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Orthogonality of some sequences of the rational functions and the Müntz polynomials

S.D. Marinković[a,*], B. Danković[a], M.S. Stanković[b], P.M. Rajković[c]

[a] Faculty of Electronic Engineering, University of Niš, 18000 Niš, Yugoslavia
[b] Faculty of Occupational Safety, University of Niš, 18000 Niš, Yugoslavia
[c] Faculty of Mechanical Engineering, University of Niš, 18000 Niš, Yugoslavia

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Abstract

In this paper we investigate the inner products in the space of rational functions and the space of Müntz polynomials. We prove that the orthogonality of a sequence in one of mentioned spaces can be induced by the orthogonality of the corresponding sequence in another space. Finally, we point to several special cases, i.e., some very different classes of well–known functions we represent on a unique way.

Keywords: Orthogonality; Rational functions; Müntz polynomials

1. Introduction

The Legendre polynomials (see, for example, [3])

$$P_n(x) = (-1)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{n+k}{k} x^k$$

satisfy the relation of orthogonality

$$\int_0^1 P_m(x) P_n(x)\,\mathrm{d}x = \frac{\delta_{mn}}{2n+1}.$$

* Corresponding author. Faculty of Electrical Engineering, University of Niš, 18000 Niš, Yugoslavia.
E-mail addresses: sladjana@elfak.ni.ac.yu (S.D. Marinković), dankovic@elfak.ni.ac.yu (B. Danković), mstan@znrfak.znrfak.ni.ac.yu (M.S. Stanković), pecar@masfak.ni.ac.yu (P.M. Rajković).

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They can be represented by Rodrigues formula

$$P_n(x) = \frac{(-1)^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^n(1-x)^n).$$

In paper [4], the authors have considered orthoexponential polynomials $p_n(t) = P_n(1 - \mathrm{e}^{-t})$ and rational functions obtained by applying Laplace transform. They have proved that, for the functions

$$W_n(s) = \mathscr{L}[p_n(t)] = (-1)^n \frac{(s-1)(s-2)\cdots(s-n)}{s(s+1)\cdots(s+n)}, \tag{1.1}$$

it exists a scalar product with respect to whom they are orthogonal. Also, the connection between the scalar products in the domains of the polynomials and the corresponding rational functions established. In paper [1] connection between some classes of the Müntz polynomials and rational functions was presented and it also considered their orthogonality.

Our purpose is to explain on a new way of previously obtained results and to study more general cases. After the preliminaries, in the second section, it was examined a general scalar product with a few parameters in the domain of the rational functions using a bilinear transform of the independent variable and a modification of the rational function corresponding to the Legendre polynomials.

In Section 3 a new operation with the Müntz polynomials is defined and the respected scalar product in the domain of the set of some classes of the Müntz polynomials is investigated.

Finally, Section 4, various classes of orthogonal functions (the Legendre polynomials, the Müntz–Legendre polynomials, some classes of the Jacobi polynomials and the Müntz polynomials respected the Malmquist rational functions) are represented on the similar way like a special cases of previously considered functions.

2. Orthogonality of a sequence of the rational functions

For the start, note that the zeros of the rational functions (1.1) can be obtained by applying the linear transform $s \to 1 - s$ to their poles. Let us substitute the linear transform by the bilinear one

$$s \;\to\; \frac{as+b}{cs-a} \quad (a^2 + bc > 0) \quad (a, b, c \in \mathbb{R}).$$

In that sense, consider the set $\mathscr{W}$ of the rational functions represented by

$$W_n(s) = \frac{\prod_{k=0}^{n-1}\left(s - \dfrac{a\bar{\alpha}_k + b}{c\bar{\alpha}_k - a}\right)}{\prod_{k=0}^{n}(s - \alpha_k)}, \tag{2.1}$$

where $\alpha_k,\ k = 0, 1, 2, \ldots$ be different complex numbers such that

$$c\alpha_i\bar{\alpha}_j - a(\alpha_i + \alpha_j) - b > 0. \tag{2.2}$$

The poles $\{\alpha_k\}$ and zeros $\{(a\bar{\alpha}_k + b)/(c\bar{\alpha}_k - a)\}$ of functions (2.1) lie on different sides of the curve

$$\Gamma = \{s \in \mathbb{C} : c\,|s|^2 - 2a\ \mathrm{Re}\,s - b = 0\}.$$

We define the mapping $(\cdot,\cdot) : \mathscr{W} \times \mathscr{W} \mapsto \mathbb{C}$ by

$$(W_n, W_m) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma W_{\max\{m,n\}}(s)\, \tilde{W}_{\min\{m,n\}}(s)\ \frac{\mathrm{d}s}{cs - a}, \tag{2.3}$$

where

$$\tilde{W}_n(s) = \overline{W_n\left(\frac{a\bar{s} + b}{c\bar{s} - a}\right)}.$$

Theorem 2.1. *Mapping* (2.3) *is the scalar product on* $\mathscr{W}$. *Moreover*, $\mathscr{W}$ *is orthogonal with respect to this scalar product and holds*

$$(W_n, W_m) = \frac{(a^2 + bc)^n}{\prod_{k=0}^{n-1} |c\alpha_k - a|^2}\ \frac{\delta_{mn}}{c|\alpha_n|^2 - 2a\ \mathrm{Re}\ \alpha_n - b}.$$

Proof. The symmetry and the linearity of the product holds immediately from the definition. For positivity, note that

$$W_n(s)\ \tilde{W}_n(s) = -\frac{(a^2 + bc)^n (c\alpha_n - a)}{\prod_{k=0}^{n} |c\alpha_k - a|^2}\ \frac{cs - a}{(s - \alpha_n)(s - \frac{a\bar{\alpha}_n + b}{c\bar{\alpha}_n - a})}$$

and

$$\begin{aligned}(W_n, W_n) &= -\frac{1}{2\pi \mathrm{i}}\ \frac{(a^2 + bc)^n (c\alpha_n - a)}{\prod_{k=0}^{n} |c\alpha_k - a|^2} \int_\Gamma \frac{\mathrm{d}s}{(s - \alpha_n)\left(s - \dfrac{a\bar{\alpha}_n + b}{c\bar{\alpha}_n - a}\right)} \\ &= -\frac{(a^2 + bc)^n (c\alpha_n - a)}{\prod_{k=0}^{n} |c\alpha_k - a|^2} \operatorname*{Res}_{s = \frac{a\bar{\alpha}_n + b}{c\bar{\alpha}_n - a}} \frac{1}{(s - \alpha_n)(s - \frac{a\bar{\alpha}_n + b}{c\bar{\alpha}_n - a})} \\ &= \frac{(a^2 + bc)^n}{\prod_{k=0}^{n-1} |c\alpha_k - a|^2}\ \frac{1}{c|\alpha_n|^2 - 2a\ \mathrm{Re}\ \alpha_n - b}.\end{aligned}$$

Suppose that $n > m$. Then

$$W_n(s)\ \tilde{W}_m(s) = -\frac{(a^2 + bc)^m (c\alpha_m - a)}{\prod_{k=0}^{m} |c\alpha_k - a|^2}\ \frac{(cs - a)\prod_{k=m+1}^{n-1} (s - \frac{a\bar{\alpha}_k + b}{c\bar{\alpha}_k - a})}{\prod_{k=m}^{n} (s - \alpha_k)}$$

and

$$(W_n, W_m) = -\frac{(a^2+bc)^m(c\alpha_m - a)}{\prod_{k=0}^{m}|c\alpha_k - a|^2}\int_\Gamma \frac{\prod_{k=m+1}^{n-1}\left(s - \frac{a\bar{\alpha}_k + b}{c\bar{\alpha}_k - a}\right)}{\prod_{k=m}^{n}(s-\alpha_k)} = 0.$$

So, we get

$$(W_n, W_m) = \frac{(a^2+bc)^n}{\prod_{k=0}^{n-1}|c\alpha_k - a|^2}\,\frac{\delta_{mn}}{c|\alpha_n|^2 - 2a\ \mathrm{Re}\ \alpha_n - b}. \qquad \square$$

The scalar product on $\mathscr{W}$ can be represented on the other way.

Theorem 2.2. *For the members of the set* $\mathscr{W}$ *holds*

$$(W_n, W_m) = \sum_{k=0}^{n}\sum_{j=0}^{m}\frac{A_{nk}\,\bar{A}_{mj}}{c\alpha_k\bar{\alpha}_j - a(\alpha_k + \bar{\alpha}_j) - b}, \tag{2.4}$$

where

$$A_{nk} = \frac{\prod_{j=0}^{n-1}\left(\alpha_k - \frac{a\bar{\alpha}_j + b}{c\bar{\alpha}_j - a}\right)}{\prod_{\substack{j=0\\ j\neq k}}^{n}(\alpha_k - \alpha_j)}. \tag{2.5}$$

Proof. Expanding the rational functions (2.1) as a linear combinations of the partial fractions and according to the definition of the function $\tilde{W}_m(s)$, we have

$$W_n(s) = \sum_{k=0}^{n}\frac{A_{nk}}{s-\alpha_k}, \qquad \tilde{W}_m(s) = -(cs-a)\sum_{j=0}^{m}\frac{\bar{A}_{mj}}{c\bar{\alpha}_j - a}\,\frac{1}{s - \frac{a\bar{\alpha}_j + b}{c\bar{\alpha}_j - a}}.$$

Hence

$$\begin{aligned} W_n(s)\,\tilde{W}_m(s) &= \left(\sum_{k=0}^{n}\frac{A_{nk}}{s-\alpha_k}\right)\left(-(cs-a)\sum_{j=0}^{m}\frac{\bar{A}_{mj}}{c\bar{\alpha}_j - a}\,\frac{1}{s - \frac{a\bar{\alpha}_j + b}{c\bar{\alpha}_j - a}}\right) \\ &= -\sum_{k=0}^{n}\sum_{j=0}^{m}\frac{A_{nk}\,\bar{A}_{mj}}{c\bar{\alpha}_j - a}\,\frac{cs-a}{(s-\alpha_k)\left(s - \frac{a\bar{\alpha}_j + b}{c\bar{\alpha}_j - a}\right)}. \end{aligned}$$

Now, the scalar product of two functions (2.2) can be represented in the form

$$(W_n, W_m) = -\frac{1}{2\pi \mathrm{i}}\sum_{k=0}^{n}\sum_{j=0}^{m}\frac{A_{nk}\,\bar{A}_{mj}}{c\bar{\alpha}_j - a}\int_\Gamma \frac{\mathrm{d}s}{(s-\alpha_k)\left(s - \frac{a\bar{\alpha}_j + b}{c\bar{\alpha}_j - a}\right)}$$

$$= -\sum_{k=0}^{n}\sum_{j=0}^{m}\frac{A_{nk}\ \bar{A}_{mj}}{c\bar{\alpha}_j - a}\ \operatorname*{Res}_{s=\frac{a\bar{\alpha}_n+b}{c\bar{\alpha}_n-a}} \frac{1}{(s-\alpha_k)\left(s-\frac{a\bar{\alpha}_j+b}{c\bar{\alpha}_j-a}\right)}$$

$$= -\sum_{k=0}^{n}\sum_{j=0}^{m}\frac{A_{nk}\ \bar{A}_{mj}}{c\bar{\alpha}_j - a}\ \frac{1}{\frac{a\bar{\alpha}_j+b}{c\bar{\alpha}_j-a} - \alpha_k},$$

wherefrom we get the statement of the theorem. □

3. Orthogonality of the Müntz polynomials

Let $\mathscr{P} = \mathscr{P}(\{\alpha_k\})$ be a set of the Müntz polynomials

$$\mathscr{P} = \left\{ p_n(x)\ :\ p_n(x) = \sum_{k=0}^{n} a_k x^{\alpha_k}\,(a_k \in \mathbb{C})\,(n \in \mathbb{N}_0)\right\}.$$

Let us define the operation $*$ over the monomials by

$$x^{\alpha} *\ x^{\beta} = x^{c\alpha\beta - a(\alpha+\beta) - b - 1}, \quad (x > 0,\ \alpha, \beta \in \mathbb{C}).$$

This way, on $\mathscr{P}$ is defined the external operation. Indeed, by applying $*$ to the Müntz polynomials $p_n(x)$ and $q_m(x)$ we join the polynomial

$$p_n(x) * q_m(x) = \sum_{k=0}^{n} a_k x^{\alpha_k} * \sum_{j=0}^{m} b_j x^{\alpha_j} = \sum_{k=0}^{n}\sum_{j=0}^{m} a_k b_j\ x^{\alpha_k} * x^{\alpha_j},$$

which, in general, is not the member of $\mathscr{P}$.

The operation $*$ is the generalization of an new operation between the Müntz polynomials involved in [1]. Also, the next statement is inspired by the similar result in [1].

Theorem 3.1. *The mapping* $(\cdot\,,\cdot) : \mathscr{P} \times \mathscr{P} \to \mathbb{C}$ *defined by*

$$(f, g)_* = \int_0^1 f(x) * \overline{g(x)}\,\mathrm{d}x, \quad (f, g \in \mathscr{P}) \tag{3.1}$$

is the scalar product on $\mathscr{P}$.

Proof. Linearity and symmetry follow immediately from the definition. For the arbitrary polynomials $p_n(x)$ and $q_m(x)$ is valid

$$(p_n, q_m)_* = \left(\sum_{k=0}^{n} a_k x^{\alpha_k}, \sum_{j=0}^{m} b_j x^{\alpha_j}\right)_*$$

$$= \int_0^1 \sum_{k=0}^{n} \sum_{j=0}^{m} a_k \bar{b}_j \ x^{\alpha_k} * x^{\bar{\alpha}_j} \, \mathrm{d}x$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{m} a_k \ \bar{b}_j \int_0^1 x^{c\alpha_k \bar{\alpha}_j - a(\alpha_k + \bar{\alpha}_j) - b - 1} \, \mathrm{d}x.$$

Because of conditions (2.2) are satisfied, we have

$$(p_n, q_m)_* = \sum_{k=0}^{n} \sum_{j=0}^{m} \frac{a_k \bar{b}_j}{c\alpha_k \bar{\alpha}_j - a(\alpha_k + \bar{\alpha}_j) - b}. \tag{3.2}$$

To examine positivity of product (3.1), we will consider the next quadratic form

$$(p_n, p_n)_* = \sum_{k=0}^{n} \sum_{j=0}^{m} \frac{a_k \ \bar{a}_j}{c\alpha_k \bar{\alpha}_j - a(\alpha_k + \bar{\alpha}_j) - b}$$

and the corresponding matrix

$$H_n = \left[\frac{1}{c\alpha_k \bar{\alpha}_j - a(\alpha_k + \bar{\alpha}_j) - b} \right]_{k,j=0}^{n}.$$

By Sylvester criterion, necessary and sufficient condition for positivity is

$$D_l = \det \ H_l > 0, \quad l = 0, 1, 2, \ldots, n.$$

Since

$$D_l = \frac{1}{\prod_{j=0}^{l} (c\bar{\alpha}_j - a)} \ \det \left[\frac{1}{\alpha_k - \frac{a\bar{\alpha}_j + b}{c\bar{\alpha}_j - a}} \right]_{k,j=0}^{l},$$

by Cauchy formula for the determinants of the mentioned form it is valid

$$D_l = \frac{1}{\prod_{j=0}^{l} (c\bar{\alpha}_j - a)} \frac{\prod_{k>j=0}^{l} (\alpha_k - \alpha_j) \left(\frac{a\bar{\alpha}_j + b}{c\bar{\alpha}_j - a} - \frac{a\bar{\alpha}_k + b}{c\bar{\alpha}_k - a} \right)}{\prod_{k,j=0}^{l} \left(\alpha_k - \frac{a\bar{\alpha}_j + b}{c\bar{\alpha}_j - a} \right)}$$

$$= \frac{1}{\prod_{j=0}^{l} (c\bar{\alpha}_j - a)} \frac{\prod_{k=j+1}^{l} \prod_{j=0}^{l-1} \frac{(a^2 + bc)|\alpha_k - \alpha_j|^2}{(c\bar{\alpha}_k - a)(c\bar{\alpha}_j - a)}}{\prod_{k=0}^{l} \prod_{j=0}^{l} \frac{c\alpha_k \bar{\alpha}_j - a(\alpha_k + \bar{\alpha}_j) - b}{c\bar{\alpha}_j - a}}.$$

Hence

$$D_l = D_{l-1} \frac{(a^2 + bc)^l}{c \, |\alpha_l|^2 - 2a \ \mathrm{Re} \ \alpha_l - b} \prod_{j=0}^{l-1} \frac{|\alpha_l - \alpha_j|^2}{|c\alpha_l \bar{\alpha}_j - a(\alpha_l + \bar{\alpha}_j) - b|^2}.$$

With respect to (2.2), it holds

$$D_0 = \frac{1}{c\,|\alpha_0|^2 - 2a\,\mathrm{Re}\,\alpha_0 - b} > 0.$$

Using the principle of mathematical induction, we obtain that $D_l > 0$ for all l. □

Now, we will consider the special Müntz polynomials

$$P_n(x) = \sum_{k=0}^{n} A_{nk} x^{\alpha_k}, \tag{3.3}$$

where the coefficients A_{nk} are defined in (2.5).

Theorem 3.2. *The sequence of the polynomials* $\{P_n(x)\}_{n\in\mathbb{N}_0}$ *defined by* (3.3) *is orthogonal with respect to the scalar product* (3.1) *and holds*

$$(P_n, P_m)_* = \frac{(a^2+bc)^n}{\prod_{k=0}^{n-1} |c\alpha_k - a|^2} \frac{\delta_{mn}}{c\,|\alpha_n|^2 - 2a\,\mathrm{Re}\,\alpha_n - b}.$$

Proof. Because of (3.2), it holds

$$(P_n, P_m)_* = \sum_{k=0}^{n}\sum_{j=0}^{m} \frac{A_{nk}\,\bar{A}_{mj}}{c\alpha_k\bar{\alpha}_j - a(\alpha_k + \bar{\alpha}_j) - b} = (W_n, W_m).$$

Knowing (2.4), we confirm that the statement is valid. □

4. Special cases

In this section we will point that the set of polynomials $\mathscr{P}$ and the set of rational functions $\mathscr{W}$ comprehend a lot of divers class of functions, which can be obtained for the special values of the parameters.

Let $c = 0$ and $a \neq 0$. Then the curve Γ separating the zeros and the poles of the rational functions (2.1) becomes the line $\mathrm{Re}\, s = -b/2a$. Specially, for $a = 1,\ b = -1$ and $\alpha_k = -k$, respected rational function (2.1) is

$$W_n(s) = \frac{(s-1)(s-2)\cdots(s-n)}{s(s+1)\cdots(s+n)},$$

i.e., the rational function respected the Legendre polynomials on the interval $(0, 1)$. The relation of orthogonality is the same as the relation given in the first section.

For $-b/a = b_0 > 0$ and $\alpha_k = -k$ the rational function (2.1) has the form

$$W_n(s) = \frac{(s-b_0)(s-b_0-1)\cdots(s-b_0-n+1)}{s(s+1)\cdots(s+n)},$$

and orthogonality in the domain of the polynomials becomes

$$(P_n, P_m)_* = \frac{\delta_{mn}}{2n + b_0},$$

i.e., it is obtained the class of the Jacobi polynomials $p_n^{(0,b_0)}(x)$ on the interval (0,1), (see [2]).

If $a = 1$, $b = -1$ and $\operatorname{Re}\alpha_k < 1/2$, respected rational function is

$$W_n(s) = \frac{(s + \bar{\alpha}_0 - 1)(s + \bar{\alpha}_1 - 1)\cdots(s + \bar{\alpha}_{n-1} - 1)}{(s - \alpha_0)(s - \alpha_1)\cdots(s - \alpha_n)}.$$

The orthogonality in the domain of the polynomials is given by

$$(P_n, P_m)_* = \frac{\delta_{nm}}{1 - \alpha_n - \bar{\alpha}_n}.$$

On this way the orthogonality of the Müntz–Legendre polynomials over the system $\{x^{-\alpha_0}, x^{-\alpha_1}, \ldots, x^{-\alpha_n}\}$ is given.

Finally, let $a = 0$ and $c \neq 0$. Then the curve Γ is the circle $|s| = b/c$. For $b = c = 1$ and $|\alpha_k| > 1$ the rational function (2.1) is

$$W_n(s) = \frac{(s - \frac{1}{\bar{\alpha}_0})(s - \frac{1}{\bar{\alpha}_1})\cdots(s - \frac{1}{\bar{\alpha}_{n-1}})}{(s - \alpha_0)(s - \alpha_1)\cdots(s - \alpha_n)},$$

i.e., Malmquist rational function of the orthogonal sequence with orthogonality relation

$$(W_n, W_m) = \frac{\delta_{mn}}{|\alpha_0\alpha_1\cdots\alpha_{n-1}|^2(|\alpha_n|^2 - 1)}.$$

In the domain of the polynomials we obtain respected sequence of the Müntz polynomials with the orthogonality relation

$$(P_n, P_m)_* = \frac{\delta_{mn}}{|\alpha_0\alpha_1\cdots\alpha_{n-1}|^2(|\alpha_n|^2 - 1)}.$$

If $b/c = b_1 > 0$ and $|\alpha_k| > \sqrt{b_1}$ we get the generalized Malmquist rational functions and respected Müntz polynomials with the orthogonality relation

$$(W_n, W_m) = (P_n, P_m)_* = \frac{b_1^n}{|\alpha_0\alpha_1\cdots\alpha_{n-1}|^2(|\alpha_n|^2 - b_1)}\,\delta_{mn}.$$

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