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FRACTIONAL INTEGRALS AND DERIVATIVES IN q-CALCULUS

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We generalize the notions of the fractional q-integral and q-derivative by introducing variable lower limit of integration. We discuss some properties and their relations. Finally, we give a q-Taylor-like formula which includes fractional q-derivatives of the function.

1. INTRODUCTION

In the theory of q-calculus (see [**5**] and [**7**]), for a real parameter $q \in \mathbb{R}^+ \setminus \{1\}$, we introduce a q-real number $[a]_q$ by

$$[a]_q := \frac{1-q^a}{1-q} \qquad (a \in \mathbb{R}) \ .$$

The q-analog of the Pochhammer symbol (q–shifted factorial) is defined by:

$$(a;q)_0 = 1 \ , \qquad (a;q)_k = \prod_{i=0}^{k-1}(1-aq^i) \qquad \big(k \in \mathbb{N} \cup \{\infty\}\big) \ .$$

Also, the q-analog of the power $(a-b)^k$ is

$$(a-b)^{(0)} = 1 \ , \qquad (a-b)^{(k)} = \prod_{i=0}^{k-1}(a-bq^i) \qquad \big(k \in \mathbb{N};\ a,b \in \mathbb{R}\big) \ .$$

There is the following relationship between them:

$$(a-b)^{(n)} = a^n\ (b/a;q)_n \qquad (a \neq 0) \ .$$

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Their natural expansions to the reals are

$$(1)\qquad (a-b)^{(\alpha)} = a^{\alpha}\,\frac{(b/a;q)_\infty}{(q^\alpha b/a;q)_\infty}\ , \qquad (a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \qquad (\alpha\in\mathbb{R})\ .$$

Notice that

$$(a-b)^{(\alpha)} = a^{\alpha}(b/a;q)_\alpha .$$

The following formulas (see, for example, [**5**] and [**4**]) will be useful:

$$(2)\qquad (a;q)_n = \left(q^{1-n}/a;q\right)_n\,(-1)^n\ a^n\ q^{\binom{n}{2}};$$

$$(3)\qquad \frac{(aq^{-n};q)_n}{(bq^{-n};q)_n} = \frac{(q/a;q)_n}{(q/b;q)_n}\ \left(\frac{a}{b}\right)^n;$$

$$(4)\qquad (a-b)^{(\alpha)} = a^{\alpha}\sum_{k=0}^{\infty}(-1)^k \begin{bmatrix}\alpha\\ k\end{bmatrix}_q q^{\binom{k}{2}}\left(\frac{b}{a}\right)^k .$$

The q-gamma function is defined by

$$(5)\qquad \Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}\ (1-q)^{1-x} = (1-q)^{(x-1)}\ (1-q)^{1-x},$$

where $x\in\mathbb{R}\setminus\{0,-1,-2,\ldots\}$. Obviously,

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x)\ .$$

We can define q-binomial coefficients with

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_q = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\beta+1)\ \Gamma_q(\alpha-\beta+1)} = \frac{(q^{\beta+1};q)_\infty\ (q^{\alpha-\beta+1};q)_\infty}{(q;q)_\infty\ (q^{\alpha+1};q)_\infty}$$

$\alpha,\beta,\alpha-\beta\in\mathbb{R}\setminus\{-1,-2,\ldots\}$. Particularly,

$$(6)\qquad \begin{bmatrix}\alpha\\ k\end{bmatrix}_q = \frac{(q^{-\alpha};q)_k}{(q;q)_k}\ (-1)^k q^{\alpha k} q^{-\binom{k}{2}} \quad (k\in\mathbb{N}).$$

The q–hypergeometric function is defined as

$${}_2\phi_1\left(\begin{matrix}a,\ b\\ c\end{matrix}\ \middle|\ q;x\right) = \sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(c;q)_n\ (q;q)_n}\ x^n\ .$$

The famous Heine transformation formula [**5**] is

$$(7)\qquad {}_2\phi_1\left(\begin{matrix}a,\ b\\ c\end{matrix}\ \middle|\ q;x\right) = \frac{(abx/c;q)_\infty}{(x;q)_\infty}\ {}_2\phi_1\left(\begin{matrix}c/a,\ c/b\\ c\end{matrix}\ \middle|\ q;abx/c\right)\ .$$

We define a q-derivative of a function $f(x)$ by

$$\big(D_qf\big)(x) = \frac{f(x)-f(qx)}{x-qx}\ \ (x\neq 0)\ , \quad \big(D_qf\big)(0) = \lim_{x\to 0}\big(D_qf\big)(x)$$

and q-derivatives of higher order:

(8) $$D_q^0 f = f\ , \qquad D_q^n f = D_q\big(D_q^{n-1} f\big) \quad (n = 1, 2, 3, \ldots)\ .$$

For an arbitrary pair of functions $u(x)$ and $v(x)$ and constants $\alpha, \beta \in \mathbb{R}$, we have linearity and product rules

$$D_q\big(\alpha\ u(x) + \beta\ v(x)\big) = \alpha\big(D_q u\big)(x) + \beta\big(D_q v\big)(x),$$
$$D_q\big(u(x) \cdot v(x)\big) = u(qx)\big(D_q v\big)(x) + v(x)\big(D_q u\big)(x)\ .$$

The q-integral is defined by

$$\big(I_{q,0} f\big)(x) = \int_0^x f(t)\,\mathrm{d}_q t = x(1-q)\sum_{k=0}^{\infty} f(xq^k)\,q^k \quad (0 \le |q| < 1),$$

and

(9) $$\big(I_{q,a} f\big)(x) = \int_a^x f(t)\,\mathrm{d}_q t = \int_0^x f(t)\,\mathrm{d}_q t - \int_0^a f(t)\,\mathrm{d}_q t.$$

However, these definitions cause troubles in research as they include the points outside of the interval of integration (see [**6**] and **10**]). In the case when the lower limit of integration is $a = xq^n$, i.e., when it is determined for some choice of x, q and positive integer n, the q-integral (9) becomes

(10) $$\int_{xq^n}^x f(t)\,\mathrm{d}_q t = x(1-q)\sum_{k=0}^{n-1} f(xq^k) q^k\,.$$

As for q-derivative, we can define an operator $I_{q,a}^n$ by

$$I_{q,a}^0 f = f, \qquad I_{q,a}^n f = I_{q,a}\big(I_{q,a}^{n-1} f\big) \quad (n = 1, 2, 3, \ldots)\ .$$

For operators defined in this manner, the following is valid:

(11) $$\big(D_q I_{q,a} f\big)(x) = f(x), \qquad \big(I_{q,a} D_q f\big)(x) = f(x) - f(a).$$

The formula for q-integration by parts is

$$\int_a^b u(x)\big(D_q v\big)(x)\,d_q x = \big[u(x)v(x)\big]_a^b - \int_a^b v(qx)\big(D_q u\big)(x)\,d_q x\ .$$

W. A. Al-Salam [**2**] and R. P. Agarwal [**1**] introduced several types of fractional q-integral operators and fractional q-derivatives. Here, we will only mention the fractional q-integral with the lower limit of integration $a = 0$, defined by

$$\big(I_q^{\eta,\alpha} f\big)(x) = \frac{x^{-(\eta+\alpha)}}{\Gamma_q(\alpha)} \int_0^x (x - tq)^{(\alpha-1)} t^{\eta} f(t)\ d_q t \qquad (\eta, \alpha \in \mathbb{R}^+)\,.$$

On the other hand, the solution of nth order q-differential equation

$$\big(D_q^n y\big)(x) = f(x), \qquad \big(D_q^k y\big)(a) = 0 \quad (k = 0, 1, \ldots, n-1),$$

can be written in the form of a multiple q-integral

$$y(x) = \big(I_{q,a}^n f\big)(x) = \int_a^x \mathrm{d}_q t \int_a^t \mathrm{d}_q t_{n-1} \int_a^{t_{n-1}} \mathrm{d}_q t_{n-2} \cdots \int_a^{t_2} f(t_1)\, \mathrm{d}_q t_1.$$

The reduction of the multiple q-integral to a single one was considered by Al-Salam [**3**]. He thought of it as a *q-analog of Cauchy's formula*:

$$(12) \qquad y(x) = \big(I_{q,a}^n f\big)(x) = \frac{1}{[n-1]_q!} \int_a^x (x - qt)^{(n-1)} f(t)\ \mathrm{d}_q t \qquad (n \in \mathbb{N}).$$

In this paper, our purpose is to consider fractional q-integrals with the parametric lower limit of integration. After preliminaries, in the third section we define the fractional q-integral in that sense. On the basis of that, the fractional q-derivative is introduced in the fourth section. Finally, in the last section, we give a q-Taylor-like formula using these fractional q-derivatives.

2. PRELIMINARIES

We will first specify some results which are useful in the sequel and which can be proved easily.

Lemma 1. *For $a, b, \alpha \in \mathbb{R}^+$ and $k, n \in \mathbb{N}$, the following properties are valid*:

$$(13) \qquad (a - bq^k)^{(\alpha)} = a^\alpha (1 - q^k b/a)^{(\alpha)},$$

$$(14) \qquad \frac{(a - bq^k)^{(\alpha)}}{(a - b)^{(\alpha)}} = \frac{(q^\alpha b/a; q)_k}{(b/a; q)_k},$$

$$(15) \qquad (q^n - q^k)^{(\alpha)} = 0 \quad (k \le n).$$

The next result will have an important role in proving the semigroup property of the fractional q–integral.

Lemma 2. *For $\mu, \alpha, \beta \in \mathbb{R}^+$, the following identity is valid*

$$(16) \qquad \sum_{n=0}^{\infty} \frac{(1 - \mu q^{1-n})^{(\alpha-1)}\ (1 - q^{1+n})^{(\beta-1)}}{(1 - q)^{(\alpha-1)}\ (1 - q)^{(\beta-1)}}\ q^{\alpha n} = \frac{(1 - \mu q)^{(\alpha+\beta-1)}}{(1 - q)^{(\alpha+\beta-1)}}\ .$$

Proof. According to the formulas (1) and (3), we have

$$\begin{aligned}(1 - \mu q^{1-n})^{(\alpha-1)} &= \frac{(\mu q^{1-n}; q)_\infty}{(\mu q^{\alpha-n}; q)_\infty} = \frac{(\mu q^{1-n}; q)_n\ (\mu q; q)_\infty}{(\mu q^{\alpha-n}; q)_n\ (\mu q^\alpha; q)_\infty} \\ &= (1 - \mu q)^{(\alpha-1)} \frac{(\mu^{-1}; q)_n}{(\mu^{-1} q^{1-\alpha}; q)_n}\ q^{(1-\alpha)n}\ .\end{aligned}$$

Applying the identity (14) to the expression $(1-q^{1+n})^{(\beta-1)}/(1-q)^{(\beta-1)}$, the sum on the left side of (16) can be written as

$$\begin{aligned} LS &= \frac{(1-\mu q)^{(\alpha-1)}}{(1-q)^{(\alpha-1)}} \sum_{n=0}^{\infty} \frac{(q^\beta;q)_n}{(q;q)_n} \frac{(\mu^{-1};q)_n}{(\mu^{-1}q^{1-\alpha};q)_n}\; q^{(1-\alpha)n} q^{\alpha n} \\ &= \frac{(1-\mu q)^{(\alpha-1)}}{(1-q)^{(\alpha-1)}}\; {}_2\phi_1\left(\begin{matrix} \mu^{-1},\ q^\beta \\ \mu^{-1}q^{1-\alpha} \end{matrix} \;\middle|\; q;q \right) . \end{aligned}$$

Using (7), we get

$$\begin{aligned} LS &= \frac{(1-\mu q)^{(\alpha-1)}}{(1-q)^{(\alpha-1)}}\; \frac{(q^{\alpha+\beta};q)_\infty}{(q;q)_\infty}\; {}_2\phi_1\left(\begin{matrix} q^{1-\alpha},\ \mu^{-1}q^{1-\alpha-\beta} \\ \mu^{-1}q^{1-\alpha} \end{matrix} \;\middle|\; q;q^{\alpha+\beta} \right) \\ &= \frac{(1-\mu q)^{(\alpha-1)}}{(1-q)^{(\alpha-1)}}\; \frac{1}{(1-q)^{(\alpha+\beta-1)}} \sum_{n=0}^{\infty} \frac{(q^{1-\alpha};q)_n\ (\mu^{-1}q^{1-\alpha-\beta};q)_n}{(q;q)_n\ (\mu^{-1}q^{1-\alpha};q)_n}\; q^{(\alpha+\beta)n} . \end{aligned}$$

According to (2) and (1), the following is valid:

$$\begin{aligned} \frac{(\mu^{-1}q^{1-\alpha-\beta};q)_n}{(\mu^{-1}q^{1-\alpha};q)_n} &= \frac{(\mu q^{\alpha+\beta-n};q)_n}{(\mu q^{\alpha-n};q)_n}\; q^{-\beta n} = \frac{(\mu q^{\alpha+\beta-n};q)_\infty}{(\mu q^{\alpha+\beta};q)_\infty}\; \frac{(\mu q^{\alpha};q)_\infty}{(\mu q^{\alpha-n};q)_\infty}\; q^{-\beta n} \\ &= \frac{(\mu q^{\alpha};q)_\infty}{(\mu q^{\alpha+\beta};q)_\infty}\; \frac{(\mu q^{\alpha+\beta-n};q)_\infty}{(\mu q^{\alpha-n};q)_\infty}\; q^{-\beta n} \\ &= \frac{(\mu q^{\alpha};q)_\infty}{(\mu q^{\alpha+\beta};q)_\infty}\; (1-\mu q^{\alpha+\beta-n})^{(-\beta)}\; q^{-\beta n} . \end{aligned}$$

Hence

$$LS = \frac{(1-\mu q)^{(\alpha+\beta-1)}}{(1-q)^{(\alpha-1)}\ (1-q)^{(\alpha+\beta-1)}} \sum_{n=0}^{\infty} \frac{(q^{1-\alpha};q)_n}{(q;q)_n}\; q^{\alpha n}(1-\mu q^{\alpha+\beta-n})^{(-\beta)} .$$

If we use formulas (6) and (4) and change the order of the summation, the last sum becomes

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(q^{1-\alpha};q)_n}{(q;q)_n}\; q^{\alpha n}(1-\mu q^{\alpha+\beta-n})^{(-\beta)} \\ &= \sum_{n=0}^{\infty} \begin{bmatrix} \alpha-1 \\ n \end{bmatrix}_q (-1)^n\; q^{-(\alpha-1)n} q^{\binom{n}{2}}\; q^{\alpha n} \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} -\beta \\ k \end{bmatrix}_q q^{\binom{k}{2}} \left(\mu q^{\alpha+\beta-n}\right)^k \\ &= \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} -\beta \\ k \end{bmatrix}_q q^{\binom{k}{2}} \left(\mu q^{\alpha+\beta}\right)^k \sum_{n=0}^{\infty} (-1)^n \begin{bmatrix} \alpha-1 \\ n \end{bmatrix}_q q^{\binom{n}{2}}\ \left(q^{1-k}\right)^n \\ &= \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} -\beta \\ k \end{bmatrix}_q q^{\binom{k}{2}} \left(\mu q^{\alpha+\beta}\right)^k \left(1-q^{1-k}\right)^{(\alpha-1)} = (1-q)^{(\alpha-1)} . \end{aligned}$$

The last relation is valid because of $\left(1-q^{1-k}\right)^{(\alpha-1)} = 0$ for $k=1,2,\dots$. Finally, the identity holds:

$$LS = \frac{(1-\mu q)^{(\alpha+\beta-1)}}{(1-q)^{(\alpha-1)}\ (1-q)^{(\alpha+\beta-1)}}\ (1-q)^{(\alpha-1)} = \frac{(1-\mu q)^{(\alpha+\beta-1)}}{(1-q)^{(\alpha+\beta-1)}} . \qquad \square$$

3. THE FRACTIONAL q-INTEGRAL

In all further considerations we assume that the functions are defined in an interval $(0,b)$ $(b>0)$, and $a \in (0,b)$ is an arbitrary fixed point. Also, the required q-derivatives and q-integrals exist and the convergence of the series mentioned in the proofs is assumed.

Generalizing the formula (12), we can define the *fractional q-integral of the Riemann-Liouville type* by

$$\big(I_{q,a}^{\alpha} f\big)(x) = \frac{1}{\Gamma_q(\alpha)} \int_a^x (x-qt)^{(\alpha-1)} f(t)\, \mathrm{d}_q t \qquad (\alpha \in \mathbb{R}^+) \;. \tag{17}$$

Using formula (4), this integral can be written as

$$\big(I_{q,a}^{\alpha} f\big)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} \alpha-1 \\ k \end{bmatrix}_q q^{\binom{k+1}{2}} x^{-k} \int_a^x t^k f(t)\, \mathrm{d}_q t \qquad (\alpha \in \mathbb{R}^+)\,.$$

Lemma 3. *For $\alpha \in \mathbb{R}^+$, the following is valid*:

$$\big(I_{q,a}^{\alpha} f\big)(x) = \big(I_{q,a}^{\alpha+1} D_q f\big)(x) + \frac{f(a)}{\Gamma_q(\alpha+1)} (x-a)^{(\alpha)} \qquad (0<a<x<b)\;.$$

Proof. Since the q-derivative over the variable t is

$$D_q\big((x-t)^{(\alpha)}\big) = -[\alpha]_q (x-qt)^{(\alpha-1)}\;,$$

and using the q-integration by parts, we obtain

$$\begin{aligned}
\big(I_{q,a}^{\alpha} f\big)(x) &= -\frac{1}{[\alpha]_q \Gamma_q(\alpha)} \int_a^x D_q\big((x-t)^{(\alpha)}\big) f(t)\, \mathrm{d}_q t \\
&= \frac{1}{\Gamma_q(\alpha+1)} \Big((x-a)^{(\alpha)} f(a) + \int_a^x (x-qt)^{(\alpha)} \big(D_q f\big)(t)\, \mathrm{d}_q t \Big) \\
&= \big(I_{q,a}^{\alpha+1} D_q f\big)(x) + \frac{f(a)}{\Gamma_q(\alpha+1)} (x-a)^{(\alpha)}\;. \qquad \square
\end{aligned}$$

Lemma 4. *For $\alpha, \beta \in \mathbb{R}^+$, the following is valid*:

$$\int_0^a (x-qt)^{(\beta-1)} \big(I_{q,a}^{\alpha} f\big)(t)\, \mathrm{d}_q t = 0 \qquad (0<a<x<b)\,.$$

Proof. Using Lemma 1 and formula (10), for $n \in \mathbb{N}_0$, we have

$$\begin{aligned}
\big(I_{q,a}^{\alpha} f\big)(aq^n) &= \frac{1}{\Gamma_q(\alpha)} \int_a^{aq^n} (aq^n - qu)^{(\alpha-1)} f(u) \mathrm{d}_q u \\
&= \frac{-a^{\alpha}(1-q)}{\Gamma_q(\alpha)} \sum_{j=0}^{n-1} (q^n - q^{j+1})^{(\alpha-1)} f(aq^j) q^j = 0\,.
\end{aligned}$$

Then, according to the definition of q-integral, it follows

$$\int_0^a (x-qt)^{(\beta-1)} \big(I_{q,a}^{\alpha} f\big)(t)\, \mathrm{d}_q t = a(1-q) \sum_{n=0}^{\infty} (x-aq^{n+1})^{(\beta-1)} \big(I_{q,a}^{\alpha} f\big)(aq^n) q^n = 0. \quad \square$$

Theorem 5. *Let $\alpha, \beta \in \mathbb{R}^+$. The q-fractional integration has the following semigroup property*

$$\big(I_{q,a}^{\beta} I_{q,a}^{\alpha} f\big)(x) = \big(I_{q,a}^{\alpha+\beta} f\big)(x) \qquad (0 < a < x < b) \ .$$

Proof. By previous lemma, we have

$$\big(I_{q,a}^{\beta} I_{q,a}^{\alpha} f\big)(x) = \frac{1}{\Gamma_q(\beta)} \int_0^x (x-qt)^{(\beta-1)} \big(I_{q,a}^{\alpha} f\big)(t)\, \mathrm{d}_q t,$$

i.e.,

$$\begin{aligned} \big(I_{q,a}^{\beta} I_{q,a}^{\alpha} f\big)(x) &= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^x (x-qt)^{(\beta-1)} \int_0^t (t-qu)^{(\alpha-1)} f(u)\, \mathrm{d}_q u \\ &\quad - \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^x (x-qt)^{(\beta-1)} \int_0^a (t-qu)^{(\alpha-1)} f(u)\, \mathrm{d}_q u \ . \end{aligned}$$

Using the result from [**1**],

$$\big(I_{q,0}^{\beta} I_{q,0}^{\alpha} f\big)(x) = \big(I_{q,0}^{\alpha+\beta} f\big)(x),$$

we conclude that

$$\big(I_{q,a}^{\beta} I_{q,a}^{\alpha} f\big)(x) = \big(I_{q,0}^{\alpha+\beta} f\big)(x) - \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^x (x-qt)^{(\beta-1)} \int_0^a (t-qu)^{(\alpha-1)} f(u)\, \mathrm{d}_q u.$$

Furthermore, we can write

$$\begin{aligned} \big(I_{q,a}^{\beta} I_{q,a}^{\alpha} f\big)(x) &= \big(I_{q,a}^{\alpha+\beta} f\big)(x) + \frac{1}{\Gamma_q(\alpha+\beta)} \int_0^a (x-qt)^{(\alpha+\beta-1)} f(t)\, \mathrm{d}_q t \\ &\quad - \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^x (x-qt)^{(\beta-1)} \int_0^a (t-qu)^{(\alpha-1)} f(u)\, \mathrm{d}_q u, \end{aligned}$$

wherefrom it follows

$$\big(I_{q,a}^{\beta} I_{q,a}^{\alpha} f\big)(x) = \big(I_{q,a}^{\alpha+\beta} f\big)(x) + a(1-q) \sum_{j=0}^{\infty} c_j f(aq^j) q^j,$$

with

$$c_j = \frac{(x-aq^{j+1})^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} - \frac{x(1-q)}{\Gamma_q(\alpha)\Gamma_q(\beta)} \sum_{n=0}^{\infty} (x-xq^{n+1})^{(\beta-1)}(xq^n - aq^{j+1})^{(\alpha-1)} q^n.$$

By using the formulas from Lemma 1 and (5), we get

$$c_j = ((1-q)x)^{\alpha+\beta-1} \times \left\{ \frac{\left(1-\frac{a}{x}\,q^{j+1}\right)^{(\alpha+\beta-1)}}{(1-q)^{(\alpha+\beta-1)}} - \sum_{n=0}^{\infty} \frac{(1-q^{n+1})^{(\beta-1)}}{(1-q)^{(\beta-1)}} \frac{\left(1-\frac{a}{x}\,q^{j+1-n}\right)^{(\alpha-1)}}{(1-q)^{(\alpha-1)}} q^{n\alpha} \right\}.$$

Putting $\mu = q^j a/x$ into (16), we see that $c_j = 0$ for all $j \in \mathbb{N}$, which completes the proof. □

Lemma 6. *For $\alpha \in \mathbb{R}^+$, $\lambda \in (-1,\infty)$, the following is valid*

$$I_{q,a}^{\alpha}\big((x-a)^{(\lambda)}\big) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)}\,(x-a)^{(\alpha+\lambda)} \qquad (0<a<x<b)\,. \tag{18}$$

Proof. For $\lambda \neq 0$, according to the definition (17), we have

$$I_{q,a}^{\alpha}\big((x-a)^{(\lambda)}\big) = \frac{1}{\Gamma_q(\alpha)}\Big(\int_0^x (x-qt)^{(\alpha-1)}(t-a)^{(\lambda)}\,\mathrm{d}_q t - \int_0^a (x-qt)^{(\alpha-1)}(t-a)^{(\lambda)}\,\mathrm{d}_q t\Big).$$

Also, the following is valid:

$$\int_0^a (x-qt)^{(\alpha-1)}(t-a)^{(\lambda)}\,\mathrm{d}_q t = a^{\lambda+1}(1-q)\sum_{k=0}^{\infty}(x-aq^{k+1})^{(\alpha-1)}(q^k-1)^{(\lambda)}q^k = 0.$$

Therefrom, by using (16), we get

$$\begin{aligned}
&\int_0^x (x-qt)^{(\alpha-1)}(t-a)^{(\lambda)}\,\mathrm{d}_q t \\
&\qquad = x^{\alpha+\lambda}(1-q)\sum_{k=0}^{\infty}(1-q^{1+k})^{(\alpha-1)}\Big(1-\frac{a}{qx}q^{1-k}\Big)^{(\lambda)} q^{(\lambda+1)k} \\
&\qquad = (1-q)\frac{(1-q)^{(\alpha-1)}(1-q)^{(\lambda)}}{(1-q)^{(\alpha+\lambda)}}(x-a)^{(\alpha+\lambda)}.
\end{aligned}$$

Using (5), we obtain the required formula.

Particularly, for $\lambda = 0$, using a q-integration by parts, we have

$$\begin{aligned}
(I_{q,a}^{\alpha}\mathbf{1})(x) &= \frac{1}{\Gamma_q(\alpha)}\int_a^x (x-qt)^{(\alpha-1)}\,\mathrm{d}_q t = \frac{1}{\Gamma_q(\alpha)}\int_a^x \frac{D_q\big((x-t)^{(\alpha)}\big)}{-[\alpha]_q}\,\mathrm{d}_q t \\
&= \frac{-1}{\Gamma_q(\alpha+1)}\int_a^x D_q\big((x-t)^{(\alpha)}\big)\,\mathrm{d}_q t = \frac{1}{\Gamma_q(\alpha+1)}\,(x-a)^{(\alpha)}\,.
\end{aligned}$$

□

4. THE FRACTIONAL q-DERIVATIVE

We define the *fractional q-derivative* by

$$(19)\qquad \big(D_{q,a}^{\alpha}f\big)(x)=\begin{cases}\big(I_{q,a}^{-\alpha}f\big)(x), & \alpha<0\\ f(x), & \alpha=0\\ \big(D_q^{\lceil\alpha\rceil}I_{q,a}^{\lceil\alpha\rceil-\alpha}f\big)(x), & \alpha>0,\end{cases}$$

where $\lceil\alpha\rceil$ denotes the smallest integer greater or equal to α.

Notice that $\big(D_{q,a}^{\alpha}f\big)(x)$ has subscript a to emphasize that it depends on the lower limit of integration used in definition (19). Since $\lceil\alpha\rceil$ is a positive integer for $\alpha\in\mathbb{R}^+$, then for $\big(D_q^{\lceil\alpha\rceil}f\big)(x)$ we apply definition (8).

Lemma 7. *For $\alpha\in\mathbb{R}\setminus\mathbb{N}_0$, the following is valid*:

$$\big(D_qD_{q,a}^{\alpha}f\big)(x)=\big(D_{q,a}^{\alpha+1}f\big)(x)\qquad(0<a<x<b)\ .$$

Proof. We will consider three cases. For $\alpha\le -1$, according to Theorem 5, we have

$$\begin{aligned}\big(D_qD_{q,a}^{\alpha}f\big)(x)&=\big(D_qI_{q,a}^{-\alpha}f\big)(x)=\big(D_qI_{q,a}^{1-\alpha-1}f\big)(x)\\&=\big(D_qI_{q,a}I_{q,a}^{-\alpha-1}f\big)(x)=\big(I_{q,a}^{-(\alpha+1)}f\big)(x)=\big(D_{q,a}^{\alpha+1}f\big)(x).\end{aligned}$$

In the case $-1<\alpha<0$, i.e., $0<\alpha+1<1$, we obtain

$$\big(D_qD_{q,a}^{\alpha}f\big)(x)=\big(D_qI_{q,a}^{-\alpha}f\big)(x)=\big(D_qI_{q,a}^{1-(\alpha+1)}f\big)(x)=\big(D_{q,a}^{\alpha+1}f\big)(x).$$

For $\alpha>0$, we get

$$(D_qD_{q,a}^{\alpha}f)(x)=(D_qD_q^{\lceil\alpha\rceil}I_{q,a}^{\lceil\alpha\rceil-\alpha}f)(x)=(D_q^{\lceil\alpha\rceil+1}I_{q,a}^{\lceil\alpha\rceil-\alpha}f)(x)=(D_{q,a}^{\alpha+1}f)(x).\quad\square$$

Theorem 8. *For $\alpha\in\mathbb{R}\setminus\mathbb{N}_0$, the following is valid*:

$$(D_qD_{q,a}^{\alpha}f)(x)-(D_{q,a}^{\alpha}D_qf)(x)=\frac{f(a)}{\Gamma_q(-\alpha)}(x-a)^{(-\alpha-1)}\quad(0<a<x<b)\ .$$

Proof. We will use formulas (11), Theorem 5, and Lemma 6, to prove the statement. Let us consider two cases. If $\alpha<0$, then

$$\begin{aligned}(D_qD_{q,a}^{\alpha}f)(x)&=(D_qI_{q,a}^{-\alpha}f)(x)=D_qI_{q,a}^{-\alpha}\big((I_{q,a}D_qf)(x)+f(a)\big)\\&=(D_qI_{q,a}^{-\alpha}I_{q,a}D_qf)(x)+f(a)(D_qI_{q,a}^{-\alpha}\mathbf{1})(x)\\&=(D_qI_{q,a}^{-\alpha+1}D_qf)(x)+f(a)D_q\Big(\frac{(x-a)^{(-\alpha)}}{\Gamma_q(-\alpha+1)}\Big)\\&=(D_qI_{q,a}I_{q,a}^{-\alpha}D_qf)(x)+f(a)\frac{[-\alpha]_q(x-a)^{(-\alpha-1)}}{\Gamma_q(-\alpha+1)}\\&=(D_{q,a}^{\alpha}D_qf)(x)+\frac{f(a)}{\Gamma_q(-\alpha)}(x-a)^{(-\alpha-1)}.\end{aligned}$$

If $\alpha > 0$, there exists $l \in \mathbb{N}_0$, such that $\alpha \in (l, l+1)$. Then, applying a similar procedure, we get

$$\begin{aligned}
(D_q D_{q,a}^{\alpha} f)(x) &= (D_q D_q^{l+1} I_{q,a}^{l+1-\alpha} f)(x) \\
&= D_q^{l+2} I_{q,a}^{l+1-\alpha}\big((I_{q,a} D_q f)(x) + f(a)\big) \\
&= (D_q^{l+1} D_q I_{q,a} I_{q,a}^{l+1-\alpha} D_q f)(x) + \frac{f(a)}{\Gamma_q(l+2-\alpha)} D_q^{l+1}\big((x-a)^{(l+1-\alpha)}\big) \\
&= (D_{q,a}^{\alpha} D_q f)(x) + \frac{f(a)}{\Gamma_q(-\alpha)}(x-a)^{(-\alpha-1)}. \qquad \square
\end{aligned}$$

5. THE FRACTIONAL q-TAYLOR-LIKE FORMULA

Many authors tried to generalize the ordinary Taylor formula in different manners. The use of the fractional calculus is of special interest in that area (see, for example [**11**] and [**8**]). Here, we will present one more generalization, based on the use of the fractional q-derivatives.

Lemma 9. *Let $f(x)$ be a function defined on an interval $(0,b)$ and $\alpha \in \mathbb{R}^+$. Then the following is valid*:

$$\big(D_{q,a}^{\alpha} I_{q,a}^{\alpha} f\big)(x) = f(x) \qquad (0 < a < x < b)\ .$$

Proof. For $\alpha > 0$, we have

$$\begin{aligned}
\big(D_{q,a}^{\alpha} I_{q,a}^{\alpha} f\big)(x) &= \big(D_q^{\lceil\alpha\rceil} I_{q,a}^{\lceil\alpha\rceil-\alpha} I_{q,a}^{\alpha} f\big)(x) = \big(D_q^{\lceil\alpha\rceil} I_{q,a}^{\lceil\alpha\rceil-\alpha+\alpha} f\big)(x) \\
&= \big(D_q^{\lceil\alpha\rceil} I_{q,a}^{\lceil\alpha\rceil} f\big)(x) = f(x)\,. \qquad \square
\end{aligned}$$

Lemma 10. *Let $\alpha \in (0,1)$. Then*

$$\big(I_{q,a}^{\alpha} D_{q,a}^{\alpha} f\big)(x) = f(x) + K(a)(x-a)^{(\alpha-1)} \qquad (0 < a < x < b)\,,$$

where $K(a)$ does not depend on x.

Proof. Let

$$A(x) = \big(I_{q,a}^{\alpha} D_{q,a}^{\alpha} f\big)(x) - f(x).$$

Applying $D_{q,a}^{\alpha}$ to the both sides of the above expression, and using Lemma 9, we get

$$\begin{aligned}
(D_{q,a}^{\alpha} A)(x) &= \big(D_{q,a}^{\alpha} I_{q,a}^{\alpha} D_{q,a}^{\alpha} f\big)(x) - D_{q,a}^{\alpha} f(x) \\
&= \big((D_{q,a}^{\alpha} I_{q,a}^{\alpha}) D_{q,a}^{\alpha} f\big)(x) - D_{q,a}^{\alpha} f(x) = 0\,.
\end{aligned}$$

On the other hand, according to Lemma 6, we obtain

$$D_{q,a}^{\alpha}\big((x-a)^{(\alpha-1)}\big) = D_q I_{q,a}^{1-\alpha}\big((x-a)^{(\alpha-1)}\big) = (D_q \mathbf{1})(x) = 0\ .$$

Hence, we conclude that $A(x)$ is a function of the form

$$A(x) = K(a)(x-a)^{(\alpha-1)}\,. \qquad \square$$

Lemma 11. *Let* $0 < a \leq c < x < b$ *and* $\alpha \in (0,1)$. *Then the following is valid*:

$$(I_{q,c}^{\alpha+k} D_{q,a}^{\alpha+k} f)(x) = \frac{(x-c)^{(\alpha+k)}}{\Gamma_q(\alpha+k+1)} (D_{q,a}^{\alpha+k} f)(c) + (I_{q,c}^{\alpha+k+1} D_{q,a}^{\alpha+k+1} f)(x), \quad (k \in \mathbb{N}_0).$$

Proof. According to Lemma 3 and Lemma 4, we have

$$\begin{aligned}(I_{q,c}^{\alpha+k} D_{q,a}^{\alpha+k} f)(x) &= (I_{q,c}^{\alpha+k+1} D_q D_{q,a}^{\alpha+k} f)(x) + \frac{(D_{q,a}^{\alpha+k} f)(c)}{\Gamma_q(\alpha+k+1)} (x-c)^{(\alpha+k)} \\ &= \frac{(D_{q,a}^{\alpha+k} f)(c)}{\Gamma_q(\alpha+k+1)} (x-c)^{(\alpha+k)} + (I_{q,c}^{\alpha+k+1} D_{q,a}^{\alpha+k+1} f)(x)\,. \qquad \square\end{aligned}$$

Now, we are ready to prove a TAYLOR type formula with fractional q-derivatives, which is the main result of this section.

Theorem 12. *Let* $f(x)$ *be defined on* $(0,b)$ *and* $\alpha \in (0,1)$. *For* $0 < a < c < x < b$, *the following is true*:

$$f(x) = \sum_{k=0}^{n-1} \frac{(D_{q,a}^{\alpha+k} f)(c)}{\Gamma_q(\alpha+k+1)} (x-c)^{(\alpha+k)} + R_n(f), \tag{20}$$

with $R_n(f) = R_0(f) - K(a)(x-a)^{(\alpha-1)} + E_n(f)$, *where*

$$R_0(f) = \frac{1}{\Gamma_q(\alpha)} \int_a^c (x-qt)^{(\alpha-1)} (D_{q,a}^{\alpha} f)(t)\, \mathrm{d}_q t\ ,$$

and $E_n(f)$ *can be represented in either of the following forms*:

$$E_n(f) = (I_{q,c}^{\alpha+n} D_{q,a}^{\alpha+n} f)(x), \tag{21}$$

$$E_n(f) = \frac{(D_{q,a}^{\alpha+n} f)(\xi)}{\Gamma_q(\alpha+n+1)} (x-c)^{(\alpha+n)} \quad (c < \xi < x). \tag{22}$$

Proof. We will deduce the proof of (21) by mathematical induction. Since

$$(I_{q,a}^{\alpha} D_{q,a}^{\alpha} f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_a^c (x-qt)^{(\alpha-1)} (D_{q,a}^{\alpha} f)(t)\, \mathrm{d}_q t + (I_{q,c}^{\alpha} D_{q,a}^{\alpha} f)(x),$$

using Lemma 10, we obtain

$$f(x) = (I_{q,c}^{\alpha} D_{q,a}^{\alpha} f)(x) + R_0(f) - K(a)(x-a)^{(\alpha-1)}.$$

According to Lemma 11, for $k=0$, we have

$$\begin{aligned}(I_{q,c}^{\alpha}D_{q,a}^{\alpha}f)(x) &= \frac{(D_{q,a}^{\alpha}f)(c)}{\Gamma_q(\alpha+1)}(x-c)^{(\alpha)} + (I_{q,c}^{\alpha+1}D_{q,a}^{\alpha+1}f)(x)\\ &= \frac{(D_{q,a}^{\alpha}f)(c)}{\Gamma_q(\alpha+1)}(x-c)^{(\alpha)} + E_1(f),\end{aligned}$$

which completes the expression for $R_1(f)$ and proves (21) for $n=1$.
Assume that (21) is valid for any $n\in\mathbb{N}$. Then, again from Lemma 11, the following holds:

$$\begin{aligned}E_n(f) = (I_{q,c}^{\alpha+n}D_{q,a}^{\alpha+n}f)(x) &= \frac{(D_{q,a}^{\alpha+n}f)(c)}{\Gamma_q(\alpha+n+1)}(x-c)^{(\alpha+n)} + (I_{q,c}^{\alpha+n+1}D_{q,a}^{\alpha+n+1}f)(x)\\ &= \frac{(D_{q,a}^{\alpha+n}f)(c)}{\Gamma_q(\alpha+n+1)}(x-c)^{(\alpha+n)} + E_{n+1}(f).\end{aligned}$$

Hence the formula (21) is valid for $n+1$. So, it is valid for each $n\in\mathbb{N}$.

The second form of remainder, (22), can be obtained by using a mean–value theorem for q–integrals [**9**]. Indeed, there exists $\xi\in(c,x)$, such that

$$\begin{aligned}E_n(f) = (I_{q,c}^{\alpha+n}D_{q,a}^{\alpha+n}f)(x) &= \frac{1}{\Gamma_q(\alpha+n)}\int_c^x (x-qt)^{(\alpha+n-1)}\big(D_{q,a}^{\alpha+n}f\big)(t)\,\mathrm{d}_qt\\ &= \frac{(D_{q,a}^{\alpha+n}f)(\xi)}{\Gamma_q(\alpha+n)}\int_c^x (x-qt)^{(\alpha+n-1)}\,\mathrm{d}_qt = \frac{(D_{q,a}^{\alpha+n}f)(\xi)}{\Gamma_q(\alpha+n)}(I_{q,c}^{\alpha+n}\mathbf{1})(x)\\ &= \frac{\big(D_{q,a}^{\alpha+n}f\big)(\xi)}{\Gamma_q(\alpha+n+1)}(x-c)^{(\alpha+n)}.\end{aligned}$$

□

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