

Design of Experiments: Factorial Designs

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Basic Concepts

- Factorial design: more than one factor is studied simultaneously.

$$2^k$$

number of factors

number of levels of each factor

2^3 design: three factors, each with two levels. Total of 8 (2^3) combinations

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Two-factor Design with Equal Number of Replicates (n')

		Factor B			
		1	2	...	c
Factor A	1	X ₁₁₁	X ₁₂₁	...	X _{1c1}
		X ₁₁₂	X ₁₂₂	...	X _{1c2}
	
		X _{11n'}	X _{12n'}	...	X _{1cn'}
	2	X ₂₁₁	X ₂₂₁	...	X _{2c1}
		X ₂₁₂	X ₂₂₂	...	X _{2c2}
	
		X _{21n'}	X _{22n'}	...	X _{2cn'}

	r	X _{r11}	X _{r21}	...	X _{rc1}
		X _{r12}	X _{r22}	...	X _{rc2}
	
		X _{r1n'}	X _{r2n'}	...	X _{rcn'}

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Notation

r : number of levels of factor A

c : number of levels of factor B

n' : number of replications for each cell

n : total number of observations ($n = rcn'$)

X_{ijk} : k - th observation for level i of factor A
level j of factor B

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Means

$$\bar{\bar{X}} = \frac{\sum_{i=1}^r \sum_{j=1}^c \sum_{k=1}^{n'} X_{ijk}}{rcn'} \quad (\text{overall or grand mean})$$

$$\bar{X}_{i..} = \frac{\sum_{j=1}^c \sum_{k=1}^{n'} X_{ijk}}{cn'} \quad (\text{mean of } i\text{-th level of factor A})$$

$$\bar{X}_{.j.} = \frac{\sum_{i=1}^r \sum_{k=1}^{n'} X_{ijk}}{rn'} \quad (\text{mean of } j\text{-th level of factor B})$$

$$\bar{X}_{ij.} = \frac{\sum_{k=1}^{n'} X_{ijk}}{n'} \quad (\text{mean of cell } i,j)$$

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Area for Grand Mean

		Factor B			
		1	2	...	c
Factor A	1	X111	X121	...	X1c1
		X112	X122	...	X1c2
	
		X11n'	X12n'	...	X1cn'
	2	X211	X221	...	X2c1
		X212	X222	...	X2c2
	
		X21n'	X22n'	...	X2cn'

	r	Xr11	Xr21	...	Xrc1
		Xr12	Xr22	...	Xrc2
	
		Xr1n'	Xr2n'	...	Xrcn'

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Area for Mean of a Level of Factor A

		Factor B			
		1	2	...	c
Factor A	1	X111	X121	...	X1c1
		X112	X122	...	X1c2
	
		X11n'	X12n'	...	X1cn'
	2	X211	X221	...	X2c1
		X212	X222	...	X2c2
	
		X21n'	X22n'	...	X2cn'

	r	Xr11	Xr21	...	Xrc1
		Xr12	Xr22	...	Xrc2
	
		Xr1n'	Xr2n'	...	Xrcn'

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Area for Mean of a Level of Factor B

		Factor B			
		1	2	...	c
Factor A	1	X111	X121	...	X1c1
		X112	X122	...	X1c2
	
		X11n'	X12n'	...	X1cn'
	2	X211	X221	...	X2c1
		X212	X222	...	X2c2
	
		X21n'	X22n'	...	X2cn'

	r	Xr11	Xr21	...	Xrc1
		Xr12	Xr22	...	Xrc2
	
		Xr1n'	Xr2n'	...	Xrcn'

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Area for Mean of a Cell

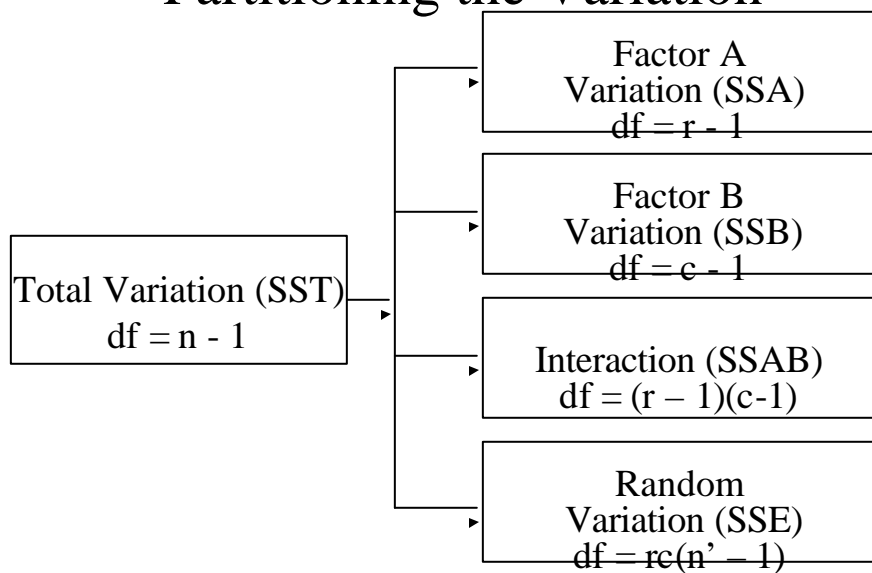
		Factor B			
		1	2	...	c
Factor A	1	X111	X121	...	X1c1
		X112	X122	...	X1c2
	
		X11n'	X12n'	...	X1cn'
	2	X211	X221	...	X2c1
		X212	X222	...	X2c2
	
		X21n'	X22n'	...	X2cn'

	r	Xr11	Xr21	...	Xrc1
		Xr12	Xr22	...	Xrc2
	
		Xr1n'	Xr2n'	...	Xrcn'

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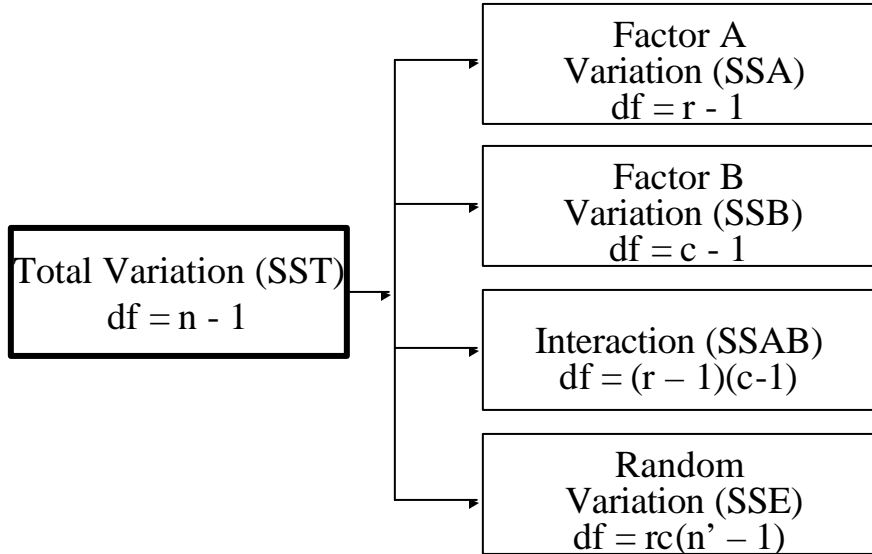
Partitioning the Variation



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Partitioning the Variation



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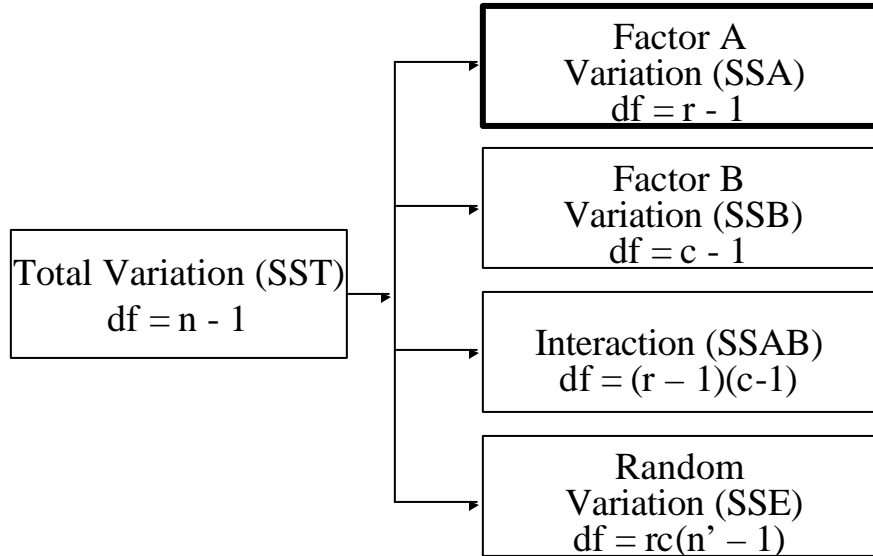
Total Variation (SST)

$$SST = \sum_{i=1}^r \sum_{j=1}^c \sum_{k=1}^{n'} \left(X_{ijk} - \bar{\bar{X}} \right)^2$$

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Partitioning the Variation



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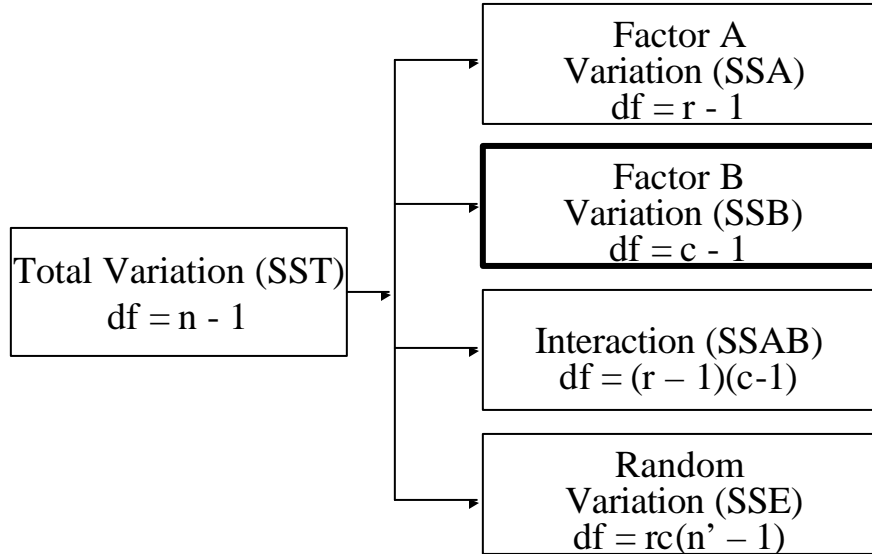
Factor A Variation (SSA)

$$SSA = cn' \sum_{i=1}^r \left(\bar{X}_{i..} - \bar{\bar{X}} \right)^2$$

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Partitioning the Variation



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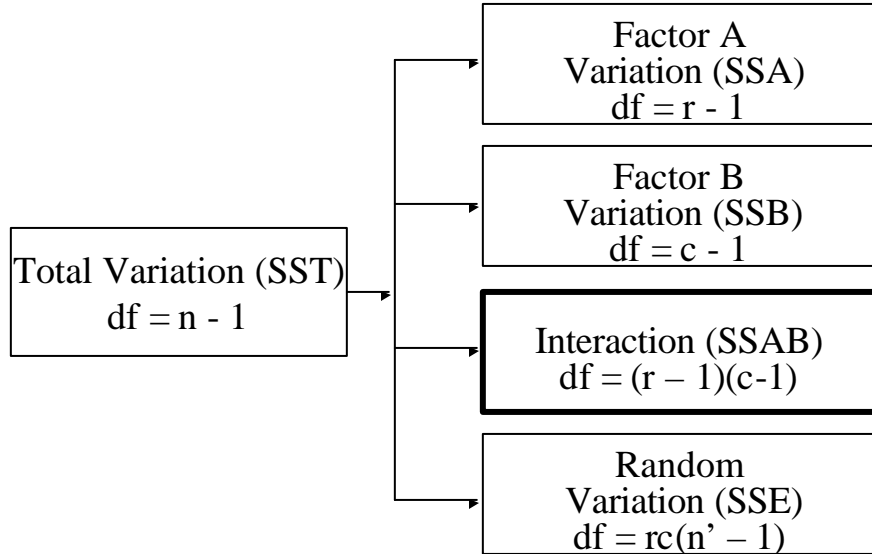
Factor B Variation (SSB)

$$SSB = rn' \sum_{j=1}^c \left(\bar{X}_{.j.} - \bar{\bar{X}} \right)^2$$

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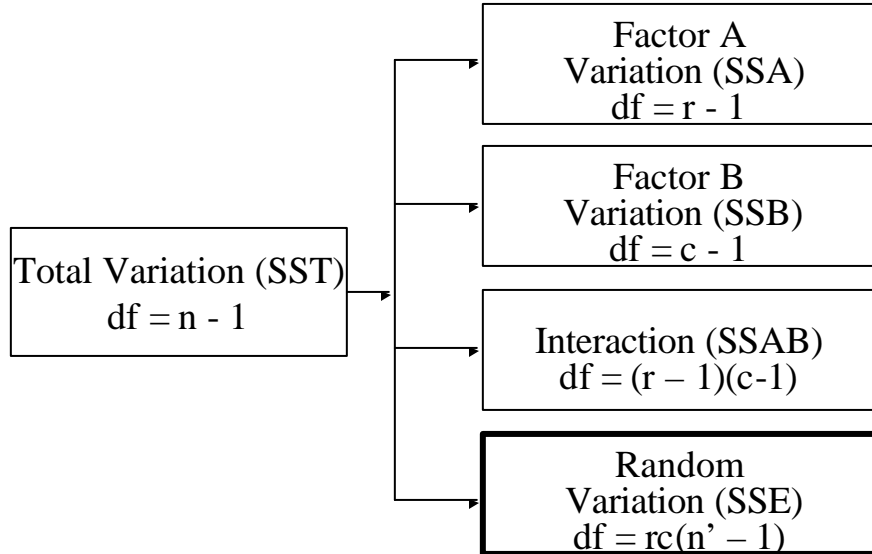
Variation due to Interaction (SSAB)

$$SSAB = n \sum_{i=1}^r \sum_{j=1}^c \left(\bar{X}_{ij.} - \bar{X}_{i..} - \bar{X}_{.j.} + \bar{\bar{X}} \right)^2$$

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Partitioning the Variation



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Random Error (SSE)

$$SSE = \sum_{i=1}^r \sum_{j=1}^c \sum_{k=1}^{n'} \left(X_{ijk} - \bar{X}_{ij.} \right)^2$$

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Mean Squares

$$MSA = \frac{SSA}{r-1}$$

$$MSB = \frac{SSB}{c-1}$$

$$MSAB = \frac{SSAB}{(r-1)(c-1)}$$

$$MSE = \frac{SSE}{rc(n'-1)}$$

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Two-Factor ANOVA Model No Difference Due to Factor A

$$H_0 : \mu_{1..} = \mu_{2..} = \dots = \mu_{r..}$$

$$H_1 : \text{Not all } \mu_{i..} \text{ (} i = 1, \dots, r \text{) are equal.}$$

$$\text{F-Test statistic for Factor A: } F = \frac{MSA}{MSE}$$

The F-test statistic follows an F distribution with (r-1) degrees of freedom in the numerator and rc(n'-1) in the denominator.

$$\text{Reject } H_0 \text{ if } F > F_u$$

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Two-Factor ANOVA Model No Difference Due to Factor B

$$H_0 : \mu_{.1}, \mu_{.2}, \dots, \mu_{.c}$$

$$H_1 : \text{Not all } \mu_{.j} \text{ (} j = 1, \dots, c \text{) are equal.}$$

$$\text{F-Test statistic for Factor B: } F = \frac{MSB}{MSE}$$

The F-test statistic follows an F distribution with (c-1) degrees of freedom in the numerator and rc(n'-1) in the denominator.

$$\text{Reject } H_0 \text{ if } F > F_u$$

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Two-Factor ANOVA Model No Interaction of Factors A and B

$$H_0 : \text{the interaction of A and B is 0.}$$

$$H_1 : \text{the interaction of A and B } \neq 0.$$

$$\text{F-Test statistic for the interaction: } F = \frac{MSAB}{MSE}$$

The F-test statistic follows an F distribution with (r-1)(c-1) degrees of freedom in the numerator and rc(n'-1) in the denominator.

$$\text{Reject } H_0 \text{ if } F > F_u$$

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Example of Two Factor Design Analysis

Response time (in msec) of a Web Site.

	1 CPU	2 CPUs
1 Server	101.0	98.0
1 Server	103.0	97.5
1 Server	102.4	99.3
1 Server	104.0	100.0
2 Servers	43.0	41.0
2 Servers	46.0	44.0
2 Servers	45.0	42.0
2 Servers	49.0	46.0

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Anova: Two-Factor With Replication

SUMMARY	1 CPU	2 CPUs	Total
<i>1 Server</i>			
Count	4	4	8
Sum	410.4	394.8	805.2
Average	102.6	98.7	100.65
Variance	1.573333	1.326667	5.588571
<i>2 Servers</i>			
Count	4	4	8
Sum	183	173	356
Average	45.75	43.25	44.5
Variance	6.25	4.916667	6.571429
<i>Total</i>			
Count	8	8	
Sum	593.4	567.8	
Average	74.175	70.975	
Variance	926.7593	881.1621	

ANOVA

Source of Variation	SS	df	MS	F	P-value	F crit
Number of Servers	12611.29	1	12611.29	3586.149	3.11E-16	4.747221
Number of CPUs	40.96	1	40.96	11.64739	0.005146	4.747221
Interaction	1.96	1	1.96	0.557346	0.469706	4.747221
Within	42.2	12	3.516667			
Total	12696.41	15				

Reject Hypothesis that there is no difference due to number of servers.

Reject Hypothesis that there is no difference due to number of CPUs.

Accept hypothesis that there is no interaction between number of servers and number of CPUs.

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