Integral Transforms and Special Functions, Vol. 16, No. 2 (2005),159-169.

THE q–ANALOGUES OF LAGUERRE POLYNOMIALS OVER A COLLECTION OF COMPLEX ORIGIN INTERVALS

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Mathematics Subject Classification: 33D45, 42C05.

Key words: Orthogonal Polynomials, Basic Hypergeometric Functions

Abstract. We construct two sequences of orthogonal polynomials with respect to the inner products which are defined by q-integrals over a collection of intervals in the complex plane. We prove that they are connected with some q-analogies of the Laguerre polynomials. For such introduced polynomials we discuss a few representations, a recurrence relation, a difference equation, a Rodrigues–type formula and a generating function. Also, a relationship between those two sequences is found.

1. Introduction

Each class of classical orthogonal polynomials has several q–analogies (see, for example, [3]). In this section we study the *little q–Laguerre polynomials* $p_n(x; a \,|\, q)$ and the *q–Laguerre polynomials* $L_n^{(\alpha)}(x; q)$ as the analogies of the Laguerre polynomials $L_n^{(\alpha)}(x)$. They are defined by

$$p_n(x; a \,|\, q) = {}_2\Phi_1\left(\begin{matrix} q^{-n}, 0 \\ aq \end{matrix} \,\middle|\, q; qx\right), \tag{1.1}$$

$$L_n^{(\alpha)}(x; q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \; {}_1\Phi_1\left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix} \,\middle|\, q; -q^{n+\alpha+1}x\right), \tag{1.2}$$

where

$${}_r\Phi_s\left(\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} \,\middle|\, q; z\right) = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_k}{(b_1, b_2, \dots, b_s; q)_k} (-1)^{(1+s-r)k} q^{(1+s-r)\binom{k}{2}} \frac{z^k}{(q; q)_k}$$

is the basic hypergeometric function, $(a;q)_r=\prod_{j=0}^{r-1}(1-aq^j)$ and $(a_1,\dots,a_n;q)_r=\prod_{j=1}^{n}(a_j;q)_r$.

The orthogonality of the little q–Laguerre and the q–Laguerre polynomials is given by the relations

$$\sum_{k=0}^{\infty}\frac{(aq)^k}{(q;q)_k}\;p_m(q^k;a\,|\,q)p_n(q^k;a\,|\,q)=\frac{(aq)^n}{(aq;q)_\infty}\frac{(q;q)_n}{(aq;q)_n}\delta_{mn}\quad(0<a<1/q),\tag{1.3}$$

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}\frac{q^{k\alpha+k}}{(-cq^k;q)_\infty}\;L_m^{(\alpha)}(cq^k;q)\;L_n^{(\alpha)}(cq^k;q)\\&=\frac{(q,-cq^{\alpha+1},-c^{-1}q^{-\alpha};q)_\infty}{(q^{\alpha+1},-c,-c^{-1}q;q)_\infty}\frac{(q^{\alpha+1};q)_n}{q^n(q;q)_n}\delta_{mn}\quad(\alpha>-1,\;c>0).\end{aligned}\tag{1.4}$$

Having in mind the definitions of the q–exponential functions

$$\begin{aligned}E_q(x)&=(-x;q)_\infty=\sum_{k=0}^{\infty}\frac{q^{\binom{k}{2}}}{(q;q)_k}\;x^k,\quad(0<|q|<1,\;x\in\mathbb{C}),\\e_q(x)&=\frac{1}{(x;q)_\infty}=\sum_{k=0}^{\infty}\frac{1}{(q;q)_k}\;x^k,\quad(0<|q|<1,\;|x|<1),\end{aligned}\tag{1.5}$$

where $(a;q)_\infty=\prod_{j=0}^{\infty}(1-aq^j)$, and the definitions of q–integral

$$\int_0^x f(z)\;d_qz:=x(1-q)\sum_{k=0}^{\infty}f(q^kx)\;q^k,\qquad\int_0^\infty f(z)\;d_qz:=(1-q)\sum_{k=-\infty}^{\infty}f(q^k)\;q^k,$$

for $0<|q|<1$ and $x>0$, it follows that $\{p_n(x;q^\alpha\,|\,q)\}_{n\in\mathbb{N}_0}$ are orthogonal with respect to the inner product

$$(f,g)_l=\int_0^1 f(x)\;\overline{g(x)}\;x^\alpha\;E_q(-qx)\;d_qx\tag{1.6}$$

and that $\{L_n^{(\alpha)}(x;q)\}_{n\in\mathbb{N}_0}$ are orthogonal with respect to the inner product

$$(f,g)_L=\int_0^\infty f(x)\;\overline{g(x)}\;x^\alpha\;e_q(-x)\;d_qx.\tag{1.7}$$

The relationship between these two q–analogies of the Laguerre polynomials is given by
(1.8)

$$p_n(x;q^{-\alpha}\,|\,q^{-1}) = \frac{(q;q)_n}{(q^{\alpha+1};q)_n}\,L_n^{(\alpha)}(-x;q),\;\; L_n^{(\alpha)}(x;q^{-1}) = \frac{(q^{\alpha+1};q)_n}{q^{n\alpha}(q;q)_n}\,p_n(-x;q^{\alpha}\,|\,q).$$

At last, let us remember (see, for example, [3]) that q-number and q-factorial are defined by

$$[a]_q = \frac{1-q^a}{1-q} \quad (a\in\mathbb{R}), \qquad [n]_q! = [n]_q\;[n-1]_q\ldots[1]_q \quad (n\in\mathbb{N})$$

and the q-derivative of a function is

$$\mathcal{D}_q F(x) = \frac{F(x)-F(qx)}{x-qx} \quad (x\neq 0).$$

In this paper, our purpose is to examine the polynomials orthogonal with respect to the inner products defined by the sum of q-integrals over a collection of the complex intervals with starting point in the origin. The similar considerations can be found in the papers of J.S. Geronimo and W.V. Assche [1], who have discussed the orthogonality of a new polynomial sequence which is obtained by some polynomial transformations of a measure and its support. Also, the orthogonality on radial rays in the complex plane was discussed in the papers of G.V. Milovanović [4] and G.V. Milovanović, P.M. Rajković and Z.M. Marjanović [5].

2. About q–orthogonality over a collection of the intervals

Let us assume that m is a positive integer, q is a real number $(0<q<1)$ and $Q=q^{1/m}$. Also, denote

$$\varphi_m(j) = \exp\Big(\mathrm{i}\frac{2\pi j}{m}\Big), \qquad j=0,1,\ldots,m-1, \quad (\mathrm{i}^2=-1).$$

The function $j\mapsto\varphi_m(j)$ has following properties:

(i) $\varphi_m(mj)=1, \qquad \overline{\varphi_m(j)}=\varphi_m(-j);$

(ii) $\varphi_m(j+k)=\varphi_m(j)+\varphi_m(k), \quad \varphi_m^N(j)=\varphi_m(Nj);$

(iii) $\displaystyle\sum_{j=0}^{m-1}\varphi_m((mn+\nu)j) = \sum_{j=0}^{m-1}\varphi_m(\nu j) = \begin{cases} m, & \nu=0\\ 0, & 1\le\nu\le m-1.\end{cases}$

We consider the polynomial $T(x) = x^m,\ x > 0$ and its inverse branches

$$T_j^{-1}(x) = \varphi_m(j)\ x^{1/m}, \quad x > 0 \qquad (0 \le j \le m-1).$$

In this way, we will define the inner products via polynomial mappings (for the idea see, for example, [1])

(2.1)
$$\langle F, G\rangle_l = \left[\frac{1}{m}\right]_q \sum_{j=0}^{m-1} \int_0^1 F\big(T_j^{-1}(z)\big)\ \overline{G\big(T_j^{-1}(z)\big)}\ z^{\gamma-1+1/m} E_q(-qz)\ d_q z,$$

(2.2)
$$\langle F, G\rangle_L = \left[\frac{1}{m}\right]_q \sum_{j=0}^{m-1} \int_0^\infty F\big(T_j^{-1}(z)\big)\ \overline{G\big(T_j^{-1}(z)\big)}\ z^{\gamma-1+1/m} e_q(-z)\ d_q z,$$

where $m\gamma + 1 > 0$. Using the property of the q-integral

$$\int_0^b F(x^m)\ d_Q x = \left[\frac{1}{m}\right]_q \int_0^b F(x)\ x^{1/m-1} d_q x \quad (b \in \{1, \infty\}) \tag{2.3}$$

the previous inner products can be rewritten as

(2.4)
$$\langle F, G\rangle_l = \sum_{j=0}^{m-1} \int_0^1 F(\varphi_m(j)z)\ \overline{G(\varphi_m(j)z)}\ z^{m\gamma} E_{Q^m}(-Q^m z^m)\ d_Q z,$$

(2.5)
$$\langle F, G\rangle_L = \sum_{j=0}^{m-1} \int_0^\infty F(\varphi_m(j)z)\ \overline{G(\varphi_m(j)z)}\ z^{m\gamma} e_{Q^m}(-z^m)\ d_Q z,$$

According to the definition of the q–integral, we get equivalent expressions:

(2.6)
$$\langle F, G\rangle_l = (1 - q^{1/m}) \sum_{j=0}^{m-1} \sum_{k=0}^{\infty} F(\varphi_m(j)q^{k/m})\ \overline{G(\varphi_m(j)q^{k/m})}\ q^{k(\gamma+1/m)} E_q(-q^{k+1}),$$

(2.7)
$$\langle F, G\rangle_L = (1 - q^{1/m}) \sum_{j=0}^{m-1} \sum_{k=-\infty}^{\infty} F(\varphi_m(j)q^{k/m})\ \overline{G(\varphi_m(j)q^{k/m})}\ q^{k(\gamma+1/m)} e_q(-q^k).$$

The inner products (2.1)-(2.4)-(2.6) and (2.2)-(2.5)-(2.7) can be represented in a common form

$$\begin{aligned}\langle F,G\rangle_* &= \sum_{j=0}^{m-1}\int_0^b F(\varphi_m(j)z)\,\overline{G(\varphi_m(j)z)}\,w(z;Q)\,d_Qz\\ &=(1-Q)\sum_{j=0}^{m-1}\sum_{k=\beta}^{\infty}F(\varphi_m(j)Q^k)\,\overline{G(\varphi_m(j)Q^k)}\,w(Q^k;Q)\,Q^k.\end{aligned}\tag{2.8}$$

In this way, the choice $b=1$, $\beta=0$, $w(z;Q)=z^{m\gamma}E_{Q^m}(-Q^m z^m)$ gives the product (2.1)-(2.4)-(2.6) and $b=\infty$, $\beta=-\infty$, $w(z;Q)=z^{m\gamma}e_{Q^m}(-z^m)$ gives (2.2)-(2.5)-(2.7).

Under the conditions $m\gamma+1>0$ there exist the sequences of the monic polynomials $\{\lambda_N^{(\gamma)}(z;Q)\}_{N\in\mathbb{N}_0}$ and $\{\Lambda_N^{(\gamma)}(z;Q)\}_{N\in\mathbb{N}_0}$ orthogonal with respect to the inner product (2.1) and (2.2) respectively. Their existence we will prove by the construction.

For the inner products (2.8) and the arbitrary pair of the functions $F(z)$ and $G(z)$ it is valid

$$\langle z^m F,G\rangle_* = \langle F,z^m G\rangle_*.$$

By the end of this section we will give some general properties related to the sequence of the polynomials $\{\lambda_N^{(\gamma)}\}_{N\in\mathbb{N}_0}$ and $\{\Lambda_N^{(\gamma)}\}_{N\in\mathbb{N}_0}$ and respective inner products without the proofs. Therefore, let $\{\pi_N\}_{N\in\mathbb{N}_0}$ denote one of mentioned sequences of polynomials, i.e. the set of the monic polynomials orthogonal with respect to the inner product (2.8), depending of the choice of the parameters and the weight function.

The polynomials $\pi_N(z)$, $N=0,1,\dots$, satisfy

$$\pi_N\big(\varphi_m(r)z\big)=\varphi_m(Nr)\pi_N(z),\tag{2.9}$$

for all $r=0,\dots,m-1$.

For $0\le M<N\le m-1$, it is valid $\langle z^N,\pi_M\rangle=0$.

From the previous facts, we conclude that the monic polynomials $\{\pi_N\}_{N\in\mathbb{N}_0}$ satisfy the recurrence relation

$$\begin{aligned}\pi_{N+m}(z)&=\left(z^m-\alpha_N^{(*)}\right)\pi_N(z)-\beta_N^{(*)}\pi_{N-m}(z),\quad N\ge m,\\ \pi_N(z)&=z^N,\quad N=0,\dots,m-1,\end{aligned}\tag{2.10}$$

where

$$\alpha_N^{(*)}=\frac{\langle z^m\pi_N,\pi_N\rangle_*}{\langle\pi_N,\pi_N\rangle_*},\ N\ge 0,\quad \beta_N^{(*)}=\begin{cases}\dfrac{\langle\pi_N,\pi_N\rangle_*}{\langle\pi_{N-m},\pi_{N-m}\rangle_*}, & N\ge m\\ 0, & N\le m-1.\end{cases}$$

The explicit form of the coefficients we will derive in the next section.

According the start values and recurrence relation, we have that the polynomials $\pi_N(z)$, $N \in \mathbb{N}_0$, can be expressed in the form

$$\pi_N(z) = \pi_{mn+\nu}(z) = z^\nu S^{(*)}_{n,\nu}(z^m; q),$$

where $S^{(*)}_{n,\nu}(t; q)$ are the monic polynomials of degree n for $n \geq 0$, $0 \leq \nu \leq m-1$.

3. Some representations of the polynomials $\lambda_N^{(\gamma)}(z;Q)$ and $\Lambda_N^{(\gamma)}(z;Q)$

The sequences of the polynomials orthogonal on several intervals are closely connected the sequences of known q–orthogonal polynomials considered in the previous section.

Theorem 3.1. *The polynomials $\lambda_N^{(\gamma)}(z;Q)$, $N \in \mathbb{N}_0$, can be represented by*

$$\lambda^{(\gamma)}_{mn+\nu}(z; q^{1/m}) = K^{(l)}_{n,\nu}(q) z^\nu p_n(z^m; q^{\gamma-1+(2\nu+1)/m} | \, q), \tag{3.1}$$

where

$$K^{(l)}_{n,\nu}(q) = (-1)^n q^{\binom{n}{2}} \; e_q(q^{n+\gamma+(2\nu+1)/m}) \; E_q(-q^{\gamma+(2\nu+1)/m}) \tag{3.2}$$

and $p_n(x; q^{\gamma-1+(2\nu+1)/m} | \, q)$ is the member of the sequence of the little q–Laguerre polynomials.

Proof. For shorten writing, let us denote $p_n(x) = p_n(x; q^{\gamma-1+(2\nu+1)/m} | \, q)$. It is enough to prove $\langle z^M, z^\nu p_n(z^m)\rangle_l = 0$ for $M < mn+\nu$ and $\langle z^{mn+\nu}, z^\nu p_n(z^m)\rangle_l \neq 0$. Really,

$$\begin{aligned}
&\langle z^M, z^\nu p_n(z^m)\rangle_l \\
&= \sum_{j=0}^{m-1} \int_0^1 \big(\varphi_m(j) z\big)^M \overline{\big(\varphi_m(j) z\big)^\nu p_n\big((\varphi_m(j) z)^m\big)} \; z^{m\gamma} E_{Q^m}(-Q^m z^m) \; d_Q z \\
&= \sum_{j=0}^{m-1} \varphi_m\big((M-\nu)j\big) \int_0^1 z^{M+\nu} p_n(z^m) \; z^{m\gamma} E_{Q^m}(-Q^m z^m) \; d_Q z.
\end{aligned}$$

Since $\sum_{j=0}^{m-1} \varphi_m\big((M-\nu)j\big) \neq 0$ only if $M \equiv \nu \pmod{m}$, then $\langle z^M, z^\nu p_n(z^m)\rangle_l = 0$, for any $M \in \mathbb{N}_0$, except for $M = Nm + \nu$ ($N \in \mathbb{N}_0$). At last, we will discuss such cases. Now, according to (2.3) and $Q^m = q$, we have

$$\begin{aligned}
&\langle z^{Nm+\nu}, z^\nu p_n(z^m)\rangle_l \\
&= m\int_0^1 z^{Nm+2\nu} p_n(z^m)\ z^{m\gamma} E_{Q^m}(-Q^m z^m)\ d_Q z \\
&= m\Big[\frac{1}{m}\Big]_q \int_0^1 x^{N+2\nu/m} p_n(x)\ x^\gamma E_q(-qx)\ x^{1/m-1}\ d_q x \\
&= m\Big[\frac{1}{m}\Big]_q \int_0^1 x^N p_n(x)\ x^{\gamma-1+(2\nu+1)/m} E_q(-qx)\ d_q x.
\end{aligned}$$

Because of the orthogonality of the little q–Laguerre polynomials, the last integral is zero for $N < n$ and it is positive for $N = n$. So, if $M = Nm + \nu < mn + \nu$, we get $\langle z^M, z^\nu p_n(z^m)\rangle_l = 0$ and $\langle z^{mn+\nu}, z^\nu p_n(z^m)\rangle_l \neq 0$. The constant $K_{n,\nu}^{(l)}(q)$ is obtained from the fact that $\lambda_N^{(\gamma)}(z; Q)$ are monic. □

Theorem 3.2. *The polynomials* $\Lambda_N^{(\gamma)}(z; Q)$, $N \in \mathbb{N}_0$, *can be represented by*

$$\Lambda_{mn+\nu}^{(\gamma)}(z; q^{1/m}) = K_{n,\nu}^{(L)}(q) z^\nu L_n^{(\gamma-1+(2\nu+1)/m)}(z^m; q), \tag{3.3}$$

where

$$K_{n,\nu}^{(L)}(q) = (-1)^n q^{-n(n+\gamma-1+(2\nu+1)/m)} e_q(q^{n+1})\ E_q(-q) \tag{3.4}$$

and $L_n^{(\gamma-1+(2\nu+1)/m)}(x; q)$ *is the member of the sequence of the q–Laguerre polynomials.*

Proof. As in the previous theorem, denoting $L_n(x) = L_n^{(\gamma-1+(2\nu+1)/m)}(x; q)$, let us prove that $\langle z^M, z^\nu L_n(z^m)\rangle_L = 0$ for $M < mn + \nu$ and $\langle z^{mn+\nu}, z^\nu L_n(z^m)\rangle_L \neq 0$. Really, since

$$\begin{aligned}
&\langle z^M, z^\nu L_n(z^m)\rangle_L \\
&\qquad = \sum_{j=0}^{m-1} \int_0^\infty \big(\varphi_m(j)z\big)^M \overline{\big(\varphi_m(j)z\big)^\nu L_n\big((\varphi_m(j)z)^m\big)}\ z^{m\gamma} e_{Q^m}(-z^m)\ d_Q z,
\end{aligned}$$

as in Theorem 3.1, there is valid $\langle z^M, z^\nu L_n(z^m)\rangle_L = 0$, for any $M \in \mathbb{N}_0$, except for $M = Nm + \nu$, $N \in \mathbb{N}_0$. In these cases, using (2.3) and $Q^m = q$ we have

$$\langle z^{Nm+\nu}, z^\nu L_n(z^m)\rangle_L = m\Big[\frac{1}{m}\Big]_q \int_0^1 x^N L_n(x)\ x^{\gamma-1+(2\nu+1)/m} e_q(-x)\ d_q x.$$

Because of the orthogonality of the q–Laguerre polynomials, if $M = Nm + \nu < mn + \nu$, we get $\langle z^M, z^\nu L_n(z^m)\rangle_L = 0$ and $\langle z^{mn+\nu}, z^\nu L_n(z^m)\rangle_L \neq 0$. As previous, the constant $K_{n,\nu}^{(L)}(q)$ is obtained from the fact that $\Lambda_N^{(\gamma)}(z;Q)$ are monic. $\square$

Based on the connection $\lambda_N^{(\gamma)}(z;Q)$ and $\Lambda_N^{(\gamma)}(z;Q)$ with $p_n(x;q^{\gamma-1+(2\nu+1)/m}|\, q)$ and $L_n^{(\gamma-1+(2\nu+1)/m)}(x;q)$ respective and their recurrence relations, we can evaluate the coefficients in recurrence relations (2.10) for both classes of the polynomials.

Corollary 3.3. *The polynomials $\lambda_N^{(\gamma)}(z;Q)$, $N \in \mathbb{N}_0$, satisfy the recurrence relation*

$$\lambda_{N+m}^{(\gamma)}(z;Q) = \left(z^m - \alpha_N^{(l)}\right)\lambda_N^{(\gamma)}(z;Q) - \beta_N^{(l)}\lambda_{N-m}^{(\gamma)}(z;Q), \quad N \geq m,$$

where

$$\alpha_{mn+\nu}^{(l)} = (1-Q)Q^{mn}\Big([m(n+\gamma)+2\nu+1]_Q + Q^{m(\gamma-1)+2\nu+1}[mn]_Q\Big),$$

$$\beta_{mn+\nu}^{(l)} = (1-Q)^2 Q^{m(2n+\gamma-2)+2\nu+1}[mn]_Q\ [m(n+\gamma-1)+2\nu+1]_Q$$

for $n \in \mathbb{N}_0$, $0 \leq \nu \leq m-1$.

Corollary 3.4. *The polynomials $\Lambda_N^{(\gamma)}(z;Q)$, $N \in \mathbb{N}_0$, satisfy the recurrence relation*

$$\Lambda_{N+m}^{(\gamma)}(z;Q) = \left(z^m - \alpha_N^{(L)}\right)\Lambda_N^{(\gamma)}(z;Q) - \beta_N^{(L)}\Lambda_{N-m}^{(\gamma)}(z;Q), \quad N \geq m,$$

where

$$\alpha_{mn+\nu}^{(L)} = (1-Q)\ Q^{-m(2n+\gamma)-2\nu-1}\Big([m(n+1)]_Q + Q\ [m(n+\gamma-1)+2\nu+1]_Q\Big),$$

$$\beta_{mn+\nu}^{(L)} = (1-Q)^2\ Q^{-m(4n+2\gamma-3)-2(2\nu+1)}\ [mn]_Q\ [m(n+\gamma-1)+2\nu+1]_Q$$

for $n \in \mathbb{N}_0$, $0 \leq \nu \leq m-1$.

Proof. The recurrence relations satisfied by polynomials $\lambda_N^{(\gamma)}(z;Q)$ and $\Lambda_N^{(\gamma)}(z;Q)$ can be obtained from the Theorem 3.1 and Theorem 3.2 and the three–term recurrence relation satisfied by the little q-Laguerre polynomials $p_n(x;a\,|\,q)$

$$\begin{aligned}-q^{-n}xp_n(x;a\,|\,q) &= (1-aq^{n+1})p_{n+1}(x;a\,|\,q)\\ &\quad - (1-aq^{n+1}+a(1-q^n))p_n(x;a\,|\,q) + a(1-q^n)p_{n-1}(x;a\,|\,q)\end{aligned}$$

and q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$

$$\begin{aligned}-q^{2n+\alpha+1}xL_n^{(\alpha)}(x;q) &= (1-q^{n+1})L_{n+1}^{(\alpha)}(x;q)\\ &\quad - \big(1-q^{n+1}+q(1-q^{n+\alpha})\big)L_n^{(\alpha)}(x;q) + q(1-q^{n+\alpha})L_{n-1}^{(\alpha)}(x;q).\ \square\end{aligned}$$

From Theorem 3.1 and Theorem 3.2 and the representation of the little q–Laguerre and the q–Laguerre polynomials (1.1) and (1.2) we can give following statements.

Theorem 3.5. *The polynomials* $\lambda_N^{(\gamma)}(z;Q)$ *and* $\Lambda_N^{(\gamma)}(z;Q)$, $N \in \mathbb{N}_0$, *can be represented by*

$$\lambda_{mn+\nu}^{(\gamma)}(z;q^{1/m}) = K_{n,\nu}^{(l)}(q)\ z^{\nu}\ {}_2\Phi_1\left(\begin{matrix} q^{-n},0 \\ q^{\gamma+(2\nu+1)/m} \end{matrix}\ \middle|\ q;\ qz^m\right),$$

$$\Lambda_{mn+\nu}^{(\gamma)}(z;q^{1/m}) = \widetilde{K}_{n,\nu}^{(L)}(q)\ z^{\nu}\ {}_1\Phi_1\left(\begin{matrix} q^{-n} \\ q^{\gamma+(2\nu+1)/m} \end{matrix}\ \middle|\ q; -q^{n+\gamma+(2\nu+1)/m}z^m\right),$$

where $K_{n,\nu}^{(l)}(q)$ *is defined by* (3.2) *and*

$$\widetilde{K}_{n,\nu}^{(L)}(q) = (-1)^n q^{-n(n+\gamma-1+(2\nu+1)/m)} e_q(q^{n+\gamma+(2\nu+1)/m})\ E_q(-q^{\gamma+(2\nu+1)/m}).$$

Theorem 3.6. *The polynomials* $\lambda_N^{(\gamma)}(z;Q)$ *and* $\Lambda_N^{(\gamma)}(z;Q)$, $N \in \mathbb{N}_0$, *can be represented by*

$$\lambda_{mn+\nu}^{(\gamma)}(z;q^{1/m}) = (-1)^n q^{\binom{n}{2}}\ e_q(q^{n+\gamma+(2\nu+1)/m}) \times \sum_{k=0}^{n} (-1)^k q^{\binom{k+1}{2}-kn} \begin{bmatrix} n \\ k \end{bmatrix}_q E_q(-q^{k+\gamma+(2\nu+1)/m})\ z^{mk+\nu},$$

$$\Lambda_{mn+\nu}^{(\gamma)}(z;q^{1/m}) = (-1)^n q^{-n(n+\gamma-1+(2\nu+1)/m)}\ e_q(q^{n+\gamma+(2\nu+1)/m}) \times \sum_{k=0}^{n} (-1)^k q^{-k(k+\gamma-1+(2\nu+1)/m)} \begin{bmatrix} n \\ k \end{bmatrix}_q E_q(-q^{k+\gamma+(2\nu+1)/m})\ z^{mk+\nu}.$$

Proof. The summation formulas can be obtained using representations (3.1)–(3.2) and (3.3)–(3.4) and the definitions of the q–hypergeometric function. □

4. Some properties of the polynomials $\lambda_N^{(\gamma)}(z;Q)$ and $\Lambda_N^{(\gamma)}(z;Q)$

In this section we will show what are some classical properties of the orthogonal polynomials look like for these two classes of the polynomials.

Theorem 4.1. *The polynomial* $\lambda^{(\gamma)}_{mn+\nu}(z;Q)$, $n\in\mathbb{N}_0$, $0\le\nu\le m-1$, *satisfies following* Q*–difference equation:*

$$A^{(l)}_{n,\nu}(z;Q)z^2\ \mathcal{D}^2_QY(z)+B^{(l)}_{n,\nu}(z;Q)z\ \mathcal{D}_QY(z)+C^{(l)}_{n,\nu}(z;Q)\ Y(z)=0$$

where

$$\begin{aligned}
A^{(l)}_{n,\nu}(z;Q)&=Q^{m(\gamma-1)+\nu+2},\\
B^{(l)}_{n,\nu}(z;Q)&=[m(\gamma-1)+\nu+2]_Q\ -Q^{m(\gamma-1)+\nu+1}\ [\nu]_Q-\frac{Q^{m(1-n)}}{1-Q}z^m,\\
C^{(l)}_{n,\nu}(z;Q)&=\frac{Q^{m(1-n)}}{1-Q}[mn+\nu]_Q\ z^m-[\nu]_Q\ [m(\gamma-1)+\nu+1]_Q.
\end{aligned}$$

Proof. The little q–Laguerre polynomial $y(x)=p_n(x;q^{\gamma-1+(2\nu+1)/m}|\ q)$ is one solution of the q–difference equation

$$\begin{aligned}
&-q^{-n+1}(1-q^n)xy(qx)\\
&\quad=q^{\gamma-1+(2\nu+1)/m}y(q^2x)+(qx-q^{\gamma-1+(2\nu+1)/m}-1)y(qx)+(1-qx)y(x).
\end{aligned}$$

Applying the relationship (3.1) we have

$$\begin{aligned}
y(x)&=\big(K_{n,\nu}(q)\big)^{-1}x^{-\nu/m}\lambda^{(\gamma)}_{mn+\nu}(x^{1/m};q^{1/m}),\\
y(qx)&=\big(K_{n,\nu}(q)\big)^{-1}q^{-\nu/m}x^{-\nu/m}\lambda^{(\gamma)}_{mn+\nu}(q^{1/m}x^{1/m};q^{1/m}),\\
y(q^2x)&=\big(K_{n,\nu}(q)\big)^{-1}q^{-2\nu/m}x^{-\nu/m}\lambda^{(\gamma)}_{mn+\nu}(q^{2/m}x^{1/m};q^{1/m}).
\end{aligned}$$

Substituting $x=z^m$ and $q=Q^m$, previous difference equation becomes

$$\begin{aligned}
Q^{m(\gamma-1)+\nu+1}\ \lambda^{(\gamma)}_{mn+\nu}(Q^2z;Q)+\big(Q^{m(1-n)}z^m-Q^{m(\gamma-1)+2\nu+1}-1\big)\ \lambda^{(\gamma)}_{mn+\nu}(Qz;Q)&\\
+Q^{\nu}(1-Q^mz^m)\ \lambda^{(\gamma)}_{mn+\nu}(z;Q)=0.&
\end{aligned}$$

Providing the formula (see [2])

$$f(q^kt)=\sum_{j=0}^{k}(-1)^j(1-q)^j\begin{bmatrix}k\\ j\end{bmatrix}_q q^{\binom{j}{2}}t^j\,\mathcal{D}^j_qf(t),$$

we get the difference equation in the required form. □

Theorem 4.2. *The polynomial* $\Lambda^{(\gamma)}_{mn+\nu}(z;Q)$, $n \in \mathbb{N}_0$, $0 \le \nu \le m-1$, *satisfies following* Q*–difference equation:*

$$A^{(L)}_{n,\nu}(z;Q) z^2\ \mathcal{D}^2_Q Y(z) + B^{(L)}_{n,\nu}(z;Q) z\ \mathcal{D}_Q Y(z) + C^{(L)}_{n,\nu}(z;Q)\ Y(z) = 0,$$

where

$$\begin{aligned}
A^{(L)}_{n,\nu}(z;Q) =& Q^{m(\gamma-1)+\nu+2}(1+Q^m z^m),\\
B^{(L)}_{n,\nu}(z;Q) =& [m(\gamma-1)+\nu+2]_Q\ - Q^{m(\gamma-1)+\nu+1}\ [\nu]_Q\\
&-\Big(\frac{Q^{m\gamma+\nu+2}}{1-Q} + Q^{m\gamma+\nu+1}[mn+\nu]_Q\Big) z^m,\\
C^{(L)}_{n,\nu}(z;Q) =& \frac{Q^{m\gamma+\nu+1}}{1-Q}[mn+\nu]_Q z^m - [\nu]_Q\ [m(\gamma-1)+\nu+1]_Q.
\end{aligned}$$

Proof. If we repeat the procedure used in the proof of the previous theorem, but now starting with the q–difference equation

$$\begin{aligned}
&- q^{\gamma+(2\nu+1)/m}(1-q^n) x y(qx)\\
&\quad = q^{\gamma-1+(2\nu+1)/m}(1+qx) y(q^2x) - \big(1 + q^{\gamma-1+(2\nu+1)/m}(1+qx)\big) y(qx) + y(x)
\end{aligned}$$

satisfied by the q–Laguerre polynomials $L^{(\gamma-1+(2\nu+1)/m)}_n(x;q)$ and the representation (3.3), we get

$$\begin{aligned}
&Q^{m(\gamma-1)+\nu+1}(1+Q^m z^m)\ \Lambda^{(\gamma)}_{mn+\nu}(Q^2 z;Q)\\
&-\big(1+Q^{m(\gamma-1)+2\nu+1}+Q^{m(n+\gamma)+2\nu+1} z^m\big)\ \Lambda^{(\gamma)}_{mn+\nu}(Qz;Q) + Q^\nu\ \Lambda^{(\gamma)}_{mn+\nu}(z;Q) = 0,
\end{aligned}$$

and, finally, the required difference relation. □

Theorem 4.3. *For the polynomials* $\lambda^{(\gamma)}_N(z;Q)$ *and* $\Lambda^{(\gamma)}_N(z;Q)$, $N \in \mathbb{N}_0$, *Rodrigues – type formulas*

$$\begin{aligned}
\lambda^{(\gamma)}_{mn+\nu}\big(z^{1/m};q^{1/m}\big) &= (-1)^n q^{n(n+\gamma-2+(2\nu+1)/m)}(1-q)^n\\
&\quad\times z^{-\gamma+1-(\nu+1)/m}\ e_q(qz)\mathcal{D}^n_{q^{-1}}\Big(z^{n+\gamma-1+(2\nu+1)/m}\ E_q(-qz)\Big),\\
\Lambda^{(\gamma)}_{mn+\nu}\big(z^{1/m};q^{1/m}\big) &= (-1)^n (1-q)^n q^{-n(n+\gamma-1+(2\nu+1)/m)}\\
&\quad\times z^{-\gamma+1-(\nu+1)/m}\ E_q(z)\mathcal{D}^n_q\Big(z^{n+\gamma-1+(2\nu+1)/m}\ e_q(-z)\Big)
\end{aligned}$$

are valid.

Proof. If in the Rodrigues – type formula for the little q–Laguerre polynomials

$$x^{\alpha} E_q(-qx)\ p_n(x; q^{\alpha} | \, q) = q^{n\alpha+\binom{n}{2}} (1-q)^n e_q(q^{\alpha+1})\ E_q(-q^{n+\alpha+1})\ \mathcal{D}^n_{q^{-1}} \Big(x^{n+\alpha} E_q(-qx) \Big)$$

we apply the representation (3.1), we get

$$\begin{aligned} &\big(K_{n,\nu}(q)\big)^{-1} z^{-\nu/m} \lambda^{(\gamma)}_{mn+\nu}(z^{1/m}; q^{1/m}) \\ &= q^{n(\gamma-1+(2\nu+1)/m)+\binom{n}{2}} (1-q)^n\ e_q(q^{\gamma+(2\nu+1)/m})\ E_q(-q^{n+\gamma+(2\nu+1)/m}) \\ &\times z^{-\gamma+1-(\nu+1)/m}\ e_q(qz) \mathcal{D}^n_{q^{-1}} \Big(z^{n+\gamma-1+(2\nu+1)/m}\ E_q(-qz) \Big). \end{aligned}$$

According to the expression (3.2) and the properties of the q-exponential functions, we get the Rodrigues – type formula for polynomials $\lambda^{(\gamma)}_N(z; Q)$. The Rodrigues – type formula for the polynomials $\Lambda^{(\gamma)}_N(z; Q)$ can be obtained as previous, using the formula

$$L^{(\alpha)}_n(x; q) = \frac{1}{[n]_q!}\ x^{-\alpha} E_q(x) \mathcal{D}^n_q \Big(x^{n+\alpha} e_q(-x) \Big).$$

and the representation (3.3)-(3.4). □

Theorem 4.4. *The generating functions for the polynomials* $\lambda^{(\gamma)}_N(z; Q)$ *and* $\Lambda^{(\gamma)}_N(z; Q)$*,* $N \in \mathbb{N}_0$*, are given by*

$$\begin{aligned} &e_q(q) \sum_{n=0}^{\infty} \sum_{\nu=0}^{m-1} E_q(-q^{n+1})\ E_q\big(-q^{n+\gamma+(2\nu+1)/m}\big) \lambda^{(\gamma)}_{mn+\nu}(z; q^{1/m})\ u^{mn+\nu} \\ &= E_q(-u^m) e_q\big((zu)^m\big) \sum_{\nu=0}^{m-1} (zu)^{\nu}\ {}_0\Phi_1 \left(\begin{matrix} - \\ q^{\gamma+(2\nu+1)/m} \end{matrix} \,\Big|\, q; q^{\gamma+(2\nu+1)/m} (zu)^m \right), \end{aligned}$$

$$\begin{aligned} &\sum_{n=0}^{\infty} \sum_{\nu=0}^{m-1} (-1)^n q^{n(n+\sigma-1)} E_q(-q^{n+1}) E_q(-q^{n+\sigma}) \Lambda^{(\gamma)}_{mn+\nu}(z; q^{1/m}) u^{mn+\nu} \\ &= E_q(-q)\ e_q(u^m)\ \sum_{\nu=0}^{m-1} (zu)^{\nu}\ E_q(-q^{\sigma})\ {}_0\Phi_1 \left(\begin{matrix} - \\ q^{\sigma} \end{matrix} \,\Big|\, q^m;\ -q^{\sigma} (zu)^m \right), \end{aligned}$$

where $\sigma = \sigma(\nu) = \gamma + (2\nu+1)/m$.

Proof. Using the generating function for the little q–Laguerre polynomials $p_n(x;a|q)$

$$\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}} E_q(-q^{n+1}) p_n(x;a\,|\,q) t^n = E_q(-q)\ e_q(xt)\ E_q(-t) {}_0\Phi_1\left(\begin{matrix} - \\ aq \end{matrix}\ \middle|\ q; aqxt\right)$$

and the relations (3.1)–(3.2) we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} E_q(-q^{n+1})\ E_q(-q^{n+\gamma+(2\nu+1)/m})\ \lambda_{mn+\nu}^{(\gamma)}(x^{1/m}; q^{1/m})\ t^n \\
&= x^{\nu/m}\ E_q(-q)\ E_q(-q^{\gamma+(2\nu+1)/m})\ e_q(xt)\ E_q(-t) \\
&\times {}_0\Phi_1\left(\begin{matrix} - \\ q^{\gamma+(2\nu+1)/m} \end{matrix}\ \middle|\ q;\ q^{\gamma+(2\nu+1)/m} xt\right),
\end{aligned}$$

for any $0 \le \nu \le m-1$. Taking $z = x^m$ and $u = t^m$ and summing for $0 \le \nu \le m-1$ we get the expansion. For the second formula we use the generating function for the q–Laguerre polynomials $L_n^{(\alpha)}(z;q)$

$$\sum_{n=0}^{\infty} E_q(-q^{n+\alpha+1})\ L_n^{(\alpha)}(x;q)\ t^n = E_q(-q^{\alpha+1})\ e_q(t)\ {}_0\Phi_1\left(\begin{matrix} - \\ q^{\alpha+1} \end{matrix}\ \middle|\ q; -q^{\alpha+1} xt\right)$$

and the representation (3.3)–(3.4). □

Theorem 4.5. *The norm of the polynomials* $\lambda_N^{(\gamma)}(z;Q)$ *and* $\Lambda_N^{(\gamma)}(z;Q)$, $N \in \mathbb{N}_0$, *are*

$$\begin{aligned}
\|\lambda_N^{(\gamma)}\|_l^2 &= m(1-Q)\ Q^{n(m(n+\gamma)+2\nu+1)} \\
&\quad \times \left(E_{Q^m}(-Q^m)\right)^2\ e_{Q^m}(Q^{m(n+\gamma)+2\nu+1})\ e_{Q^m}(Q^{m(n+1)}), \\
\|\Lambda_N^{(\gamma)}\|_L^2 &= m(1-Q)\ Q^{-2mn(n+\gamma)-2n(2\nu+1)+mn} \\
&\quad \times \left(E_{Q^m}(-Q^m)\right)^2\ e_{Q^m}(Q^{m(n+1)})\ e_{Q^m}(-1)\ e_{Q^m}(-Q^m) \\
&\quad \times e_{Q^m}(Q^{m(n+\gamma)+2\nu+1})\ E_{Q^m}(Q^{m\gamma+2\nu+1})\ E_{Q^m}(Q^{-m(\gamma+1)-2\nu-1}).
\end{aligned}$$

Proof. The norms of the polynomials are obtained using the definition of the inner products (2.1)-(2.4)-(2.6) and (2.2)-(2.5)-(2.7) and Theorem 3.1 and Theorem 3.2. □

Finally, we will give the simple, but not trivial connection between the involving class of the polynomials.

Theorem 4.6. *For the polynomials* $\lambda_{mn+\nu}^{(\gamma)}(z;Q)$ *and* $\Lambda_{mn+\nu}^{(\gamma)}(z;Q)$, $n \in \mathbb{N}_0$, $0 \le \nu \le m-1$, *holds*

$$\varphi_m(\nu/2)\ \lambda_{mn+\nu}^{(\gamma)}(z;Q^{-1}) = (-1)^n\ \Lambda_{mn+\nu}^{(\gamma)}\big(\varphi_m(1/2)\ z;Q\big).$$

Proof. Using Theorem 3.1 and Theorem 3.2 and the connection between the little q–Laguerre and the q–Laguerre polynomials given by (1.8), we obtain

$$(-1)^{\nu/m}\Lambda_{mn+\nu}^{(\gamma)}\big(z;Q^{-1}\big) = (-1)^n\lambda_{mn+\nu}^{(\gamma)}\big((-1)^{1/m}\ z;Q\big).$$

Since $(-1)^{1/m} = \varphi_m(j+1/2)$ for any $0 \le j \le m-1$, we have

$$\varphi_m\big(\nu(j+1/2)\big)\ \Lambda_{mn+\nu}^{(\gamma)}\big(z;Q^{-1}\big) = (-1)^n\lambda_{mn+\nu}^{(\gamma)}\big(\varphi_m(j+1/2)\ z;Q\big)$$

and according to (2.9) and the properties of $\varphi_m(j)$, we get the required relation–ship.

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