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Influence of randomly varying damping coefficient on the dynamic stability of continuous systems

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Abstract

A method for the determination of sufficient conditions for the almost sure asymptotic stability of some continuous systems, when the damping coefficient is random time-dependent function is studied. In this case, the probabilistic property of the derivative process of the damping coefficient is taken into account. The problem is solved by means Liapunov direct method, and tested on a simply supported elastic beam, compressed by time-dependent stochastic axial force. Regions of almost sure asymptotic stability as functions of the constant part of viscous damping coefficient and magnitude of an axial force are given.

Keywords: Almost sure stability; Liapunov direct method; Gaussian process

1. Introduction

In recent studies of dynamic stability of continuous systems, the constant viscous damping coefficient is presumed. Actually, it is time-dependent, or precisely, it is a random function of time.

Taking this fact as a starting point of the stability of discrete systems, the bounds of the regions of asymptotic stability, mean-square stability and almost sure stability are determined. In the case when the damping coefficient is a random time-dependent function, it is more interesting to find a condition of the almost sure stability.

One of the first papers dedicated to this problem is given by Infante (1968), where the stability theorem of linear discrete systems is defined, based on eigenvalue properties of the quadratic forms. Infante's results are extended by Kozin and Wu (1973), where probability density of excitation processes system parameters are known.

Using "best" quadratic Liapunov function from the previous paper, Kozin (1972) investigated almost sure stability of continuous systems with a constant damping coefficient. In the case when the damping coefficient is time-dependent stochastic function Kozin's "best" functional based on the "best" quadratic function does not lead us to the problem solution. One of the ways to overcome that problem is proposed in this paper.

Ariaratnam and Ly (1989) and Ariaratnam and Xie (1989), generalized earlier Infante (1968), Kozin and Wu's (1973) results, considering a linear system where system parameters are ergodic processes.

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The continuous systems considered in this paper are governed by a linear partial differential equation with random coefficients. These coefficients are assumed to be stationary and ergodic in the stochastic case. It is desired to obtain sufficient conditions for almost sure asymptotic stability of the equilibrium state of the system.

As an example, an isotropic, elastic beam with an ergodic, random damping coefficient is considered. The condition of the almost sure asymptotic stability was determined by Liapunov direct method. Liapunov functional was constructed as a sum of modified kinetic energy and elastic energy of the system. Regions of the almost sure asymptotic stability as functions of constant deterministic forces and one part of the damping coefficient are given.

2. Problem formulation

Let us consider a continuous dynamic system which occupies a bounded domain $\boldsymbol{\Omega}$ with the boundary $\boldsymbol{C}$ in one, two or three-dimensional space $\{\boldsymbol{x}\}$. Let us designate by $w(\boldsymbol{x},t)$ the displacement of the system from an equilibrium state which, for simplicity, is taken as $w(\boldsymbol{x},t)\equiv 0$, t is time ($t\geqslant 0$), and it is assumed that this displacement is governed by a linear partial differential equation whose form is:

$$\frac{\partial^2 w}{\partial t^2}+2\big(\varsigma+g(t)\big)\frac{\partial w}{\partial t}+\Im w+\sum_{i=1}^{p}\big[f_{i0}+f_i(t)\big]\Im_i w=0,\quad \boldsymbol{x}\in\boldsymbol{\Omega},\ t\geqslant 0 \tag{1}$$

with homogeneous time-independent boundary conditions:

$$\aleph w=0,\quad \boldsymbol{x}\in\boldsymbol{C}. \tag{2}$$

In this formulation $\Im$, $\Im_i$ and $\aleph$ are linear spatial differential operators, ζ is a positive constant, functions $g(t)$, $f_1(t),\ldots,f_p(t)$ are measurable, strictly stationary functions which satisfy an ergodic property insuring the equality of time and ensemble averages, and $f_{10},\ldots,f_{p0}$ are constants.

For zero initial conditions, differential equation (1) possesses trivial solution $w(\boldsymbol{x},t)\equiv 0$, which represents the equilibrium state of the system. For non-zero initial conditions in the form:

$$w(\boldsymbol{x},0)=w_0(\boldsymbol{x}),\quad \frac{\partial w(\boldsymbol{x},0)}{\partial t}=v_0(\boldsymbol{x}),\quad \boldsymbol{x}\in\boldsymbol{\Omega}, \tag{3}$$

the solution $w(\boldsymbol{x},t)$ of differential equation (1) will represent perturbation of the equilibrium state of the system.

The operators $\Im$ and $\Im_i$ ($i=1,\ldots,p$) contain only self-adjoint terms with constant coefficients. Hence, whenever w_1 and w_2 satisfy the boundary conditions (2):

$$\int_{\Omega} w_1\Im w_2\,\mathrm{d}\Omega=\int_{\Omega} w_2\Im w_1\,\mathrm{d}\Omega,\qquad \int_{\Omega} w_1\Im_i w_2\,\mathrm{d}\Omega=\int_{\Omega} w_2\Im_i w_1\,\mathrm{d}\Omega,\quad i=1,\ldots,p. \tag{4}$$

We assume that the solution of Eq. (1) exists and belongs to an appropriate Hilbert space. The purpose of the present paper is to derive the criteria for solving the following problem: will deviations from the unperturbed state (equilibrium state) be sufficiently small in a certain mathematical sense. To estimate a perturbed solution of Eq. (1) we introduce a measure of distance $\|\cdot\|$ of the solution of Eq. (1) with non-trivial initial conditions (3) from the trivial one. Following Caughey and Gray (1965), we shell say that the trivial solution of Eq. (1) is almost surely asymptotically stable if a measure of distance between the perturbated solution and the trivial one tends to zero with probability one as time tends to infinity:

$$P\Big\{\lim_{t\to\infty}\big\|w(\cdot,t)\big\|=0\Big\}=1. \tag{5}$$

3. Stability analysis

The transformation of the form:

$$w(\boldsymbol{x},t)=u(\boldsymbol{x},t)\exp\left\{-\int_0^t g(\tau)\,\mathrm{d}\tau\right\} \tag{6}$$

converts differential equation (1) to:

$$\frac{\partial^2 u}{\partial t^2}+2\varsigma\frac{\partial u}{\partial t}+\left\{\Im+\sum_{i=1}^{p}\big[f_{i0}+f_i(t)\big]\Im_i\right\}u-\varphi(t)u=0,\quad \boldsymbol{x}\in\boldsymbol{\Omega},\ t\geqslant 0, \tag{7}$$

where

$$\varphi(t) = g^2(t) + 2\varsigma g(t) + \dot{g}(t). \tag{8}$$

We construct the Liapunov functional as a sum of modified kinetic energy and elastic energy of the system:

$$\boldsymbol{V}(u, v) = \frac{1}{2} \int_{\Omega} \left[(v + \varsigma u)^2 + \varsigma^2 u^2 + u \left(\Im u + \sum_{i=1}^{p} f_{0i} \Im_i u \right) \right] \mathrm{d}\Omega, \tag{9}$$

where $v = \partial u / \partial t$. The functional is positive-definite if:

$$\int_{\Omega} u \left(\Im u + \sum_{i=1}^{p} f_{0i} \Im_i u \right) \mathrm{d}\Omega \geqslant k \int_{\Omega} u^2 \, \mathrm{d}\Omega, \tag{10}$$

where $k \geqslant 0$. Then the square root of functional $\boldsymbol{V}$ can be chosen as the measure $\|u(\boldsymbol{x}, t)\| = \sqrt{\boldsymbol{V}}$ of the distance of the disturbed solution from the initial state.

The time derivative of the functional (9) is:

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = -\int_{\Omega} \left\{ \varsigma v^2 + v \sum_{i=1}^{p} f_i(t) \Im_i u + \varsigma u \left[\Im u + \sum_{i=1}^{p} \bigl(f_{i0} + f_i(t) \bigr) \Im_i u \right] - u(v + \varsigma u)\varphi(t) \right\} \mathrm{d}\boldsymbol{x}. \tag{11}$$

Let a scalar function $\lambda(t)$ be such that:

$$\frac{\dot{\boldsymbol{V}}}{\boldsymbol{V}} \leqslant \lambda(t). \tag{12}$$

Integration of this expression yields:

$$\boldsymbol{V}(t) \leqslant \boldsymbol{V}(0) \exp\left\{ t \left[\frac{1}{t} \int_0^t \lambda(\tau) \, \mathrm{d}\tau \right] \right\} \tag{13}$$

and when $t \to \infty$, relation (13) will be satisfied if:

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \lambda(\tau) \, \mathrm{d}\tau < 0 \tag{14}$$

or, if processes $g(t)$, $\dot{g}(t)$, $f_1(t), \ldots, f_p(t)$ are ergodic and stationary:

$$\mathrm{E}\bigl\{\lambda(t)\bigr\} < 0, \tag{15}$$

where E is the operator of the mathematical expectation. Based on relation (13) we conclude that $\lim_{t \to \infty} \|u\|^2 = \lim_{t \to \infty} \boldsymbol{V} = 0$ and we obtain the following estimation:

$$\begin{aligned} \bigl\| w(\boldsymbol{x}, t) \bigr\|^2 &= \bigl\| u(\boldsymbol{x}, t) \bigr\|^2 \exp\left\{ -2 \int_0^t g(\tau) \, \mathrm{d}\tau \right\} = \boldsymbol{V}(u, v) \exp\left\{ -2 \int_0^t g(\tau) \, \mathrm{d}\tau \right\} \\ &\leqslant \boldsymbol{V}_0 \exp\left\{ \int_0^t \bigl[\lambda(\tau) - 2g(\tau) \bigr] \mathrm{d}\tau \right\}. \end{aligned} \tag{16}$$

According to (16), the trivial solution $w \equiv 0$ of Eq. (1) will be almost surely asymptotically stable if:

$$\mathrm{E}\bigl\{\lambda(t) - 2g(t)\bigr\} < 0. \tag{17}$$

If $\mathrm{E}\{g(t)\} = 0$, relation (17) is reduced to (15).

4. Example

Let us consider a thin, simply supported beam which is subjected to axial compressions and time-varying damping which are assumed to be an ergodic random processes. The equation of transverse motion of the beam has the form:

$$\frac{\partial^2 w}{\partial t^2} + 2\big(\varsigma + g(t)\big)\frac{\partial w}{\partial t} + \frac{\partial^4 w}{\partial z^4} + \big(f_0 + f(t)\big)\frac{\partial^2 w}{\partial z^2} = 0, \tag{18}$$

with boundary conditions:

$$\left.\begin{matrix} z=0 \\ z=1 \end{matrix}\right\}, \quad w = 0, \quad \frac{\partial^2 w}{\partial z^2} = 0. \tag{19}$$

By using transformation (6) differential equation (18) reduces to:

$$\frac{\partial^2 u}{\partial t^2} + 2\varsigma\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial z^4} + \big(f_0 + f(t)\big)\frac{\partial^2 u}{\partial z^2} - u\varphi(t) = 0. \tag{20}$$

Liapunov functional has the form:

$$\boldsymbol{V} = \frac{1}{2}\int_0^1\left[(v + \varsigma u)^2 + \varsigma^2 u^2 + \left(\frac{\partial^2 u}{\partial z^2}\right)^2 - f_0\left(\frac{\partial u}{\partial z}\right)^2\right]\mathrm{d}z \tag{21}$$

and its time derivative is:

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = -\int_0^1\left[\varsigma v^2 + \varsigma u\frac{\partial^4 u}{\partial z^4} + \varsigma f_0 u\frac{\partial^2 u}{\partial z^2} + f(t)\frac{\partial^2 u}{\partial z^2}(v + \varsigma u) - \varphi(t)u(v + \varsigma u)\right]\mathrm{d}z. \tag{22}$$

Using the extremum properties of the minimal eigenvalues:

$$\int_0^1\left(\frac{\partial^2 u}{\partial z^2}\right)^2\mathrm{d}z \geqslant \pi^2\int_0^1\left(\frac{\partial u}{\partial z}\right)^2\mathrm{d}z \tag{23}$$

the functional (21) will be positive-definite if:

$$f_0 \leqslant \pi^2, \tag{24}$$

which represents Euler condition for the static stability of the beam.

As the maximum is a particular case of the stationary point, we put:

$$\delta(\lambda\boldsymbol{V} - \dot{\boldsymbol{V}}) = 0. \tag{25}$$

By using the associated Euler–Lagrange equations we obtain:

$$\begin{aligned} &(\lambda + 2\varsigma)v + f(t)\frac{\partial^2 u}{\partial z^2} + \big(\lambda\varsigma - \varphi(t)\big)u = 0, \\ &\lambda\left[\varsigma(v + 2\varsigma u) + \frac{\partial^4 u}{\partial z^4} + f_0\frac{\partial^2 u}{\partial z^2}\right] + 2\varsigma\left(\frac{\partial^4 u}{\partial z^4} + f_0\frac{\partial^2 u}{\partial z^2}\right) + f(t)\left(\frac{\partial^2 v}{\partial z^2} + 2\varsigma\frac{\partial^2 u}{\partial z^2}\right) - \varphi(t)(v + 2\varsigma u) = 0. \end{aligned} \tag{26}$$

From (26) we can eliminate one of the unknowns, say v, yielding the fourth-order equation in u:

$$\begin{aligned} &(\lambda + 2\varsigma)^2\left(\frac{\partial^4 u}{\partial z^4} + f_0\frac{\partial^2 u}{\partial z^2}\right) + f(t)\left[\big(\lambda\varsigma + 4\varsigma^2 + \varphi(t)\big)\frac{\partial^2 u}{\partial z^2} - f(t)\frac{\partial^4 u}{\partial z^4}\right] \\ &\quad + \big(\lambda\varsigma - \varphi(t)\big)\left[\big(\lambda\varsigma + 4\varsigma^2 + \varphi(t)\big)u - f(t)\frac{\partial^2 u}{\partial z^2}\right] = 0. \end{aligned} \tag{27}$$

According to the boundary conditions (19) we can write the solution in the form:

$$u(z,t) = \sum_{m=1}^{\infty} T_m(t)U_m(z) = \sum_{m=1}^{\infty} T_m(t)\sin\alpha_m z, \tag{28}$$

where $\alpha_m = m\pi$, and from (27) we obtain the unknown function:

$$\lambda_m(t) = -2\varsigma \pm \frac{|2\varsigma^2 + \alpha_m^2 f(t) + \dot{g}(t) + 2\varsigma g(t) + g^2(t)|}{\sqrt{\alpha_m^2(\alpha_m^2 - f_0) + \varsigma^2}}. \tag{29}$$

The largest region of almost sure asymptotic stability is if:

$$\lambda(t) = \max_m \lambda_m(t) = \max_m \left\{ -2\varsigma + \frac{|2\varsigma^2 + \alpha_m^2 f(t) + \dot{g}(t) + 2\varsigma g(t) + g^2(t)|}{\sqrt{\alpha_m^2(\alpha_m^2 - f_0) + \varsigma^2}} \right\}. \tag{30}$$

5. Numerical results and discussion

It is well known (Kozin and Wu, 1973), that if joint probability density $p(f, g, \dot{g})$ of the excitation processes is available, stability regions can be enlarged. This improvement is shown for system with ergodic Gaussian coefficients. We assume that $f(t)$, $g(t)$ and $\dot{g}(t)$, with respect to their nature, are distributed independent Gaussian random processes with jointly probability density of the form:

$$p(f, g, \dot{g}) = \frac{1}{(2\pi)^{3/2}\sigma_f \sigma_g \sigma_{\dot{g}}} \exp\left\{ -\frac{f_{(t)}^2}{2\sigma_f^2} - \frac{g_{(t)}^2}{2\sigma_g^2} - \frac{\dot{g}_{(t)}^2}{2\sigma_{\dot{g}}^2} \right\} \tag{31}$$

so that for any integrable function $\mathrm{H}(f, g, \dot{g})$

$$\mathrm{E}\{\mathrm{H}(f, g, \dot{g})\} = \iiint_{-\infty}^{\infty} \mathrm{H}(f, g, \dot{g}) p(f, g, \dot{g})\, \mathrm{d}f\, \mathrm{d}g\, \mathrm{d}\dot{g}. \tag{32}$$

Changing to new variables defined by:

$$\xi = \frac{f}{\sqrt{2}\sigma_f}, \quad \eta = \frac{g}{\sqrt{2}\sigma_g}, \quad \theta = \frac{\dot{g}}{\sqrt{2}\sigma_{\dot{g}}} \tag{33}$$

Eq. (17) becomes:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{H}(\sigma_f, \sigma_g, \sigma_{\dot{g}}, \xi, \eta, \theta) \exp(-\xi^2 - \eta^2 - \theta^2)\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\theta < 2\varsigma, \tag{34}$$

where:

$$H(\sigma_f, \sigma_g, \sigma_{\dot{g}}, \xi, \eta, \theta) = \frac{|2\varsigma^2 + \sqrt{2}m^2\pi^2\sigma_f \xi + 2\sqrt{2}\sigma_g \varsigma \eta + 2\sigma_g^2 \eta^2 + \sqrt{2}\sigma_{\dot{g}} \varsigma|}{\sqrt{m^2\pi^2(m^2\pi^2 - f_0) + \varsigma^2}}. \tag{35}$$

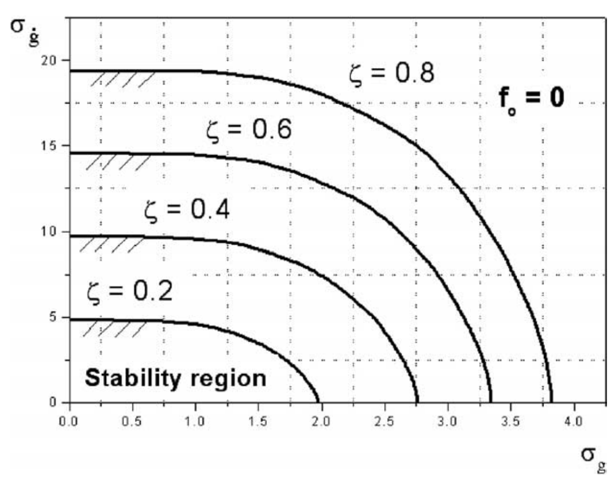

Fig. 1. Stability regions of the non-compressed beam as a function of the constant part of the damping coefficient.

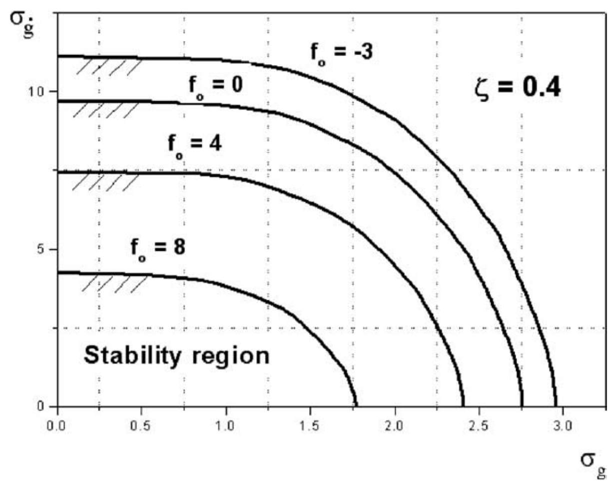

Fig. 2. Stability regions of the beam subjected only to the deterministic compressive force.

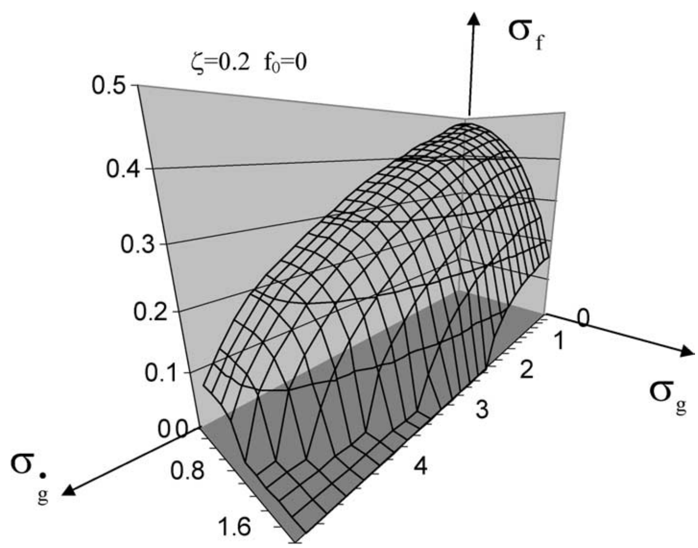

Fig. 3. Stability region of the beam subjected to the stochastic compressive force when $\zeta = 0.2$.

Solving inequality (34) was performed by extending Pavlović et al. (2001) numerical calculation, by using the corresponding Gauss–Hermite quadratures. Almost sure stability regions are given as the functions of variances processes $\sigma_f, \sigma_g, \sigma_{\dot{g}}$, constant component of viscous damping and compressive force f_0.

In Figs. 1 and 2 stability regions are shown when $f(t) = 0$, i.e. the compressive force has only the deterministic term f_0. We can notice that the increase of the deterministic component of viscous damping and decrease of the compressive force enlarged almost sure stability regions.

In Figs. 3 and 4 three dimensional surface bounded stability space.

The results obtained in this paper can be of importance in the study of the dynamic stability of viscous damped elastic structures in a randomly fluctuating supersonic flow field (with Mach number near unity).

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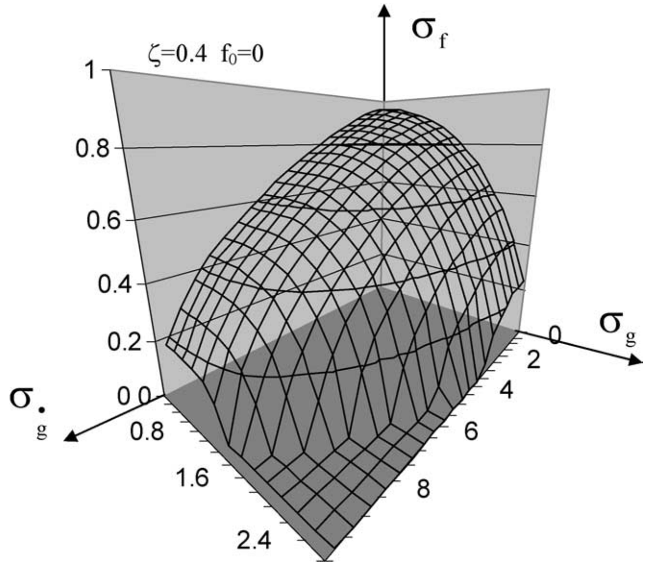

Fig. 4. Stability region of the beam subjected to the stochastic compressive force when $\zeta = 0.4$.

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