Modeling errors in long-haul optical fiber transmission systems by using instantons and Edgeworth expansion

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***Abstract*— In this work we use a new approach to model error events in long-haul optical fiber transmission systems. Existing approaches for obtaining probability density functions (PDFs) relay on numerical simulations or analytical approximations. Numerical simulations make far tails of the PDFs difficult to obtain, while analytical approximations are often inaccurate, as they neglect nonlinear interaction between pulses and noise.**

Our approach combines the instanton method from statistical mechanics, to model far tails of the PDFs, with numerical simulations to refine the middle part of the PDFs. We combine the two methods by using an orthogonal polynomial expansion constructed specifically for this problem. We demonstrate the approach on an example of a specific submarine transmission system.

I. Introduction

Investigation of error statistics in high speed optical fiber communication systems is a fundamental task. The nonlinear nature of the propagation of light, nonlinear inter-symbol interference (ISI) between neighboring pulses coupled with noise makes this task very challenging.

Due to ISI it is necessary to study probability density functions (PDFs) of signal samples corresponding to longer bit configurations and not just individual bits. Existing approaches for modelling these PDFs rely either on extensive numerical simulations [1], [2] or on simplified and often inaccurate analytical approximations ([3] and references therein).

Numerical approaches approximate middle part of distributions well, but far tails of the PDFs are very hard to obtain numerically. Unfortunately, tails of the PDFs are very important, since acceptable bit error-rates in communication industry are low $\sim 10^{-9} - 10^{-12}$, so error events fall into far tails of distributions.

All the existing analytical approaches neglect non-linear interaction between pulses and noise (which is implicitly incorporated in numerical modelling), therefore leading to PDFs approximations that are applicable only under very sever restrictions in terms of system speed, distance and types of fiber used (see [4], [5]).

Several recent papers [6], [7], [5], [9] used Karhunen-Loève series expansion (KL) to determine PDFs. In the cases when covariance function is known this method works well [5]. In the case of a general optical fiber communication system covariance between received pulses is not known and the KL expansion approach has several drawbacks. Firstly a covariance matrix needs to be approximated numerically (in which case Karhunen-Loève expansion is in fact the Principle Component Analysis (PCA)) and numerical calculation of eigen-quantities is often unstable. Secondly, a separate set of eigen quantities needs to be numerically calculated for each bit configuration (for more details about problems related with this approach see [10], [6], and appendix of [5]).

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In this paper we use method of optimal fluctuations or *instantons* to model far tails, and numerical simulations to refine the middle part of PDFs. We combine these two approaches by using Edgeworth expansion with orthogonal polynomials specially constructed for this problem.

The method of optimal fluctuations was originally developed in statistical physics [11]. This approach is very similar to numerical saddle point approaches used in [8] but in contrast to that paper we calculate the saddle point, i.e, "the most damaging" noise configuration analytically.

Edgeworth expansion is a statistical method for approximating unknown PDFs [12]. An unknown distribution $w(x)$ is approximated by successive improvements of known starting approximation $u(x)$ by numerically (or experimentally) obtained moments of $w(x)$. In the existing literature this method is almost always used with a Gaussian distribution as $u(x)$ [12], probably because in this case it involves widely known Hermite polynomials (this special case of Edgeworth expansion is referred to as Gram-Charlier expansion). Recently use of Gram-Charlier expansion was suggested [3] in the context of optical communication systems.

In this work we use PDFs derived by instanton method as $u(x)$. By using these PDFs, we obtain better asymptotic properties of the approximate distributions, which leads to faster and more accurate approximation of unknown PDFs. This requires derivation of a special family of orthogonal polynomials, which is given in Section III.

In the Section IV we apply the proposed method to find PDFs for a single mode system with parameters corresponding to submarine systems.

II. Probability Density Functions obtained by instanton method

The derivation of the instanton approximation for the PDFs follows the method derived in [16], and it is already presented in [13], so due to space limitations it will not be repeat it here.

Final formula is:

$$P(E|s) \approx C \exp\left[-\frac{1}{2Nz}\left(\sqrt{E} - \sqrt{\int_{-T/2}^{T/2} |A_s(t,z)|^2}\right)^2\right] \quad (1)$$

where C is normalization constant, N is noise intensity, z is propagation distance, T is the size of a time slot and s is a bit configuration surrounding the center bit that gives $A_s(t,z)$ via non-linear interaction of the pulses in the absence of noise.

The asymptotic behavior of PDFs is not Gaussian as often assumed in the existing literature. We note that this can be concluded from the study in [4] and it is explicitly stated in [5]. Similar approximations for PDFs have been derived in different ways, see for example [2], [14].

At the end of this section, we would like to mention that instanton method allows any noise statistics as long as it is uncorrelated. Also, the calculations do not depend to specific pulse shaping.

III. Edgeworth expansion

In this section we use Edgeworth expansion [12] to refine PDFs derived in the previous section.

An unknown distribution $w(x)$ can be represented as

$$w(x) = u(x)\left[\sum_{i=1}^{\infty} C_i P_i(x)\right] \quad (2)$$

where $u(x)$ is a starting approximate distribution and $P_i(x), n \in N$ is a family of polynomials orthogonal with respect to the weight $u(x)$. Let $P_i(x) = \sum_{k=1}^{i} a_k x^k$. By multiplying Eq. 2, by $P_i(x)$ and integrating over the domain of orthogonality (in our case $x \in (0, +\infty)$ since x represents energy) we get $\sum_{k=1}^{i} a\eta_k = C_i$ where $\eta_k, k \in N$ are moments of the distribution $w(x)$. We can obtain a finite number j of these moments numerically or experimentally, therefore deriving an approximate PDF $\tilde{w}_j(x)$ by truncating infinite sum Eq. 2 to j terms. These approximate PDFs $\tilde{w}_j(x), j \in N$ are guaranteed to converge to $w(x)$ uniformly [12].

In order to use PDFs given in Eq. 1 as starting distributions we need polynomials $P_n(x;m;p), n \in N$ orthogonal with respect to the weight

$$u(x;m;p) = \mathrm{e}^{-m(\sqrt{x}-p)^2}, \quad x \in (0, +\infty) \qquad (m, p > 0). \quad (3)$$

These polynomials can be seen as generalization of Laguerre polynomials and to the best of our knowledge have not been studied before. Here we shall briefly explain how to construct them; their properties will be studied in a separate publication which is in preparation. Moment of the distribution $u(x;m;p)$ can be written as

$$\mu_n(m;p) = \frac{2}{m^{2n+2}} \int_{-p}^{+\infty} (t+p)^{2n+1} \mathrm{e}^{-t^2}\, dt.$$

These moments can be calculated via recurrent relation derived by partial integration rule.

Knowing all the moments $\mu_n(m;p)$ we can calculate the polynomials by:

$$P(x;m;p) = \frac{\det(E_n)}{m^{(n+1)(3n-2)} \mu_1^n(p)}, \quad (4)$$

where the matrix $E_n = [e_{i,j}]_{(n+1)\times(n+1)}$ has the elements $e_{i,j} = m^{2(n-j)} \mu_{i+j-1}$ $(j = 1, 2, \ldots, n)$ $e_{i,n+1} = m^{2(n+i)-4} x^{i-1}$, $(i = 1, 2, \ldots, n+1)$.

IV. Numerical results

In this section we shall illustrate the developed method. We considered the system that consists of periodically distributed sections of fiber with positive D_+ and negative dispersion D_- separated by amplifiers (EDFA). One *span* consist of one section of fiber with positive dispersion, one section of fiber with negative dispersion, and corresponding amplifiers.

The transmission of a signal through the fiber is modelled by the nonlinear Schrödinger equation (NLSE) [15]. In the system simulator, propagation of pulses through the system, i.e. solving NLSE, was done numerically by the split-step Fourier method.

The parameters of positive dispersion D_+ and negative dispersion D_- fibers are given in Table I. Pre-compensation of -330 ps/nm and corresponding post-compensation were also applied. The RZ modulation format has duty cycle of 33%, and the launched power was set to -6dBm. EDFAs with noise figure of 8dB were deployed after every fiber section. The nonlinear distance of this system is roughly 6000km. As expected, for distances below this number the instanton approximation itself approximates the true PDF well. To illustrate this, in the Fig. 1, we plot both PDFs obtained by instanton method and histograms obtained numerically, for two bit configurations: (i) with zero in the center slot $s = 0110110$ and (ii) with "1" at the center slot $s = 0001000$. Both bit configurations were propagated through 100 spans, that is 5000km.

TABLE I

Parameters of the fibers used

Parameters	D_+ fiber	D_- fiber
Dispersion [ps/(nm km)]	20	-40
Dispersion Slope [ps/(nm^2 km)]	0.06	-0.12
Effective Cross-sectional Area [μm^2]	110	50
Nonlinear refractive index [m^2/W]	2.2×10^{-20}	2.2×10^{-20}
Attenuation Coefficient [dB/km]	0.19	0.25
Length (in one span) [km]	33.4	16.7

However, when the propagation distance is longer the instanton approximation is not sufficient. As Fig. 2 and Fig. 3 show, after 300 spans (approximately two and a half nonlinear distances) neglecting of nonlinear interaction between noise and pulses in the instanton approximation makes this approximation overly "optimistic", i. e, too narrow. After polynomial correction is added, the refined PDFs show negligible differences to numerically obtained histograms. Note that only polynomials up to fourth order are needed to refine the PDF for "0110110", and only first two polynomials are needed for "0001000". This is in sharp difference with what is reported in [3], where, for a similar system, more than a dozen of Hermite polynomials are needed to sufficiently improve starting Gaussian distribution (also presented on figures).

Of course, question arises how many polynomials are needed to approximate an unknown distribution sufficiently

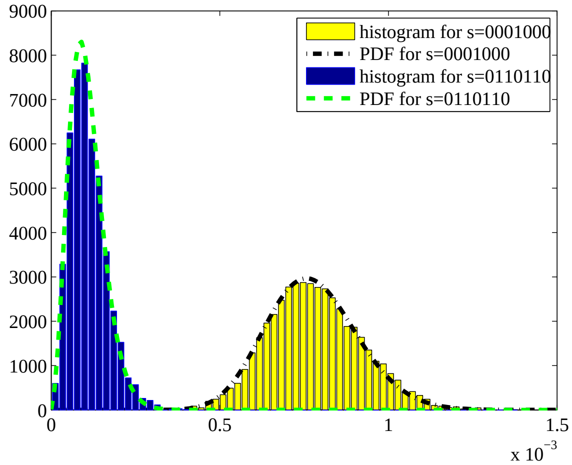

Fig. 1. Probability density functions after 100 spans

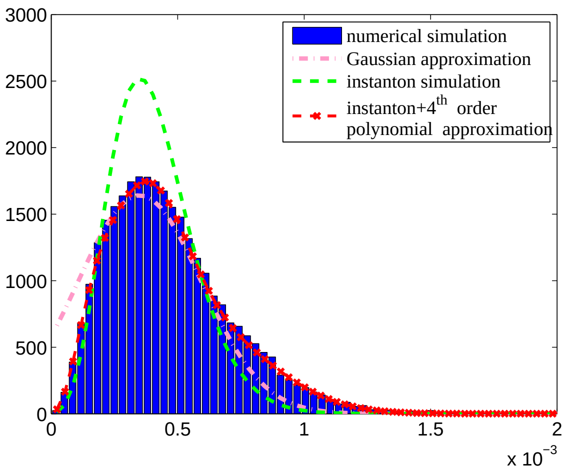

Fig. 2. Comparison of various PDF approximations for energy of bit pattern “0110110” after 300 spans

well. From the engineering point of view, satisfactory answer can be to add higher order polynomials until the moment difference between two consecutive refinements falls under certain threshold.

V. Summary

We developed an approach for approximating probability density functions that is both practical and accurate. The instanton method gives right asympotic behavior for the tails of distributions. Use of the parameterized family of orthogonal polynomials saves from extensive numerical calculations, making this approach applicable for high speed applications.

Method is also very general (it is not restricted to specific pulse shaping, bit rate, propagation distance) and therefore applicable to a wide range of systems.

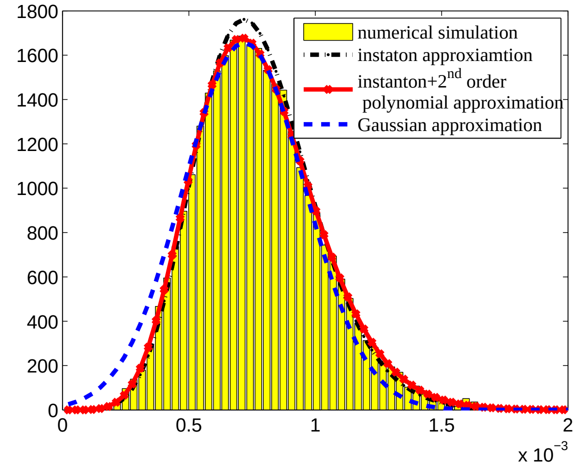

Fig. 3. Comparison of various PDF approximations for energy of bit pattern “0001000” after 300 spans

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References

[1] I. Djordjevic, M. Ivkovic, B. Vasic, I. Gabitov, “Achievable information rates for high-speed long-haul optical transmission”, *J. Lightwave Technol.*, vol. 23 no.11, pp. 3755-3763, Nov. 2005.

[2] O.E.Agazzi, M.R. Hueda, H.S. Carrer, D.E. Crivelli, “Maximum-likelihood sequence estimation in dispersive optical channels” J. Lightwave Technology, vol. 23, issue 2, pp. 749 - 763, Feb. 2005.

[3] M. Nazarathy, “Accurate evaluation of bit-error rates of optical communication systems using the Gram-Chariler series”, *IEEE Trans. on Communications*, vol. 54, no.2, Feb. 2006.

[4] P.A. Humblet, M. Azizoglu, ”On the bit error rate in lightwave systems with optical amplifiers,” *J. Lightwave Tech.* vol.9, 1576-82, Nov. 1991.

[5] N. Alić *et al.* “Signal statistics and maximum likelihood sequence estimation in intensity modulated fiber optic links containing a single optical preamplifier”, *Opt. Exp*, vol. 13, pp. 4568-4579, Jun. 2005.

[6] T. Freckmann, J. Spiedel, “Viterbi equalizer with analytically calculated branch meterics for optical ASK and DBPSK,” *Photonic Technology Letters*, vol. 18, no. 1, pp. 277-279, Jan. 2006.

[7] F. Buchali, G. Thielecke, H. Blow, “Viterbi equalizer for mitigation of distortions from chromatic dispersion and PMD at 10 Gb/s,” in Proc. Opt. Fiber Comm. Conf., vol. 1, Los Angeles, CA, Feb. 2004.

[8] C.L. Ho, “Calculating the performance of optical communication systems with modal noise by saddlepoint method,” *J. Lightw. Technol.*, no. 9, pp. 18201825, Sep. 1995.

[9] E. Forestieri, “Evaluating the error probability in lightwave systems with chromatic dispersion, arbitrary pulse shape and pre- and postdetection filtering,” *J. Lightw. Technol.*, vol. 18, no. 11, pp. 1493-1503, Nov. 2000.

[10] E.R. Dougherty, *Random Processes for Image and Signal Processing*. SPIE Opt. Eng. Press, IEEE Press, New York 1998.

[11] I. M. Lifshitz, “Energy spectrum structure and quantum states of disordered condensed systems,”*Usp. Fiz. Nauk*, vol. 83 no.4, 1964.

[12] J. Kolassa *Series Approximation Methods in Statistics,* 2nd ed. Springer-Verlag, New York 1997.

[13] M. Ivkovic, I. Djordjevic, B. Vasic, “A soft decision decoding scheme for long-haul optical transmission systems based on the instanton approach”, accepted for presentation at GLOBECOM’06.

[14] T. Foggi *et al.* “Maximum likelihood sequence detection with closed-form metrics in OOK optical systems impaired by GVD and PMD”, J. Lightwave Technology vol. 24. no. 8., pp 3073-3087, August 2006.

[15] G. P. Agrawal, *Nonlinear Fiber Optics*. San Diego, CA: Academic, 2001.

[16] V. Chernyak, *et al.* “PMD-induced fluctuations of bit-error rate in optical fiber systems”, *J. Lightwave Technol.* vol. 22, no.4, pp.1155-68, 2004