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On q-orthogonal polynomials over a collection of complex origin intervals related to little q-Jacobi polynomials

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Abstract We construct the sequence of orthogonal polynomials with respect to an inner product which is defined by q-integrals over a collection of intervals in the complex plane. We prove that they are connected with little q-Jacobi polynomials. For such polynomials we discuss a few representations, a recurrence relation, a difference equation, a Rodrigues-type formula and a generating function.

Keywords Basic hypergeometric functions · Orthogonal polynomials

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1 Introduction

We will start with well-known facts from q-calculus (see, for example, [1, 3, 5]). For a real number $q \in (0, 1)$, the basic number $[a]_q$ is given by

$$[a]_q = \frac{1 - q^a}{1 - q} \quad (a \in \mathbb{R}),$$

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the q-derivative of a function is

$$\mathcal{D}_q F(x) = \frac{F(x) - F(qx)}{x - qx} \quad (x \neq 0)$$

and q-integral over a complex finite interval is defined by

$$\int_0^c F(x)\, d_q x := c(1-q) \sum_{k=0}^{\infty} F(cq^k) q^k \quad (c \in \mathbb{C}).$$

The generalization of the gamma function $\Gamma(x)$ is given by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} \quad (1-q)^{1-x},$$

where $(a;q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j)$. The basic hypergeometric function is defined by

$${}_r\Phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} \middle| q; z\right) = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_k}{(b_1, b_2, \ldots, b_s; q)_k} (-1)^{(1+s-r)k} q^{(1+s-r)\binom{k}{2}} \frac{z^k}{(q;q)_k},$$

where $(a;q)_\lambda = (a;q)_\infty/(aq^\lambda;q)_\infty$ and $(a_1, a_2, \ldots, a_n; q)_\lambda = \prod_{j=1}^{n} (a_j;q)_\lambda$.

The *little q-Jacobi polynomials* are part of the Askey-scheme of hypergeometric orthogonal polynomials [5] and are defined by

$$p_n(x;\, a,\, b|\, q) = {}_2\Phi_1\left(\begin{matrix} q^{-n},\, abq^{n+1} \\ aq \end{matrix} \middle| q;\, qx\right).$$

Their orthogonality is given by the next relation

$$\sum_{k=0}^{\infty} \frac{(bq;\, q)_k}{(q;\, q)_k} (aq)^k p_m(q^k; a,\, b|\, q)\, p_n(q^k;\, a,\, b|\, q)$$
$$= \frac{(abq^2;\, q)_\infty}{(aq;\, q)_\infty} \frac{(1-abq)\,(aq)^n}{(1-abq^{2n+1})} \frac{(q,\, bq;\, q)_n}{(aq,\, abq;\, q)_n} \delta_{mn}$$

for $0 < a < q^{-1}$, $b < q^{-1}$. If $a = q^\alpha$ and $b = q^\beta$, the inner product on the left-hand side can be expressed by q-integral

$$(p_m,\, p_n) = \frac{(q^{\beta+1};\, q)_\infty}{(1-q)\,(q;\, q)_\infty} \int_0^1 p_m(x;\, q^\alpha,\, q^\beta|\, q)\, p_n\,(x;\, q^\alpha,\, q^\beta|\, q)\, w(x)\, d_q x,$$

where

$$w(x) \;=\; x^\alpha \frac{(qx;\, q)_\infty}{(q^{\beta+1}x;\, q)_\infty} = x^\alpha (qx;\, q)\beta.$$

In the next sections, we will define a new inner product and establish the connections of the corresponding orthogonal polynomials with little q-Jacobi polynomials.

Similar problems were discussed by some other authors. Likewise, in the paper [2], J.S. Geronimo and W.V. Assche have discussed the orthogonality of a new polynomial sequence which is obtained by some polynomial transformations of a measure and its support. J.A. Charris, M.E.H. Ismail and S. Monsalve [4] examined the orthogonality of the polynomial sequence defined by blocks of recurrence relations and established the connection with polynomial mappings. Also, the orthogonality on radial rays in the complex plane was discussed in the papers of G.V. Milovanović [6] and G.V. Milovanović, P.M. Rajković and Z.M. Marjanović [7].

2 About q-orthogonality over the collection of intervals

Let us assume that N is a positive integer, q is a real number $(0 < q < 1)$ and $Q = q^{1/N}$. We start with the sequence of the little q-Jacobi polynomials $p_n(x; q^{1/N-1}, 1|q)$ which is orthogonal with respect to the inner product

$$(f, g)_0 = \int_0^1 f(x)g(x)x^{1/N-1}d_qx.$$

Also, let us denote

$$\varphi_N(j) = \exp\left(i\frac{2\pi j}{N}\right), \quad j = 0, 1, \ldots, N-1, \quad (i^2 = -1).$$

Then, it is valid

Lemma 2.1. *The function $j \mapsto \varphi_N(j)$ has following properties:*

(1) $\varphi_N(N) = 1, \quad \overline{\varphi_N(j)} = \varphi_N(-j)$;
(2) $\varphi_N(j+l) = \varphi_N(j) \cdot \varphi_N(l), \quad \varphi_N^M(j) = \varphi_N(jM)$;
(3) $\sum_{j=0}^{N-1} \varphi_N((Nn+\nu)j) = \sum_{j=0}^{N-1} \varphi_N(\nu j) = \begin{cases} N, & \nu = 0 \\ 0, & 1 \le \nu \le N-1. \end{cases}$

We consider the polynomial $T(x) = x^N$ over the interval $E_0 = (0, 1]$. Its inverse branches are

$$T_j^{-1}(x) = \varphi_N(j)x^{1/N}, \quad x \in (0, 1] \quad (0 \le j \le N-1)$$

and the sets

$$E_j = T_j^{-1}(E_0) = (0, \varphi_N(j)] \quad (0 \le j \le N-1)$$

are the intervals which connect the origin and N-th roots of unity. We will define the inner product

$$\langle F, G\rangle = \left[\frac{1}{N}\right]_q \sum_{j=0}^{N-1} \int_{E_0} F\big(T_j^{-1}(x)\big)\,\overline{G\big(T_j^{-1}(x)\big)}\,x^{1/N-1}d_qx. \tag{2.1}$$

Lemma 2.2. *For the q-integral it is valid*

$$\int_0^c F(x) d_q x = c \int_0^1 F(cx) d_q x \quad (c \in \mathbb{C}),$$

$$\int_0^1 F(x^N) d_Q x = \left[\frac{1}{N}\right]_q \int_0^1 F(x) \quad x^{1/N-1} d_q x.$$

Using these properties of the q-integral, the inner product (2.1) can be also written in the two following forms

$$\langle F,\ G\rangle = \sum_{j=0}^{N-1} \varphi_N(-j) \int_{E_j} F(x)\, \overline{G(x)}\, d_Q x \tag{2.2}$$

$$\langle F,\ G\rangle = \sum_{j=0}^{N-1} \int_{E_0} F\left(\varphi_N(j)x\right)\, \overline{G\left(\varphi_N(j)x\right)}\, d_Q x. \tag{2.3}$$

It is easy to see that it is valid $\langle z^N F,\ G\rangle = \langle F,\ z^N G\rangle$, where $F(z)$ and $G(z)$ are an arbitrary pair of functions.

The inner product (2.3) is positive-definite because of $\|F\|^2 = \langle F, F\rangle > 0$, except for $F(x) \equiv 0$. It implies the existence of the sequence of the monic orthogonal polynomials $\{P_m(z)\}$ which satisfies

$$\langle P_m,\ P_n\rangle = \delta_{mn} \|P_m\|^2 \quad (m, n \in \mathbb{N}_0).$$

We can construct this sequence by Gram–Schmidt orthogonalization.

If we evaluate the moments

$$\mu_{i,k} = \langle z^i, z^k\rangle = \begin{cases} \dfrac{N}{[i+k+1]_Q}, & i \equiv k (\mathrm{mod}\ N) \\ 0, & \text{others} \end{cases}$$

and denote the moment-determinants by

$$\Delta_0 = 1, \quad \Delta_m = \det\, [\mu_{i,k}]_{i,k=0}^{m-1}, \quad m \geq 1,$$

then these polynomials can be expressed in the form

$$P_0(z) = 1, \quad P_m(z) = \frac{1}{\Delta_m} \begin{vmatrix} \mu_{00} & \mu_{10} & \cdots & \mu_{m-1,0} & 1 \\ \mu_{01} & \mu_{11} & & \mu_{m-1,1} & z \\ \vdots & & & & \\ \mu_{0,m} & \mu_{1,m} & & \mu_{m-1,m} & z^m \end{vmatrix}, \quad m \geq 1.$$

This sequence is unique and the norms are $\|P_m\|^2 = \Delta_{m+1}/\Delta_m$. Precisely,

$$\|P_{Nn+\nu}\|^2 = \begin{cases} \dfrac{N}{|2\nu+1|_Q}, & n=0 \\[2ex] \dfrac{NQ^{n(N(n-1)+2\nu+1)}}{[2Nn+2\nu+1]_Q} \dfrac{\prod\limits_{i=1}^{n}[Ni]_Q^2}{\prod\limits_{i=0}^{n-1}[Ni+2\nu+1]_Q^2}, & n \geq 1. \end{cases}$$

By the definition of the q-analogous of the Γ-function, we have

$$\|P_{Nn+\nu}\|^2 = \begin{cases} \dfrac{N[1/N]_q}{[(2\nu+1)/N]_q}, & n=0 \\ \dfrac{N[1/N]_q\, q^{n((n-1)+(2\nu+1)/N)}}{[2n+(2\nu+1)/N]_q} \left(\dfrac{[n]_q!\Gamma_q((2\nu+1)/N)}{\Gamma_q(n+(2\nu+1)/N)}\right)^2, & n \geq 1. \end{cases}$$

Lemma 2.3. *The polynomials* $\{P_m(z)\}$ *satisfy*

$$P_m(\varphi_N(j)z) = \varphi_N(mj)P_m(z), \quad j = 0, \dots, N-1.$$

Lemma 2.4. *The first N members of the orthogonal polynomial sequence are*

$$P_m(z) = z^m, \quad m = 0, \dots, N-1.$$

Theorem 2.5. *The monic polynomials* $\{P_m(z)\}_{m=0}^{+\infty}$ *satisfy the recurrence relation*

$$\begin{aligned} P_{m+N}(z) &= (z^N - \alpha_m)\, P_m(z) - \beta_m P_{m-N}(z), \quad m \geq 0, \\ P_m(z) &= z^m, \quad m = 0, \dots, N-1, \end{aligned} \tag{2.4}$$

where

$$\alpha_m = \frac{\langle z^N P_m, P_m\rangle}{\langle P_m, P_m\rangle}, \quad m \geq 0, \quad \beta_m = \begin{cases} \dfrac{\langle P_m, P_m\rangle}{\langle P_{m-N}, P_{m-N}\rangle}, & m \geq N \\ 0, & m \leq N-1. \end{cases}$$

The explicit form of coefficients will be derived in the next section. But it should be emphasized that they are real because of $(z^N P_m, P_m) = (P_m, z^N P_m)$.

3 Some representations of the polynomials $\{P_m(z)\}$

According to the last section, we easily get to the next conclusion.

Lemma 3.1. *Every polynomial $P_{Nn+\nu}(z)$ can be expressed in the form*

$$P_{Nn+\nu}(z) = z^\nu S_n^{(\nu)}(z^N; q), \qquad (n \in \mathbb{N}_0,\ 0 \le \nu \le N-1)$$

where $S_n^{(\nu)}(t;q)$ are the monic polynomials of degree n.

Theorem 3.2. *The polynomial $P_{Nn+\nu}(z)$ can be represented by*

$$P_{Nn+\nu}(z) = K_{n,\nu}(q)\, z^\nu\, p_n(z^N; q^{(2\nu+1)/N-1}, 1|q), \tag{3.1}$$

where

$$K_{n,\nu}(q) = (-1)^n q^{\binom{n}{2}} \frac{\left(\Gamma_q(n+(2\nu+1)/N)\right)^2}{\Gamma_q\left((2\nu+1)/N\right)\Gamma_q\left(2n+(2\nu+1)/N\right)} \tag{3.2}$$

and $p_n(x; q^{(2\nu+1)/N-1}, 1|\ q)$ is the member of the sequence of the little q-Jacobi polynomials.

Proof: For any $n, l \in \mathbb{N}_0$ and $\nu \in \{0, 1, \ldots, N-1\}$ we have

$$\langle P_{Nn+\nu}, P_{Nl+\nu}\rangle = (1-Q)\sum_{j=0}^{N-1}\sum_{k=0}^{\infty} P_{Nn+\nu}(\varphi_N(j)Q^k)\,\overline{P_{Nl+\nu}(\varphi_N(j)Q^k)}Q^k.$$

According to Lemma 2.1 and Lemma 3.1, we yield

$$\begin{aligned}
&\langle P_{Nn+\nu}, P_{Nl+\nu}\rangle \\
&\quad = (1-Q)\sum_{j=0}^{m-1}\sum_{k=0}^{\infty}\varphi_N((Nn+\nu)j)P_{Nn+\nu}(Q^k)\overline{\varphi_N((Nl+\nu)j)P_{Nl+\nu}(Q^k)}Q^k \\
&\quad = \quad N(1-Q)\sum_{k=0}^{\infty} P_{Nn+\nu}(Q^k)\overline{P_{Nl+\nu}(Q^k)}Q^k.
\end{aligned}$$

Since the polynomials $P_m(z)$ have all real coefficients, taking $Q = q^{1/N}$, the previous relation can be written as

$$\begin{aligned}
\langle P_{Nn+\nu}, P_{Nl+\nu}\rangle &= N(1-q^{1/N})\sum_{k=0}^{\infty}(q^{k/N})^\nu S_n^{(\nu)}(q^k;q)\,(q^{k/N})^\nu S_l^{(\nu)}(q^k;\ q)\,q^{k/N} \\
&= N(1-q^{1/N})\sum_{k=0}^{\infty} S_n^{(\nu)}(q^k;\ q)\,S_l^{(\nu)}(q^k;\ q)\,q^{k((2\nu+1)/N-1)}\,q^k \\
&= N\left[\frac{1}{N}\right]_q \int_0^1 S_n^{(\nu)}(x;\ q)\,S_l^{(\nu)}(x;\ q)\,x^{(2\nu+1)/N-1} d_q t.
\end{aligned}$$

Because of the orthogonality of the polynomials $P_m(z)$, we have

$$\int_0^1 S_n^{(\nu)}(x;\, q)\, S_l^{(\nu)}(x;\, q\,)\, x^{(2\nu+1)/N-1} d_q x \;=\; \frac{[N]_Q}{N}\, \| P_{Nn+\nu} \|^2\, \delta_{nl}.$$

It means that the sequence of the polynomials $S_n^{(\nu)}(x;q)$ is orthogonal with respect to the same inner product as little q-Jacobi polynomials $p_n(x;q^{(2\nu+1)/N-1)},1|q)$. So, because of the uniqueness of the sequence of the orthogonal polynomials and the fact that $S_n^{(\nu)}(x;q)$ are monic, we get the coefficient $K_{n,\nu}(q)$ and, finally, the representation (3.1–2). □

Corollary 3.3. *The coefficients in recurrence relation (2.3) for* $m = Nn + \nu, n \in \mathbb{N}_0, 0 \le \nu \le N-1$, *can be represented by*

$$\alpha_{Nn+\nu} = Q^{Nn} \times \frac{[Nn+2\nu+1]_Q^2[N(2n-1)+2\nu+1]_Q + Q^{2\nu+1-N}[Nn]_Q^2[N(2n+1)+2\nu+1]_Q}{[N(2n-1)+2\nu+1]_Q\,[2Nn+2\nu+1]_Q[N(2n+1)+2\nu+1]_Q}$$

$$\beta_{Nn+\nu} = Q^{2N(n-1)+2\nu+1} \times \frac{[N(n-1)+2\nu+1]_Q^2[Nn]_Q^2}{[N(2n-2)+2\nu+1]_Q\,[N(2n-1)+2\nu+1]_Q^2\,[2Nn+2\nu+1]_Q}.$$

Proof: The statement follows from Theorem 3.2 and the recurrence relation for the little q-Jacobi polynomials $y_n(x) = p_n(x;a,\ b \mid q)$

$$-xy_n(x) = A_n y_{n+1}(x) - (A_n + C_n)y_n(x) + C_n y_{n-1}(x),$$

where

$$A_n = q^n \frac{(1-aq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-\ abq^{2n+2})}, \quad C_n = aq^n \frac{(1-q^n)(1-bq^n)}{(1-abq^{2n})(1-\ abq^{2n+1})}.$$

□

The proof of the next statement follows from Theorem 3.2 and the definition of the little q-Jacobi polynomials by the basic hypergeometric function.

Theorem 3.4. *The polynomial* $P_m(z)$, *with the index* $m = Nn + \nu$, *where* $n \in \mathbb{N}_0$ *and* $0 \le \nu \le N-1$, *can be represented by*

$$P_{Nn+\nu}(z) = K_{n,\nu}(q)\, z^\nu\, {}_2\Phi_1\left(\begin{matrix} q^n,\ q^{(2\nu+1)/N+n} \\ q^{(2\nu+1)/N} \end{matrix} \middle| \, q;\, qz^N \right),$$

where $K_{n,\nu}(q)$ is defined by (3.2).

Theorem 3.5. *Every polynomial $P_{Nn+\nu}(z)$ can be expressed in the explicit form*

$$P_{Nn+\nu}(z) = (-1)^n q^{\binom{n}{2}} \frac{\left(\Gamma_q(n+(2\nu+1)/N)\right)}{\Gamma_q\,(2\nu+(2\nu+1)/N)} \times \sum_{k=0}^{n} (-1)^k q^{\binom{k+1}{2}-kn} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\Gamma_q\,(n+k+(2\nu+1)/N)}{\Gamma_q\,(k+(2\nu+1)/N)} . z^{Nk+\nu}$$

Proof: For the little q-Jacobi polynomial $p_n(x; q^{(2\nu+1)/N-1},\ 1|\, q)$, it is valid (see [5])

$$p_n(x; q^{(2\nu+1)/N-1},\ 1|\, q) = {}_2\Phi_1\left(\begin{matrix} q^{-n}, q^{(2\nu+1)/N+n} \\ q^{(2\nu+1)/N} \end{matrix} \middle| q; qx \right) = \sum_{k=0}^{\infty} \frac{(q^{-n}, q^{(2\nu+1)/N+n}; q)_k}{(q^{(2\nu+1)/N}; q)_k} \frac{q^k x^k}{(q;q)_k}$$

$$= 1 + \sum_{k=1}^{\infty} \left(\prod_{i=0}^{k-1} \frac{(1-q^{-n+i})(1-q^{(2\nu+1)/N+n+i})}{(1-q^{1+i})(1-q^{(2\nu+1)/N+i})} \right) q^k x^k$$

$$= 1 + \sum_{k=1}^{n} \frac{(-1)^k q^{-kn+\binom{k}{2}} \prod_{i=0}^{k-1} [n-i]_q}{\prod_{i=0}^{k-1} [i+1]_q} \frac{\prod_{i=0}^{k-1} [(2\nu+1)/N+n+i]_q}{\prod_{i=0}^{k-1} [(2\nu+1)/N+i]_q} q^k x^k$$

$$= 1 + \sum_{k=1}^{n} (-1)^k q^{-kn+\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\Gamma_q((2\nu+1)/N)\Gamma_q((2\nu+1)/N+n+k)}{\Gamma_q((2\nu+1)/N+n)\Gamma_q((2\nu+1)/N+k)} q^k x^k$$

$$= \frac{\Gamma_q((2\nu+1)/N)}{\Gamma_q((2\nu+1)/N+n)} \sum_{k=0}^{n} (-1)^k q^{-kn+\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\Gamma_q((2\nu+1)/N+n+k)\Gamma_q}{((2\nu+1)/N+k)} q^k x^k.$$

Putting $x = z^N$ and using Theorem 3.2, we get the required formula. □

4 Some properties of the polynomials $\{P_m(z)\}$

From the theory of the orthogonal polynomials it is well-known that all zeros of the little q-Jacobi polynomials $p_n(x;\ a,\ b|\, q)$ are simple and lie in the interval (0, 1) under the conditions $0 < q < 1, 0 < a < q^{-1}, b < q^{-1}$.

By the representations from the third section, we easily conclude that all zeros of the polynomial $P_m(z)$, orthogonal with respect to (2.1), are located on the support-intervals.

Moreover, they are simple with a possible exception of a multiple zero in the origin $z = 0$ of the order ν if $m \equiv \nu \pmod{N}$.

Theorem 4.1. *Every polynomial $P_m(z)(m = Nn + \nu, n \in \mathbb{N}_0, 0 \le \nu \le N-1)$ satisfies the following Q-difference equation*

$$A_{n,\nu}(z;\,Q)z^2D_Q^2P_{Nn+\nu}(z) + B_{n,\nu}(z;Q)zD_QP_{Nn+\nu}(z) + C_{n,\nu}(z;Q)P_{Nn+\nu}(z) \\ = (1-Q)T_{n,\nu}(z;Q)z\,D_QP_{Nn+\nu}(z),$$

where

$$A_{n,\nu}(z;\,Q) = Q^{2-N}(Q^{2N}z^N - 1),$$
$$B_{n,\nu}(z;\,Q) = [2]_QQ^Nz^N - [2-N]_Q,$$
$$C_{n,\nu}(z;\,Q) = Q^{-Nn-\nu}(Q^{Nn}[\nu]_Q[1-N+\nu]_Q - [Nn+\nu]_Q[Nn+\nu+1]_QQ^Nz^N),$$
$$T_{n,\nu}(z;\,Q) = Q^{-Nn-\nu}(Q^{Nn}[\nu]_Q[1-N+\nu]_Q - [Nn+\nu]_Q[Nn+\nu+1]_Q).$$

Proof: One solution of the q-difference equation (see [5])

$$q^{(1-n)}(1-q^n)(1-q^{n+(2\nu+1)/N})ty(qt) \\ = q^{(2\nu+1)/N-1}(q^2t-1)(y(q^2t)-y(qt)) - (qt-1)(y(qt)-y(t))$$

is the little q-Jacobi polynomial $y(t) = p_n(t;\,q^{(2\nu+1)/N-1},\ 1|\,q)$. Applying the relation (3.1), we have

$$y(t) = (K_{n,\nu}(q))^{-1}t^{-\nu/N}P_{Nn+\nu}(t^{1/N}),$$
$$y(qt) = (K_{n,\nu}(q))^{-1}q^{-\nu}t^{-\nu/N}P_{Nn+\nu}(q^{1/N}t^{1/N}),$$
$$y(q^2t) = (K_{n,\nu}(q))^{-1}q^{-2\nu}t^{-\nu/N}P_{Nn+\nu}(q^{2/N}t^{1/N}).$$

Having in mind that $Q = q^{1/N}$, substituting $t = z^N$ in q-difference equation, it becomes

$$Q^{N(1-n)-\nu}(1-Q^{Nn})(1-Q^{Nn+2\nu+1})z^NP_{Nn+\nu}(Qz) \\ = Q^{\nu+1-N}(Q^{2N}z^N-1)(Q^{-\nu}P_{Nn+\nu}(Q^2z) - P_{Nn+\nu}(Qz)) \\ -(Q^Nz^N-1)(Q^{-\nu}P_{Nn+\nu}(Qz) - P_{Nn+\nu}(z)).$$

According to [3], we have

$$P_{Nn+\nu}(Q^kz) = \sum_{j=0}^{k}(-1)^j(1-Q)^j\begin{bmatrix}k\\j\end{bmatrix}_QQ^{\binom{j}{2}}z^j\mathcal{D}_Q^jP_{Nn+\nu}(z),$$

where from the last identity takes the required form. □

Theorem 4.2. *For the polynomial $P_m(z)$ satisfies the Rodrigues-type formula*

$$P_{Nn+\nu}(z^{1/N}) = (-1)^n q^{n(n-2+(2\nu+1)/N)} \frac{\Gamma_q(n+(2\nu+1)/N)}{\Gamma_q(2n+(2\nu+1)/N)} \times z^{1-(\nu+1)/N}\, \mathcal{D}^n_{q^{-1}}\left(z^{n-1+(2\nu+1)/N}\,(qz;q)_n\right)$$

is valid.

Proof: The Rodrigues-type formula for the little q-Jacobi polynomials $p_n(x;\, q^{(2\nu+1)/N-1},\, 1|\, q)$ is

$$\begin{aligned} & x^{(2\nu+1)/N-1} p_n(x; q^{(2\nu+1)/N-1}, 1|\, q) \\ &= \frac{q^{n((2\nu+1)/N-1)+\binom{n}{2}}(1-q)^n}{(q^{(2\nu+1)/N}; q)_n} \mathcal{D}^n_{q^{-1}}\left((qx;q)_n\, x^{n-1+(2\nu+1)/N}\right) \\ &= q^{n((2\nu+1)/N-1)+\binom{n}{2}} \frac{\Gamma_q((2\nu+1)/N)}{\Gamma_q(n+(2\nu+1)/N)}\, \mathcal{D}^n_{q^{-1}}\left((qx;q)_n\, x^{n-1+(2\nu+1)/N}\right). \end{aligned}$$

According to Theorem 3.2, we can write

$$p_n(x;\, q^{(2\nu+1)/N-1},\ 1|\, q) = (K_{n,\nu}(q))^{-1} x^{-\nu/N} P_{Nn+\nu}(x^{1/N}),$$

and by simplifying the previous identity, we get the formula. □

Theorem 4.3. *The generating function for the polynomials $\{P_m(z)\}$ is given by*

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{\nu=0}^{N-1} \frac{(1-q)^{2n}}{([n]_q!)^2} \frac{\Gamma_q\left((2\nu+1)/N\right)\Gamma_q\left(2n+(2\nu+1)/N\right)}{\left(\Gamma_q\left(n+(2\nu+1)/N\right)\right)^2} P_{Nn+\nu}(z) t^{Nn+\nu} \\ &= \sum_{\nu=0}^{N-1} (zt)^\nu\ {}_0\Phi_1\left(\begin{matrix} - \\ q^{(2\nu+1)/N} \end{matrix}\middle| q;\, q^{(2\nu+1)/N} z^N t^N\right)\ {}_2\Phi_1\left(\begin{matrix} z^{-N}, 0 \\ q \end{matrix}\middle| q; z^N t^N\right). \end{aligned}$$

Proof: Starting from the generating function for the little q-Jacobi polynomials $p_n(x; q^{(2\nu+1)/N-1}, 1 \mid q)$, which is given by

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}}(1-q)^{2n}}{([n]_q!)^2}\ p_n(x; q^{(2\nu+1)/N-1}, 1|\, q)\, u^n \\ &= {}_0\Phi_1\left(\begin{matrix} - \\ q^{(2\nu+1)/N} \end{matrix} \mid q;\, q^{(2\nu+1)/N} xu\right) {}_2\Phi_1\left(\begin{matrix} x^{-1}, 0 \\ q \end{matrix} \mid q;\, xu\right) \end{aligned}$$

and using the relations (3.1–2), we have

$$\sum_{n=0}^{\infty} \frac{(1-q)^{2n}}{([n]_q!)^2} \frac{\Gamma_q((2\nu+1)/N)\Gamma_q(2n+(2\nu+1)/N)}{(\Gamma_q(n+(2\nu+1)/N))^2} x^{-\nu/N} P_{Nn+\nu}(x^{1/N})\, u^n$$
$$= {}_0\Phi_1\left(\begin{matrix} - \\ q^{(2\nu+1)/N} \end{matrix} \;\middle|\; q;\, q^{(2\nu+1)/N} xu \right) {}_2\Phi_1\left(\begin{matrix} x^{-1}, 0 \\ q \end{matrix} \;\middle|\; q;\, xu \right)$$

for any $0 \le \nu \le N-1$. Taking $z = x^N$ and $u = t^N$ and summing by ν from 0 to $N-1$, we get the desirable expansion. □

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