Continuous Random Variables

Daniel A. Menascé, Ph.D.
Department of Computer Science
George Mason University

Relevant Functions

- Probability density function (pdf) of r.v. X: $f_X(x)$

$$P[a \le X \le b] = \int_a^b f_X(x)dx$$

- Cumulative distribution function (CDF):

$$F_X(x) = P[X \le x]$$

- Tail of the distribution (reliability function):

$$R_X(x) = P[X > x] = 1 - F_X(x)$$

Moments

- k-th moment: $E[X^k] = \int_{-\infty}^{+\infty} x^k f_X(x)dx$
- Expected value (mean): first moment

$$m = E[X] = \int_{-\infty}^{+\infty} x f_X(x)dx$$

- k-th central moment:

$$E[(X-m)^k] = \int_{-\infty}^{+\infty} (x-m)^k f_X(x)dx$$

- Variance: second central moment

$$s^2 = E[(X-m)^2] = \int_{-\infty}^{+\infty} (x-m)^2 f_X(x)dx$$

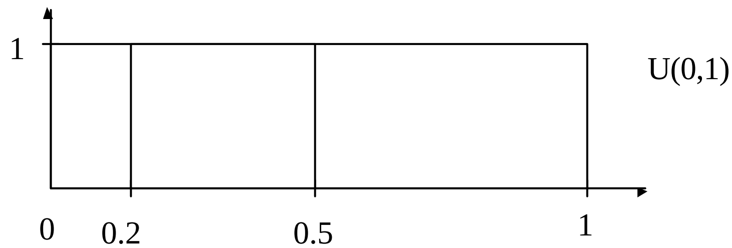
The Uniform Distribution

- pdf: $f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$

- Mean: $m = \frac{a+b}{2}$

- Variance: $s^2 = \frac{(b-a)^2}{12}$

The Uniform Distribution



P[0.2<X<0.5]=(0.5-0.2)x1.0=0.3

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The Normal Distribution $N(m, s)$

- Many natural phenomena follow a normal distribution.
- The normal distribution can be used to approximate the binomial and the Poisson distributions.
- Two parameters: mean and standard deviation.

$$f_X(x) = \frac{1}{\sqrt{2p}\, s} e^{-(1/2)[(x-m)/s]^2}$$

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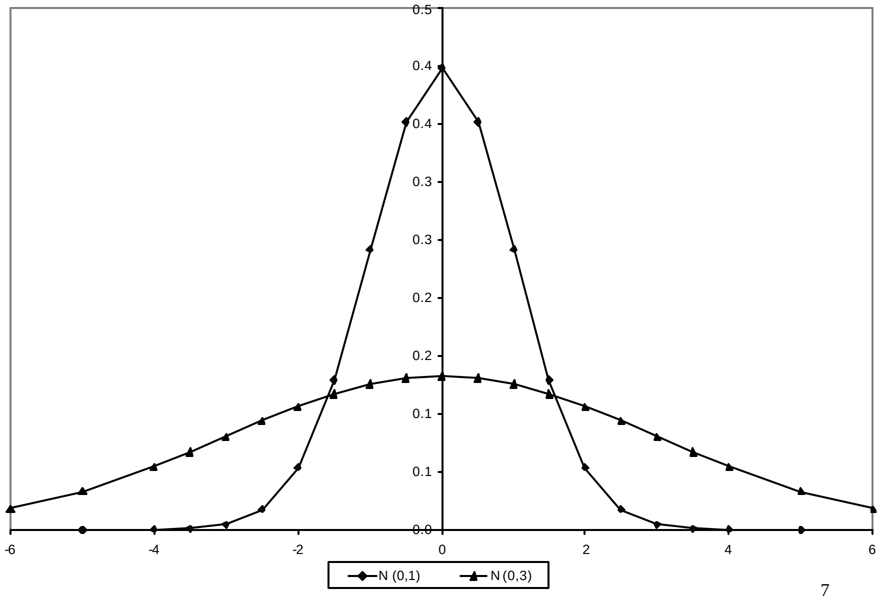
The Normal Distribution $N(m, s)$



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The Standard Normal Distribution

- To use tables for computing values related to the normal distribution, we need to standardize a normal r.v. as

standard normal score

$$Z = \frac{X - m}{s}$$

- Given X, compute a Z value z.
- Find the area value in a Table (Prob [0<Z<z]).

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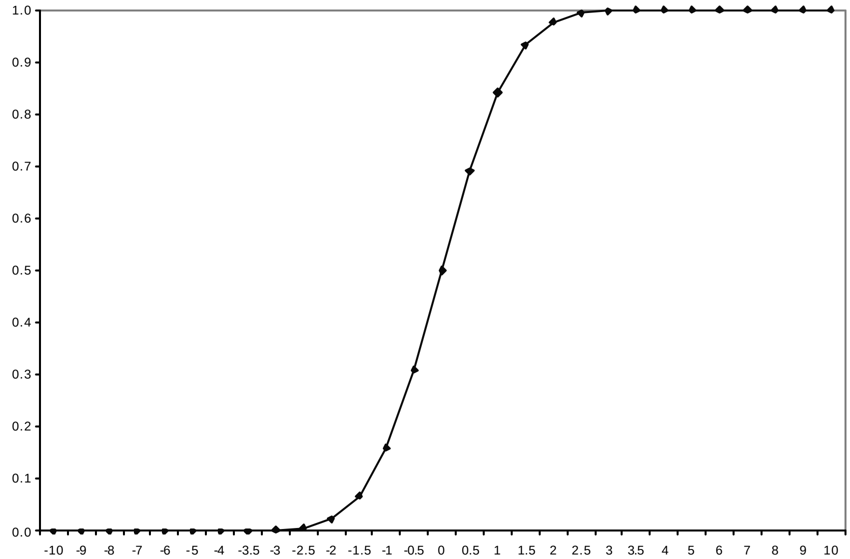
Normal CDF



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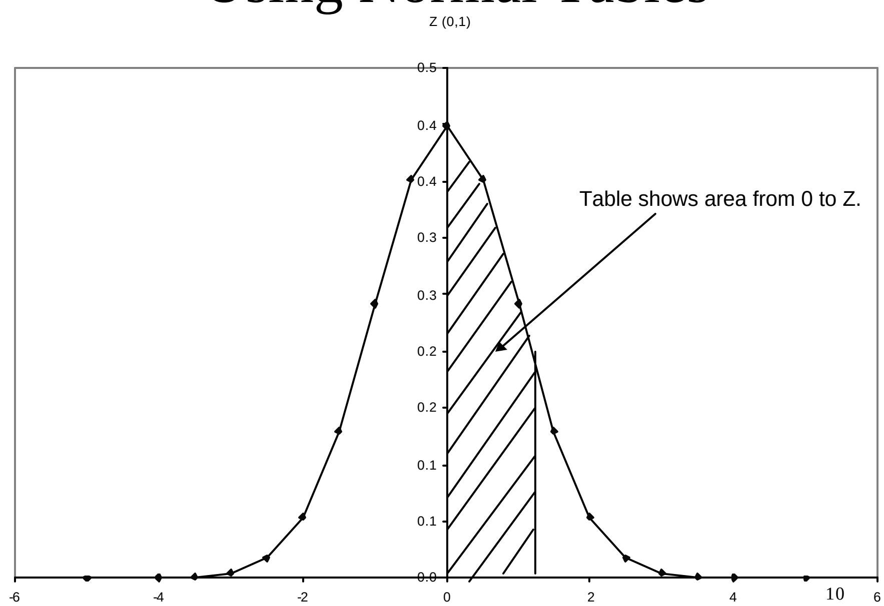
Using Normal Tables



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The Normal as an Approximation to the Binomial Distribution

- The normal can approximate the binomial if the variance of the binomial

$$np(1-p) \geq 10$$

- Binomial: $m = np$

$$s = \sqrt{np(1-p)}$$

- Transformation: $Z = \dfrac{X - np}{\sqrt{np(1-p)}}$

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The Normal as an Approximation to the Binomial Distribution

- Consider a binomial r.v. X with average 50 and variance 25. What is $P[50 \leq X \leq 60]$?
- Transformation: $Z = \dfrac{X - 50}{\sqrt{25}} = \dfrac{60 - 50}{5} = 2.0$
- Using the table, the area between 50 and 60 for Z=2.0 is 0.4772. So,

$$P[50 \leq X \leq 60] = 0.4772$$

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The Normal as an Approximation to the Poisson Distribution

- The normal can approximate the Poisson if the $\lambda > 5$.
- Poisson: $$m = l$$ $$s = \sqrt{l}$$
- Transformation: $$Z = \frac{X - l}{\sqrt{l}}$$

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The Lognormal Distribution

- Obtained when a right skewed variable is transformed using natural logarithms and and the transformed values are normally distributed.

$$f_X(x) = \frac{1}{x\sqrt{2p}\,s_{\ln X}} e^{-(1/2)[(\ln X - m_{\ln X})/s_{\ln X}]^2} \qquad x > 0$$

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The Lognormal distribution

- Mean: $E[X] = e^{m_{\ln X} + s^2_{\ln X}/2}$
- Standard Deviation: $s = \sqrt{e^{2m_{\ln X} + s^2_{\ln X}} .(e^{s^2_{\ln x}} - 1)}$

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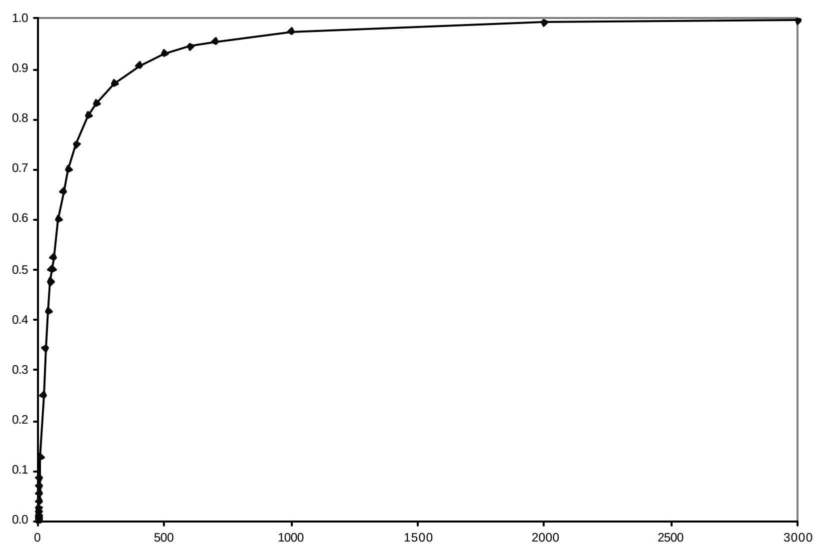
Lognormal CDF



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The Exponential Distribution

- Widely used in queuing systems to model the inter-arrival time between requests to a system.
- If the inter-arrival times are exponentially distributed then the number of arrivals in an interval *t* has a Poisson distribution and vice-versa.

$$f_X(x) = l e^{-l.x} \qquad F_X(x) = 1 - e^{-l.x} \qquad x \geq 0$$

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The Exponential Distribution

- Mean and Standard Deviation:

$$m = s = 1/l$$

- The COV is 1. The exponential is the only continuous r.v. with COV=1.
- The exponential distribution is "memoryless." The distribution of the residual time until the next arrival is also exponential with the same mean as the original distribution.

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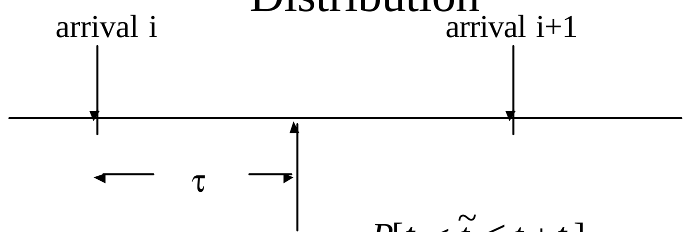
Memoryless Property of the Exponential Distribution



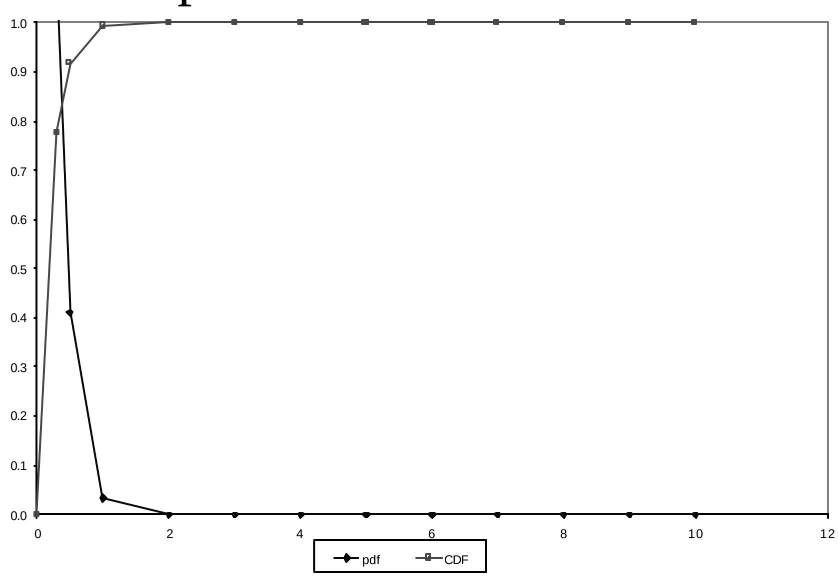
$$P[\tilde{t} \le t + t \mid \tilde{t} > t] = \frac{P[t < \tilde{t} \le t + t]}{P[\tilde{t} > t]}$$

$$= \frac{P[\tilde{t} \le t + t] - P[\tilde{t} \le t]}{P[\tilde{t} > t]}$$

$$= \frac{1 - e^{-l.(t+t)} - (1 - e^{-l.t})}{1 - (1 - e^{-l.t})}$$

$$= 1 - e^{-l.t}$$

Exponential Distribution



Pareto Distribution

- A case of a heavy-tailed distribution.
- The probability of large values is not negligible.

$$f_X(x) = \frac{a}{x^{1+a}} \quad a > 0 \quad , \quad x \geq 1$$

$$F_X(x) = 1 - \frac{1}{x^a} \quad a > 0 \quad , \quad x \geq 1$$

- Mean: $\frac{a}{a-1} \qquad a > 1$

- Variance: $\frac{a}{(a-1)^2(a-2)} \qquad a > 2$

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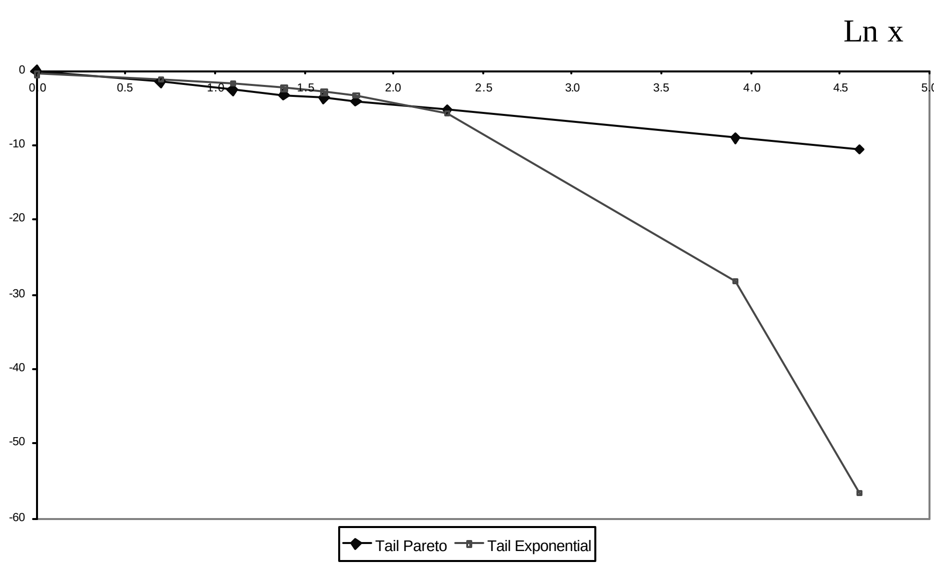
Tail of the Pareto and Exponential Distributions



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