Properties of Regular Languages

- *Pumping Lemma.* Every regular language satisfies the pumping lemma. If somebody presents you with fake regular language, use the pumping lemma to show a contradiction.

- *Closure properties.* Building automata from components through operations, e.g. given L and M we can build an automaton for $L \cap M$.

- *Decision properties.* Computational analysis of automata, e.g. are two automata equivalent.

- *Minimization techniques.* We can save money since we can build smaller machines.

The Pumping Lemma Informally

Suppose $L_{01} = \{0^n 1^n : n \geq 1\}$ were regular.

Then it would be recognized by some DFA A, with, say, k states.

Let A read 0^k. On the way it will travel as follows:

ϵ	p_0
0	p_1
00	p_2
$\ldots$	$\ldots$
0^k	p_k

$\Rightarrow \exists i < j : p_i = p_j$ Call this state q.

Now you can fool A:

If $\widehat{\delta}(q, 1^i) \in F$ the machine will foolishly accept $0^j 1^i$.

If $\widehat{\delta}(q, 1^i) \notin F$ the machine will foolishly reject $0^i 1^i$.

Therefore L_{01} cannot be regular.

- Let's generalize the above reasoning.

Theorem 4.1.

The Pumping Lemma for Regular Languages.

Let L be regular.

Then $\exists n, \forall w \in L : |w| \geq n \Rightarrow w = xyz$ such that

1. $y \neq \epsilon$

2. $|xy| \leq n$

3. $\forall k \geq 0,\ xy^k z \in L$

Proof: Suppose L is regular

The L is recognized by some DFA A with, say, n states.

Let $w = a_1 a_2 \ldots a_m \in L,\ m > n$.

Let $p_i = \widehat{\delta}(q_0, a_1 a_2 \cdots a_i)$.

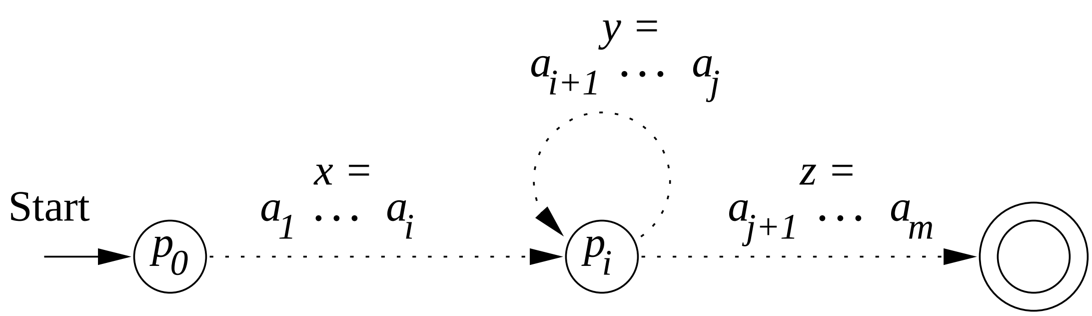
$\Rightarrow \exists i < j : p_i = p_j$

Now $w = xyz$, where

1. $x = a_1 a_2 \cdots a_i$

2. $y = a_{i+1} a_{i+2} \cdots a_j$

3. $z = a_{j+1} a_{j+2} \ldots a_m$



Evidently $xy^k z \in L$, for any $k \geq 0$. $Q.E.D.$

Example: Let L_{eq} be the language of strings with equal number of zero's and one's.

Suppose L_{eq} is regular. Then $w = 0^n 1^n \in L$.

By the pumping lemma $w = xyz$, $|xy| \leq n$, $y \neq \epsilon$ and $xy^k z \in L_{eq}$

$$w = \underbrace{000\cdots}_{x}\underbrace{\cdots 0}_{y}\underbrace{0111\cdots 11}_{z}$$

In particular, $xz \in L_{eq}$, but xz has fewer 0's than 1's.

Suppose $L_{pr} = \{1^p : p \text{ is prime }\}$ were regular.

Let n be given by the pumping lemma.

Choose a prime $p \geq n + 2$.

$$w = \overbrace{\underbrace{111 \cdots}_{x}\underbrace{\cdots 1}_{\substack{y \\ |y|=m}}\underbrace{1111 \cdots 11}_{z}}^{p}$$

Now $xy^{p-m}z \in L_{pr}$

$|xy^{p-m}z| = |xz| + (p - m)|y| =$
$p - m + (p - m)m = (1 + m)(p - m)$
which is not prime unless one of the factors is 1.

- $y \neq \epsilon \Rightarrow 1 + m > 1$

- $m = |y| \leq |xy| \leq n, \;\; p \geq n + 2$
$\Rightarrow p - m \geq n + 2 - n = 2.$