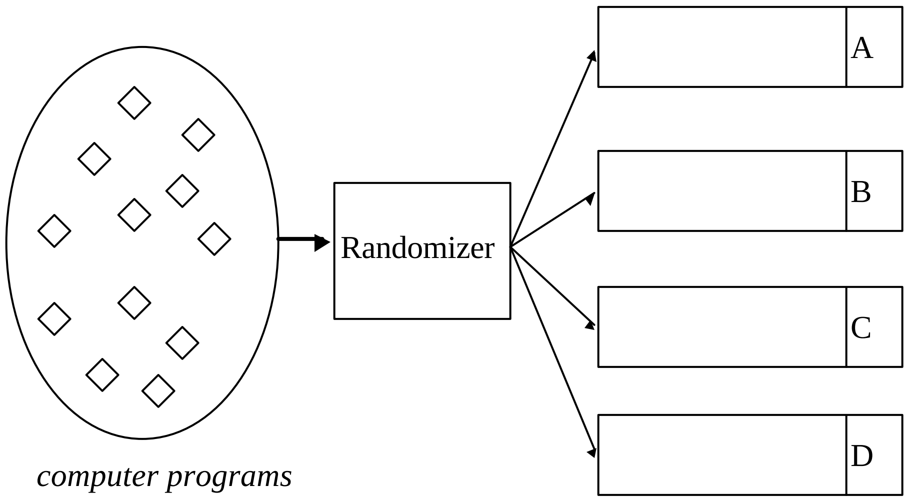
Randomized Block Model



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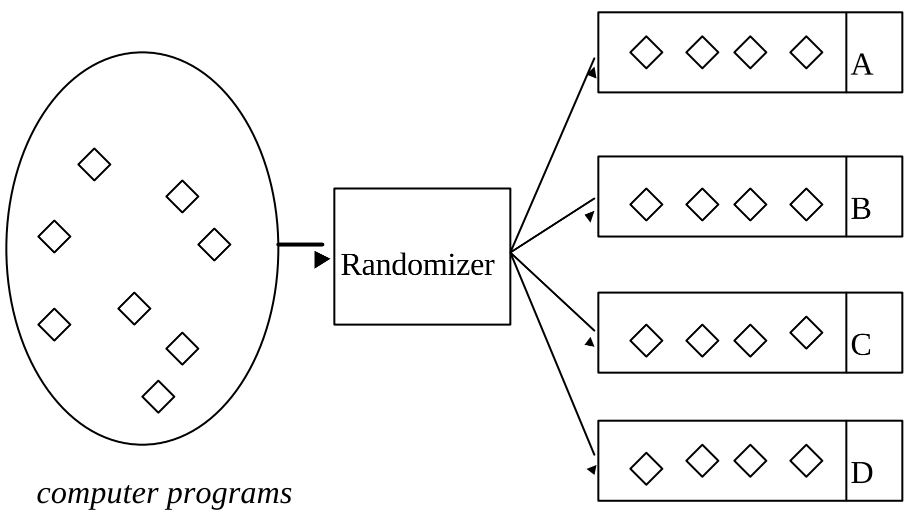
Randomized Block Model



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Page Replacement Algorithm

	A	B	C	D	
block 1	11.0	12.0	18.0	11.0	*program 1*
block 2	13.0	14.0	19.0	12.0	*program 2*
block 3	17.0	18.4	23.4	16.5	*program 3*
block 4	14.0	14.9	20.0	12.5	*program 4*
block 5	15.0	16.0	21.0	13.5	*program 5*

r = 5
c = 4

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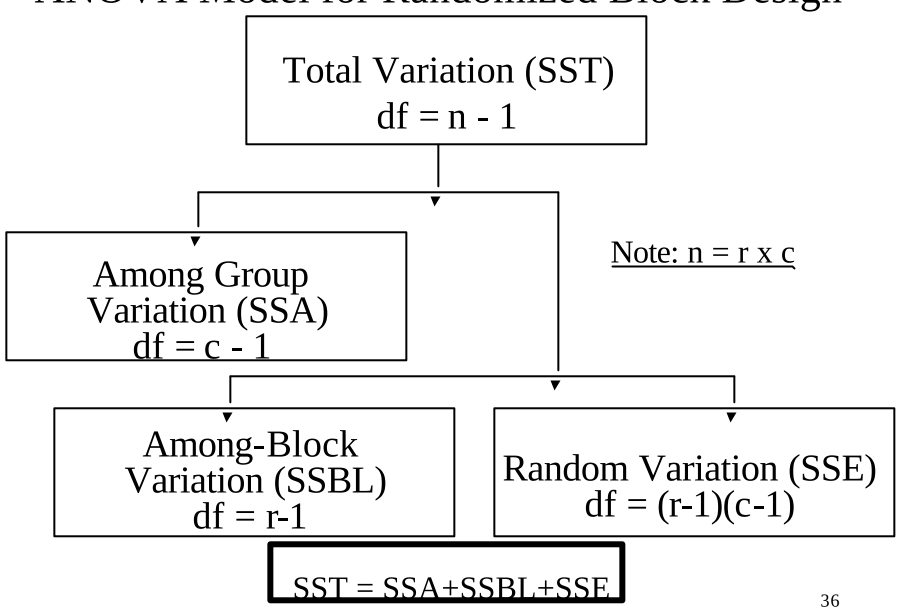
ANOVA Model for Randomized Block Design



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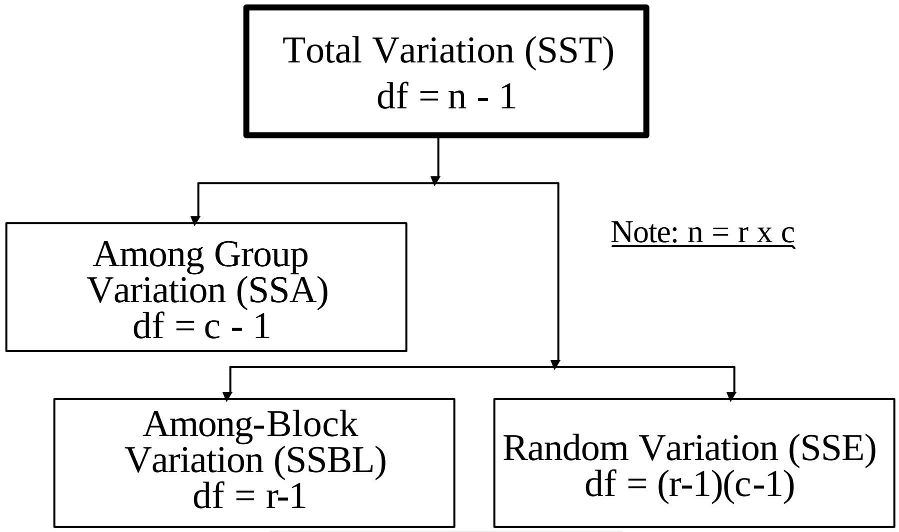
ANOVA Model for Randomized Block Design



Note: n = r x c

SST = SSA+SSBL+SSE

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SST (Sum of Squares Total)

$$SST = \sum_{j=1}^{c}\sum_{i=1}^{r}\left(X_{ij} - \overline{\overline{X}}\right)^2$$

where

$$\overline{\overline{X}} = \frac{\sum_{j=1}^{c}\sum_{i=1}^{r} X_{ij}}{rc}$$: overall or grand mean.

X_{ij} : observation in i-th block and level j.

r : number of blocks.

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	Page Replacement Algorithm				
	A	B	C	D	
block 1	11.0	12.0	18.0	11.0	*program 1*
block 2	13.0	14.0	19.0	12.0	*program 2*
block 3	17.0	18.4	23.4	16.5	*program 3*
block 4	14.0	14.9	20.0	12.5	*program 4*
block 5	15.0	16.0	21.0	13.5	*program 5*

Grand Mean = 15.61

SST = $(11.0-15.61)^2+(13.0-15.61)^2+\ldots+(15.0-15.61)^2+$

…

$(11.0-15.61)^2+(12.0-15.61)^2+\ldots+(13.5-15.61)^2$

= 232.44

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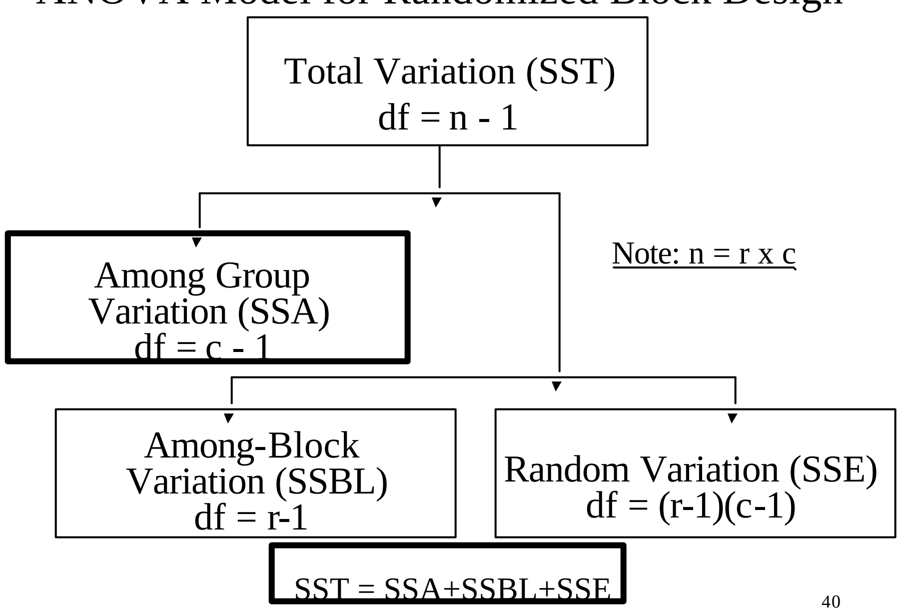
ANOVA Model for Randomized Block Design



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SSA (Sum of Squares Among Group)

$$SSA = r\sum_{j=1}^{c}\left(\overline{X}_{.j} - \overline{\overline{X}}\right)^2$$

where

$$\overline{X}_{.j} = \frac{\sum_{i=1}^{r} X_{ij}}{r}: \text{ group mean.}$$

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Page Replacement Algorithm

	A	B	C	D	
block 1	11.0	12.0	18.0	11.0	*program 1*
block 2	13.0	14.0	19.0	12.0	*program 2*
block 3	17.0	18.4	23.4	16.5	*program 3*
block 4	14.0	14.9	20.0	12.5	*program 4*
block 5	15.0	16.0	21.0	13.5	*program 5*
Mean	14.0	15.1	20.3	13.1	

Grand Mean 15.61

SSA = 5 * [$(14.0-15.61)^2+(15.1-15.61)^2+\ldots+(13.1-15.61)^2$]
= 155.018

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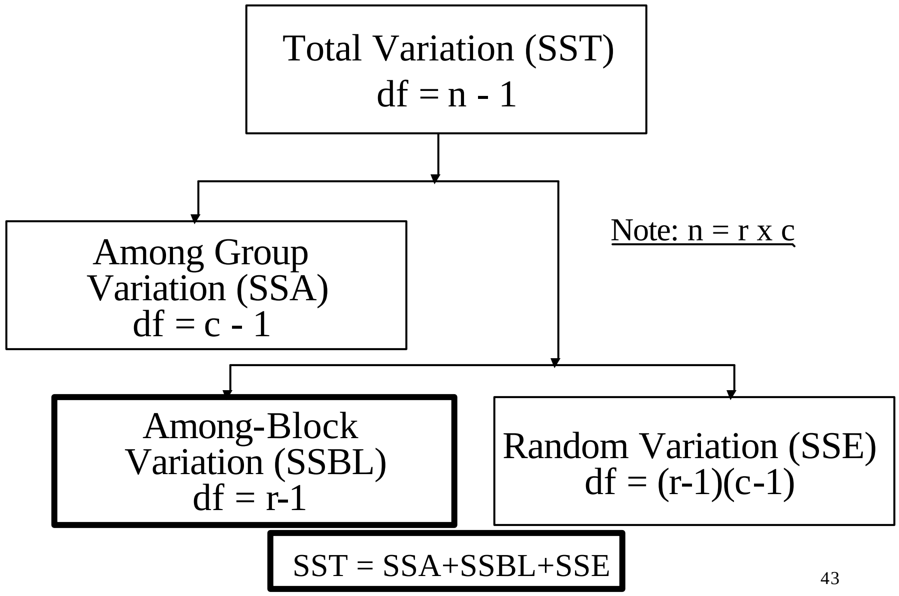
ANOVA Model for Randomized Block Design



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SSBL (Sum of Squares Among Blocks)

$$SSBL = c\sum_{i=1}^{r}\left(\overline{X}_{i.} - \overline{\overline{X}}\right)^2$$

where

$$\overline{X}_{i.} = \frac{\sum_{j=1}^{c} X_{ij}}{c}: \text{ block mean.}$$

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Page Replacement Algorithm

	A	B	C	D		Mean
block 1	11.0	12.0	18.0	11.0	*program 1*	13.0
block 2	13.0	14.0	19.0	12.0	*program 2*	14.5
block 3	17.0	18.4	23.4	16.5	*program 3*	18.8
block 4	14.0	14.9	20.0	12.5	*program 4*	15.4
block 5	15.0	16.0	21.0	13.5	*program 5*	16.4
Mean	14.0	15.1	20.3	13.1		

Grand Mean 15.61

$$SSBL = 4 * [(13.0-15.61)^2+(14.5-15.61)^2+\ldots+(16.4-15.61)^2]$$
$$= 76.133$$

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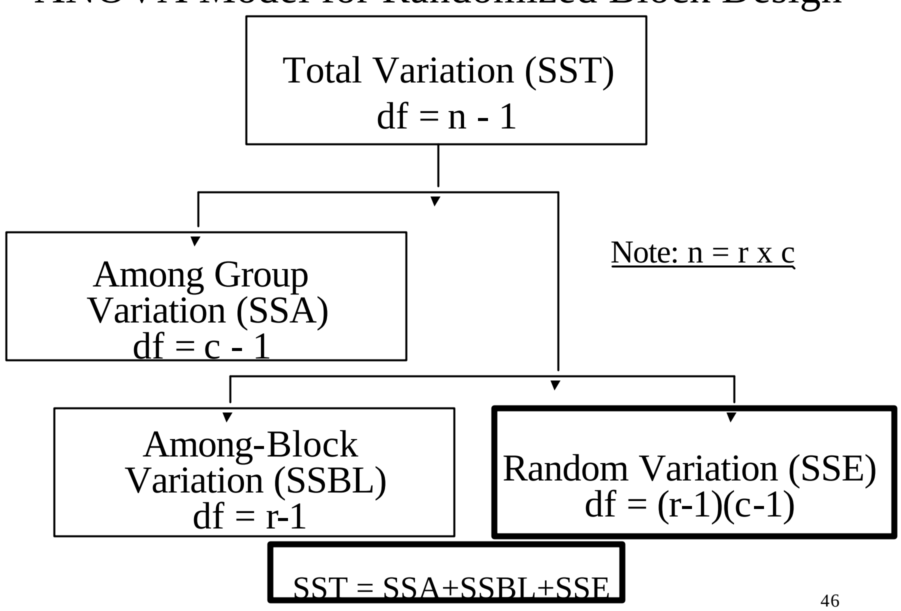
ANOVA Model for Randomized Block Design



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SSE (Random Error)

$$SSE = \sum_{j=1}^{c}\sum_{i=1}^{r}\left(X_{ij} - X_{.j} - X_{i.} + \overline{\overline{X}}\right)^2$$

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Page Replacement Algorithm

	A	B	C	D		Mean
block 1	11.0	12.0	18.0	11.0	*program 1*	13.0
block 2	13.0	14.0	19.0	12.0	*program 2*	14.5
block 3	17.0	18.4	23.4	16.5	*program 3*	18.8
block 4	14.0	14.9	20.0	12.5	*program 4*	15.4
block 5	15.0	16.0	21.0	13.5	*program 5*	16.4
Mean	14.0	15.1	20.3	13.1		

Grand Mean 15.61

SSE = $(11.0-14.0-13.0-15.61)^2+(13.0-14.0-14.5+15.61)^2+$

….

$(21.0-20.3-16.4+15.61)^2+(13.5-13.1-16.4+15.61)^2$

= 1.287

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ANOVA Model: Mean Squares

$$MSA = \frac{SSA}{c-1}$$

$$MSBL = \frac{SSBL}{r-1}$$

$$MSE = \frac{SSE}{(r-1)(c-1)}$$

The mean squares are variances!

If there are no real differences among the c groups, MSA, MSBL, and MSE provide estimates for the variance inherent in the data.

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ANOVA Hypothesis Testing

$$H_0 : m_{.1} = m_{.2} = \ldots = m_{.c}$$

$$H_1 : \text{Not all } m_{.j} \ (j = 1, \ldots, c) \text{ are equal.}$$

F-Test statistic: $F = \frac{MSA}{MSE}$

The F-test statistic follows an F distribution with (c-1) degrees Of freedom in the numerator and (r-1)(c-1) in the denominator.

Reject H_0 if $F > Fu$

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Page Replacement Algorithm

	A	B	C	D		Mean
block 1	11.0	12.0	18.0	11.0	*program 1*	13.0
block 2	13.0	14.0	19.0	12.0	*program 2*	14.5
block 3	17.0	18.4	23.4	16.5	*program 3*	18.8
block 4	14.0	14.9	20.0	12.5	*program 4*	15.4
block 5	15.0	16.0	21.0	13.5	*program 5*	16.4
Mean	14.0	15.1	20.3	13.1		

Grand Mean	15.61
SSA	155.018
SSE	1.287
SSBL	76.133
SST	232.438
MSA	51.67267
MSBL	19.03325
MSE	0.10725
F	481.80 ← F=MSA/MSE
df numer.	3
df denom.	12
Fu	7.23 (from table) ←

F > Fu ⇨ reject Ho

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Anova: Two-Factor Without Replication

SUMMARY	*Count*	*Sum*	*Average*	*Variance*
Row 1	4	52	13	11.333
Row 2	4	58	14.5	9.667
Row 3	4	75.3	18.825	9.949
Row 4	4	61.4	15.35	10.590
Row 5	4	65.5	16.375	10.563
Column 1	5	70	14	5
Column 2	5	75.3	15.06	5.638
Column 3	5	101.4	20.28	4.292
Column 4	5	65.5	13.1	4.425

ANOVA

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Rows (blocks)	76.133	4	19.03325	177.4662	1.46E-10	3.25916
Columns (groups)	155.018	3	51.67267	481.79643	9.11E-13	3.4903
Error	1.287	12	0.10725			
Total	232.438	19				

MSA (51.67267); MSE (0.10725); F =MSA/MSE

Since the p-value is less than $\alpha = 0.05$, reject Ho.
Since F > F critical, reject Ho.

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Estimated Relative Efficiency (RE)

$$RE = \frac{\overbrace{(r-1)MSBL}^{\text{SSBL}} + r(c-1)MSE}{\underbrace{(rc-1)}_{\text{n-1}}MSE}$$

- Used to assess if blocking results in an increase in precision in comparing the different groups.

MSA	51.67267
MSBL	19.03325
MSE	0.10725

RE	38.2

- If blocking is not used, we would need 38.2 times as many observations to obtain the same precision in comparing the groups.

Multiple Comparisons: The Tukey-Kramer Procedure

- Obtain the critical range:

$$\text{critical range} = q_u \sqrt{\frac{MSE}{r}}$$

where q_u is the upper-tail critical value from a *Studentized range** distribution with c degrees of freedom in the numerator and (r-1)(c-1) degrees of freedom in the denominator.

* See table A9.

Multiple Comparisons: The Tukey-Kramer Procedure with PhStat

Tukey Kramer Multiple Comparisons

Group	Sample Mean
1	14
2	15.06
3	20.28
4	13.1

Comparison	Absolute Difference	Critical Range	Result
Group 1 to Group 2	1.06	0.615124	**Means are different**
Group 1 to Group 3	6.28	0.615124	**Means are different**
Group 1 to Group 4	0.9	0.615124	**Means are different**
Group 2 to Group 3	5.22	0.615124	**Means are different**
Group 2 to Group 4	1.96	0.615124	**Means are different**
Group 3 to Group 4	7.18	0.615124	**Means are different**

Intermediate Calculations	
MSE	0.10725
r	5
c	4

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