

The Pumping Lemma

The Burning Question...

- We've looked at a number of regular languages
- I know that you are just dying to know...
 - Is there a language L that is not regular?
 - To answer this, we'll use what is known as The Pumping Lemma.

The Pumping Lemma

- The pumping lemma formalizes the idea that if a string from a RL is long enough, eventually at least one state on its FA will have to be repeated on the path that accepts the string.
 - Implies that there is a Kleene star in there somewhere!
- Continually looping on this state will produce an infinite number of strings in the language

The Pumping Lemma

- Statement of the pumping lemma
 - Let L be a regular language.
 - Then there exists a constant n (which varies for different languages), such that for every string $x \in L$ with $|x| \geq n$, x can be expressed as $x = uvw$ such that:
 1. $|v| > 0$
 2. $|uv| \leq n$
 3. For all $k \geq 0$, the string $uv^k w$ is also in L .

The Pumping Lemma

- What this means
 - For a long enough string x in L :
 - We can express x as the concatenation of three smaller strings
 - The middle string can be “pumped” (repeated) any number of times (including 0 = deleting) and the resulting string will be in L .

Pumping Lemma

- Proof of the pumping lemma
 - Since L is regular, there is a FA $M = (Q, \Sigma, q_0, A, \delta)$ that accepts L .
 - Assume M has n states.
 - Consider a string x with $|x| = m \geq n$.
 - Express $x = a_1 a_2 a_3 \dots a_m$ where each $a_i \in \Sigma$.
 - Define p_i to be the state M is in after reading i characters:
 - $p_i \stackrel{\Delta}{=} (q_0, a_1 a_2 \dots a_i)$
 - $p_0 = q_0$

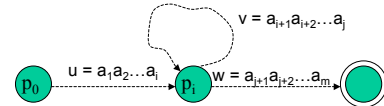
Pumping Lemma

- Proof of the pumping lemma
 - Since $|x| \geq n$, and we only have n states, one state on its path must be visited more than once.
 - There exists integers i and j , $0 \leq i < j \leq n$ such that $p_i = p_j$
 - Then $x = uvw$
 - $u = a_1 a_2 \dots a_i$
 - $v = a_{i+1} a_{i+2} \dots a_j$
 - $w = a_{j+1} a_{j+2} \dots a_m$

Pumping Lemma

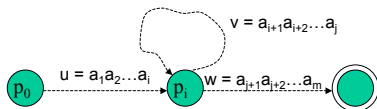
- Proof of pumping lemma

- Then $x = uvw$
 - $u = a_1 a_2 \dots a_i$
 - $v = a_{i+1} a_{i+2} \dots a_j$
 - $w = a_{j+1} a_{j+2} \dots a_m$



Pumping Lemma

- Proof of pumping lemma
 - You can loop (pump) on the v loop 0 or more times and there will still be a path to the accepting state.



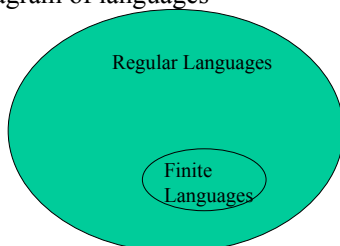
Pumping Lemma

- So what good is the pumping lemma?
- It can be used to answer that burning question:
 - Is there a language L that is not regular?

Non-regular languages

- Venn-diagram of languages

Is there something out here?



Pumping lemma

- The real strength of the pumping lemma is proving that languages are not regular
 - Proof by contradiction
 - Assume that the language to be tested is regular
 - Use the pumping lemma to come to a contradiction
 - Original assumption about the language being regular is false
- You cannot prove a language to be regular using the Pumping Lemma!!!!

Pumping lemma

- The Pumping Lemma game
 - To show that a language L is not regular
 - Assume L is regular
 - Choose an “appropriate” string x in L
 - In terms of n (number of states in DFA)
 - Express $x = uvw$ following rules of pumping lemma
 - Show that $uv^k w$ is not in L , for some k
 - The above contradicts the Pumping Lemma
 - Our assumption that L is regular is wrong
 - L must not be regular

Pumping Lemma



Pumping lemma

- Example:
 - $L = \{x \in \{0,1\}^* \mid 0^i 1^i, i \geq 0\}$
 - Ex: 000111, 0011, ϵ , 00001111

Pumping lemma

- Example
 - Let's play!
 - Assume that L is regular.
 - Then there is an FA, M that accepts L .
 - Let n be the number of states in M



Pumping lemma

- Example:
 - $L = \{x \in \{0,1\}^* \mid 0^i 1^i, i \geq 0\}$
 - Let's play
 - Choose an appropriate string $x \in L$
 - Let $x = 0^n 1^n$
 - Apply Pumping Lemma to x
 - $x = uvw$
 - $|uv| \leq n$
 - $|v| > 0$

Pumping lemma

- $x = uvw = 0^n 1^n$
 - 00 ... 0 11 ... 1
 - Since $|uv| \leq n$, uv must consist entirely of 0s and, as such, v must also consist entirely of 0s.
 - $v = 0^j$ for some $j \leq n$

Pumping lemma

- $x = uvw = 0^n 1^n = 0^i 0^j 0^k 1^n$ $i + j + k = n$
 - Let's pump!
 - By the Pumping Lemma
 - uv^2w is also in L
 - $uv^2w = 0^i 0^{2j} 0^k 1^n$
 - Certainly $i + 2j + k \neq n$
 - uv^2w has more 0's than 1's
 - Thus $uv^2w \notin L$ CONTRADICTION!

Pumping lemma

- We arrived at a contradiction,
 - Thus our original assumption that L is regular must be incorrect
 - Thus L is not regular.
- Note that we need to find only 1 string x that fails in order for the proof by contradiction to work.
 - The key is finding the x that won't work
- Questions?

Pumping Lemma

- Let's try another example:
 - $L = \{x \in \{0,1\}^* \mid 0^i x, |x| \leq i\}$
 - Ex: 0001, 0010, ϵ , 0000101
 - Let's play!
 - Assume that L is regular.
 - Then there is an FA, M that accepts L .
 - Let n be the number of states in M

Pumping Lemma

- Another Example:
 - $L = \{x \in \{0,1\}^* \mid 0^i x, |x| \leq i\}$
 - Let's play
 - Choose an appropriate string $x \in L$
 - Let $x = 0^n 1^n$
 - Apply Pumping Lemma to x
 - $x = uvw$
 - $|uv| \leq n$
 - $|v| > 0$

Pumping Lemma

- $x = uvw = 0^n 1^n$
 - 00 ... 0 11 ... 1
- Since $|uv| \leq n$, uv must consist entirely of 0s and, as such, v must also consist entirely of 0s.
 - $v = 0^j$ for some $j \leq n$

Pumping Lemma

- $x = uvw = 0^n 1^n = 0^i 0^j 0^k 1^n$ $i + j + k = n$
 - Let's (un)pump!
 - By the Pumping Lemma
 - uv^0w is also in L
 - $uv^0w = uw = 0^i 0^k 1^n$
 - Certainly $n > i + k$
 - The length of the prefix of 0s is less than the suffix x
 - Thus $uv^0w \notin L$ CONTRADICTION!

Pumping Lemma

- We arrived at a contradiction,
 - Thus our original assumption that L is regular must be incorrect
 - Thus L is not regular.
- Note that we need to find only 1 string x that fails in order for the proof by contradiction to work.
 - We can show x not to work by pumping 0 times
- Questions?

Non-regular languages

- Informal notion of what regular languages can't express:
 - Counting and comparing
 - Any operation that implies the use of a stack
 - Pal
 - xx^r

Pumping Lemma

- Let's try another example:
 - L = set of palindromes over $\{a,b\}$
 - Strings that read the same forwards and backwards
 - "Madam I'm Adam"
 - Ex: aa, abba, abbbba, ϵ
 - Let's play!
 - Assume that L is regular.
 - Then there is an FA, M that accepts L .
 - Let n be the number of states in M

Pumping Lemma

- Another Example:
 - L = set of palindromes over $\{a,b\}$
 - Let's play
 - Choose an appropriate string $x \in L$
 - Let $x = a^n b a^n$
 - Apply Pumping Lemma to x
 - $x = uvw$
 - $|uv| \leq n$
 - $|v| > 0$

Pumping Lemma

- $x = uvw = a^n b a^n$
 - aa ... a b aa ... a
- Since $|uv| \leq n$, uv must consist entirely of a and, as such, v must also consist entirely of a .
 - $v = a^j$ for some $j \leq n$

Pumping Lemma

- $x = uvw = a^n b a^n = a^i a^j a^k b a^n$ $i + j + k = n$
 - Let's pump!
 - By the Pumping Lemma
 - uv^2w is also in L
 - uv^2w has more than n a 's
 - Number of a 's following b is still n
 - Thus uv^2w cannot be a palindrome
 - Thus $uv^2w \notin L$ CONTRADICTION!

Pumping Lemma

- We arrived at a contradiction,
 - Thus our original assumption that L is regular must be incorrect
 - Thus L is not regular.
- Note that we need to find only 1 string x that fails in order for the proof by contradiction to work.
- Questions?

Pumping Lemma

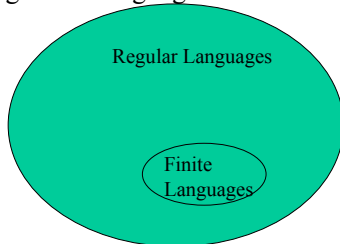
- Summary
 - The pumping lemma formalizes the idea that if a string is longer enough, eventually at least one state on the DFA will have to be repeated on the path that accepts the string.
 - Continually looping on this state will produce an infinite number of strings in the language
 - Used to show that languages are not regular
 - Has other uses as we'll see next time.

Non-regular languages

- Venn-diagram of languages

Is there
something
out here?

YES



Next time

- Decision properties / algorithms for RLs
- Problem Session...
 - For homeworks: past and present
- Questions?