Context-Free Grammars & Context-Free Languages

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What you'll learn

1. **Context-free grammars:**
   - Structure
   - Interpretation (multiple equivalent ones)
     - Recursive inference
     - Reductions
     - Derivations & standard derivations
     - Parse trees
   - Proof of equivalence

2. **The relation between regular and context-free languages:**
   - context-free languages are richer (strictly)

Regular languages are too restrictive

The examples concerning the pumping lemma for regular languages show that e.g.

- palindromes (“madamimadam”),
- $\{0^n 1^n\}$,
- the set of arithmetic expressions with *correctly paired brackets*

are not regular.

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Need some (limited) counting mechanism!

How to construct palindromes

All palindromes over $\{0, 1\}$ can be constructed by the following rules:

1. ε is a palindrome.
2. 0 is a palindrome.
3. 1 is a palindrome.
4. If w is a palindrome then so is $0w0$.
5. If w is a palindrome then so is $1w1$.

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All palindromes over $\{0, 1\}$ can be constructed by the following rules:

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4. If w is a palindrome then so is $0w0$. $P \to 0P0$
5. If w is a palindrome then so is $1w1$. $P \to 1P1$

The reddish things are called “production rules”.

Context-free grammars

A context-free grammar (CFG) consists of

1. a finite set V of *variables* (a.k.a. *nonterminals* or *nonterminal symbols*),
2. a finite set T of *terminals* (or *terminal symbols*),
3. a finite set P of *productions* (a.k.a. *rules*),
4. a designated *start symbol* (a.k.a. *axiom*) $S \in V$,

Thus, a CFG G is a 4-tuple $G = (V, T, P, S)$ formed of its variables V, terminals T, productions P, and start symbol S.

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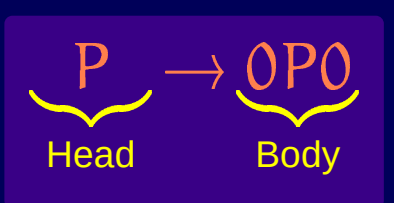
2. a finite set T of *terminals* (or *terminal symbols*), which provides the alphabet of the language defined by the grammar,
3. a finite set P of *productions* (a.k.a. *rules*), that provide the recursive definitions of the languages associated to the elements of V,
4. a designated *start symbol* (a.k.a. *axiom*) $S \in V$, that declares which of the $|V|$ languages defined by the grammar is the one we are actually interested in.

Thus, a CFG G is a 4-tuple $G = (V, T, P, S)$ formed of its variables V, terminals T, productions P, and start symbol S.

Productions

A production consists of

1. A variable $A \in V$ that is (partially) defined by the production, called the *head* of the production,
2. the production symbol $\rightarrow$ (or sometimes ::=),
3. a string α over $V \cup T$, called the *body* of the production.



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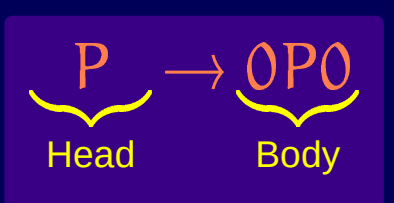
3. a string α over $V \cup T$, called the *body* of the production.
   The body represents *one possible* way of building strings belonging to the language of the head variable A: We obtain a word of the language of A by replacing any variable A' occuring in α by a word (over T) known to be in the language of A'.



Recursive inference

A word w belongs to the language of a CFG $G = (V, T, P, S)$ iff it can be *inferred* to belong to the language of S by the following recursive inference rules:

1. If P contains a production $A \to v$ with $v \in T^*$ then we can infer that v is in the language of A **(base inference)**

2. If we have inferred $w_1, \ldots, w_n \in T^*$ to be in the languages of A_1 to A_n and P contains a production $A \to v_1 A_1 v_2 A_2 \ldots v_n A_n v_{n+1}$ with $v_i \in T^*$ then we can infer that $v_1 w_1 v_2 w_2 \ldots v_n w_n v_{n+1}$ is in the language of A **(recursive inference)**

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Notation:

- The language $L(G)$ of a CFG $G = (V, T, P, S)$ is $\{w \in T^* \mid w \text{ can be inferred to be in } G\}$.
- For $A \in V$, $L_G(A)$ is $L(G')$ with $G' \stackrel{\text{def}}{=} (V, T, P, A)$.

Reductions

A word $w \in \mathrm{T}^*$ belongs to the language of a CFG $\mathrm{G} = (\mathrm{V}, \mathrm{T}, \mathrm{P}, S)$ iff it can be *reduced* to S in a finite number of steps by the following reduction rule:

- If P contains a production $\mathrm{B} \to \beta$ (with $\beta \in (\mathrm{V} \cup \mathrm{T})^*$) then $\alpha\beta\gamma \in (\mathrm{V} \cup \mathrm{T})^*$ can be reduced to $\alpha\mathrm{B}\gamma$.

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Notation:

- If $\alpha \in (V \cup T)^*$ can be reduced to S in a finite number of steps then α is called a sentential form of G.
- A sentential form w with $w \in T^*$ is called a sentence of G.

Derivations

A word $w \in \mathrm{T}^*$ belongs to the language of a CFG $\mathrm{G} = (\mathrm{V}, \mathrm{T}, \mathrm{P}, S)$ iff it can be *derived* from S in a finite number of steps by the following derivation rule:

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Notation:

- If β is a derivation of α (wrt. G) then we write $\alpha \Rightarrow_G \beta$.
- If β can be derived from α in a finite number (maybe 0) of steps, then we write $\alpha \Rightarrow_G^* \beta$.
- If $S \Rightarrow_G \alpha$ then α is called a sentential form of G.
- A sentential form w with $w \in T^*$ is called a sentence of G.

Leftmost derivations

A word $w \in T^*$ belongs to the language of a CFG $G = (V, T, P, S)$ iff it can be *derived through leftmost derivation* from S in a finite number of steps by the following leftmost derivation rule:

- If P contains a production $B \to \beta$ and $\alpha \in T^*$ then $\alpha B \gamma \in (V \cup T)^*$ gives rise to the *leftmost* derivation $\alpha\beta\gamma$.

Notation:

- If β is a leftmost derivation of α (wrt. G) then we write $\alpha \overset{lm}{\Rightarrow} \beta$.
- If $S \overset{lm}{\Rightarrow} \alpha$ then α is called a left sentential form of G.

Rightmost derivations

A word $w \in T^*$ belongs to the language of a CFG $G = (V, T, P, S)$ iff it can be *derived through rightmost derivation* from S in a finite number of steps by the following rightmost derivation rule:

- If P contains a production $B \to \beta$ and $\gamma \in T^*$ then $\alpha B \gamma \in (V \cup T)^*$ gives rise to the *rightmost* derivation $\alpha\beta\gamma$.

Notation:

- If β is a rightmost derivation of α (wrt. G) then we write $\alpha \overset{rm}{\Rightarrow} \beta$.
- If $S \overset{rm}{\Rightarrow} \alpha$ then α is called a right sentential form of G.

Parse trees

A parse tree for a CFG $G = (V, T, P, S)$ is a tree satisfying

1. each leaf is labeled by either a variable $A \in V$, a terminal $t \in T$, or by ε;
2. each interior node is labeled by a variable $A \in V$;
3. if a node is labeled A and has sons X_1 to X_n (from left to right) then $A \to X_1 \dots X_n \in P$,

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2. each interior node is labeled by a variable $A \in V$;
3. if a node is labeled A and has sons X_1 to X_n (from left to right) then $A \to X_1 \dots X_n \in P$, where all $X_i \neq \varepsilon$ if $n > 1$.

Yields of parse trees

The yield of a parse tree is the word $\alpha \in (V \cup T)^*$ obtained by concatenating all leafs of the parse tree from left to right.

A word $w \in T^*$ belongs to the language of a CFG $G = (V, T, P, S)$ iff w is the yield of a parse tree for G with root S.

Why termed context-free?

The context-free grammars are termed “context-free” because reductions/derivations don’t take account of the context:

$$\beta \Rightarrow_G \delta \quad \text{iff} \quad \alpha\beta\gamma \Rightarrow_G \alpha\delta\gamma$$

Context-free languages

Def: The context-free languages are exactely those languages that can be defined by a context-free grammar.

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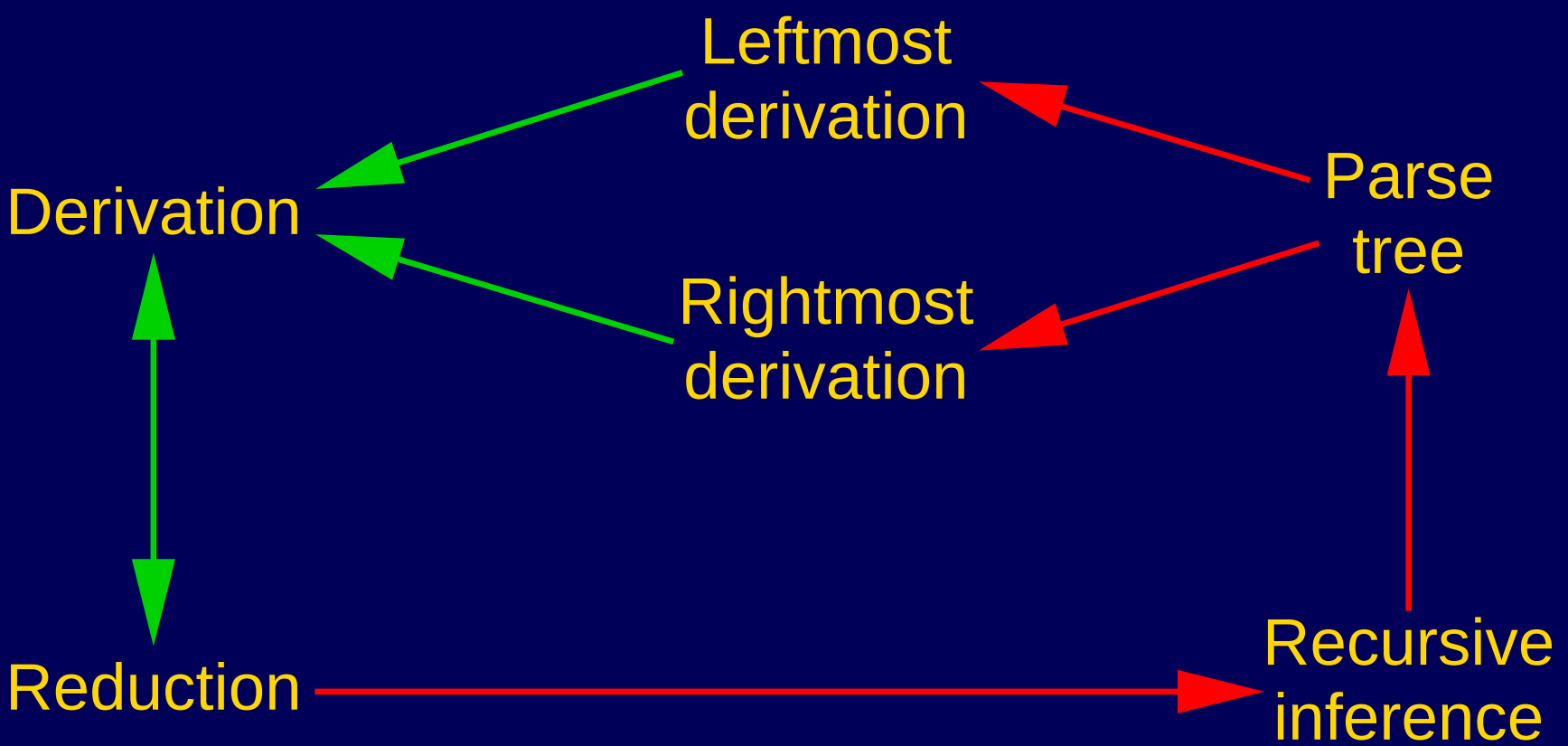
Qu.: What is “the language” defined by a CFG, given that we have multiple mechanisms for using CFGs, namely recursive inference, reductions, (restricted) derivations, parse trees?

Thm: Given a CFG G, the languages defined by G via

1. recursive inference,
2. reductions,
3. derivations,
4. leftmost derivations,
5. rightmost derivations,
6. yields of parse trees

coincide.

Structure of proof



Formal statements: Parse tree to leftm. der.

Thm: Let $G = (V, T, P, S)$ be a CFG, $A \in V$, and $w \in T^*$.

If there is a parse tree with root A and yield w then $A \overset{lm}{\Rightarrow}{}^*_G w$.

N.B.: Above statement holds for $w \in T^*$. It is in general not true for arbitrary yields of parse trees.

Prf: We perform an induction on the height of the parse tree.

Formal statements: Rec. inf. to parse tree

Thm: Let $G = (V, T, P, S)$ be a CFG, $A \in V$, and $w \in T^*$.
If there is a recursive inference showing that w is in the language of A then there is a a parse tree with root A and yield w.

Prf: We perform an induction on the number of steps in the recursive inference.

Formal statements: Reductions to rec. inf.

Thm: Let $G = (V, T, P, S)$ be a CFG, $A \in V$, and $w \in T^*$.
If there is a reduction sequence reducing w to A (i.e., $w \rightsquigarrow_G^* A$) then there is a recursive inference that determines w to be in the language of A.

N.B.: Above statement applies only to $w \in T^*$. It can't be generalized because the recursive inference procedure starts its recursion with productions that have terminal body, i.e. can infer properties of terminal strings only.

Prf: We perform an induction on the length of the reduction sequence.

Regular implies context-free

Thm: If $\mathbb{L}$ is regular then $\mathbb{L}$ is context-free ($=$ definable by a CFG).

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Thm: If L is regular then L is context-free ($=$ definable by a CFG).

Prf: (Sketch) We map REs to CFGs: Given RE E, we take a variable $[\mathrm{F}]$ for each subexpression F of E and generate productions $\mathrm{P}_{[\mathrm{F}]}$ for $[\mathrm{F}]$ such that $\mathrm{L}_{\mathrm{G}}([\mathrm{F}]) = \mathrm{L}(\mathrm{F})$.

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$$
\begin{array}{rcl}
\emptyset & \rightsquigarrow & \text{no production} \\
\varepsilon & \rightsquigarrow & [\varepsilon] \to \varepsilon \\
a & \rightsquigarrow & [a] \to a \\
F_1 + F_2 & \rightsquigarrow & \left\{ \begin{array}{l} [F_1 + F_2] \to [F_1] \\ [F_1 + F_2] \to [F_2] \end{array} \right. \\
F_1 F_2 & \rightsquigarrow & [F_1 F_2] \to [F_1][F_2] \\
F^* & \rightsquigarrow & \left\{ \begin{array}{l} [F^*] \to \varepsilon \\ [F^*] \to [F][F^*] \end{array} \right.
\end{array}
$$

Take the CFG $G = (\underbrace{\{[F] \mid F \text{ subexp. of } E\}}_{V}, \underbrace{\Sigma_E}_{T}, \underbrace{\bigcup P_{[F]}}_{P}, \underbrace{[E]}_{S})$.

Context-free does not imply regular

As the palindromes are context-free, yet not regular, there is a context-free language that is not regular (in fact that are infinitely many).
Hence:

Thm: The context-free languages are a strict superset of the regular languages.