CS 252

Theory of Automata, Formal Languages, and Computation

Lecture 10: Closure Properties for Regular Languages

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Spring 2004

Outline

- Closure of regular languages under boolean operations
  - Union
  - Complementation
  - Intersection
  - Difference
- Closure of regular languages under reversal
- Closure of regular languages under homomorphisms and inverse homomorphisms

Closure Properties

- There are many operations that, when applied to regular languages result in regular languages
  - Union, intersection, difference, reversal, …
- Such properties of regular languages are called ***closure properties***
  - e.g. closure of regular languages over intersection
- Closure properties are very useful for designing complex automata
  - For example, given two different languages we can systematically construct an automaton that recognizes exactly the intersection of these languages

Closure of Regular Languages Under Boolean Operations

- Let L and M be languages over alphabet Σ. Then $L \cup M$ is the language that contains all strings that are in either or both of L and M.

- Let L and M be languages over alphabet Σ. Then $L \cap M$ is the language that contains all strings that are in both L and M.

- Let L be a language over alphabet Σ. Then $\overline{L}$, the complement of L, is the set of strings in Σ^* that are not in L.

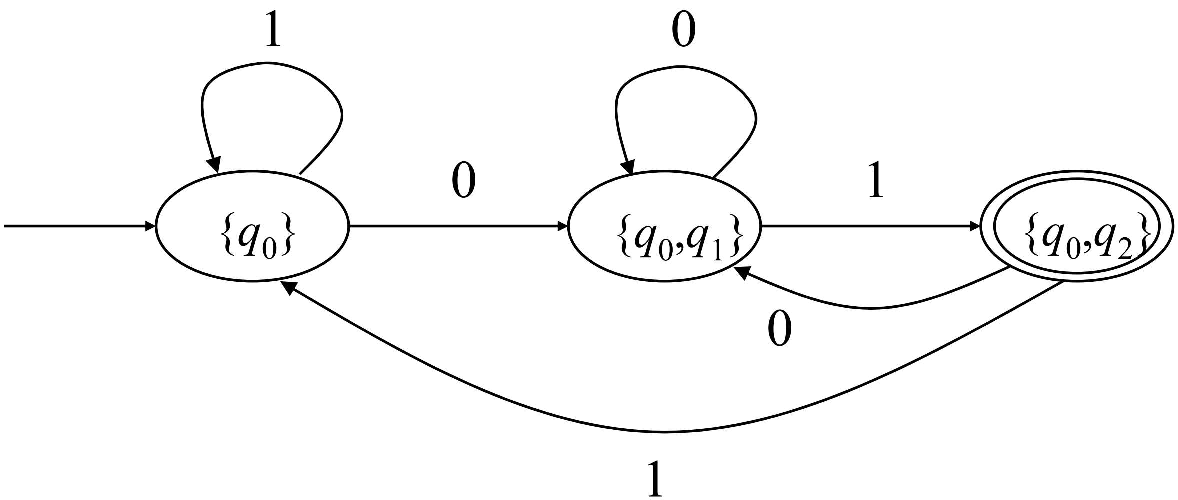
Closure Under Union

- ***Theorem***

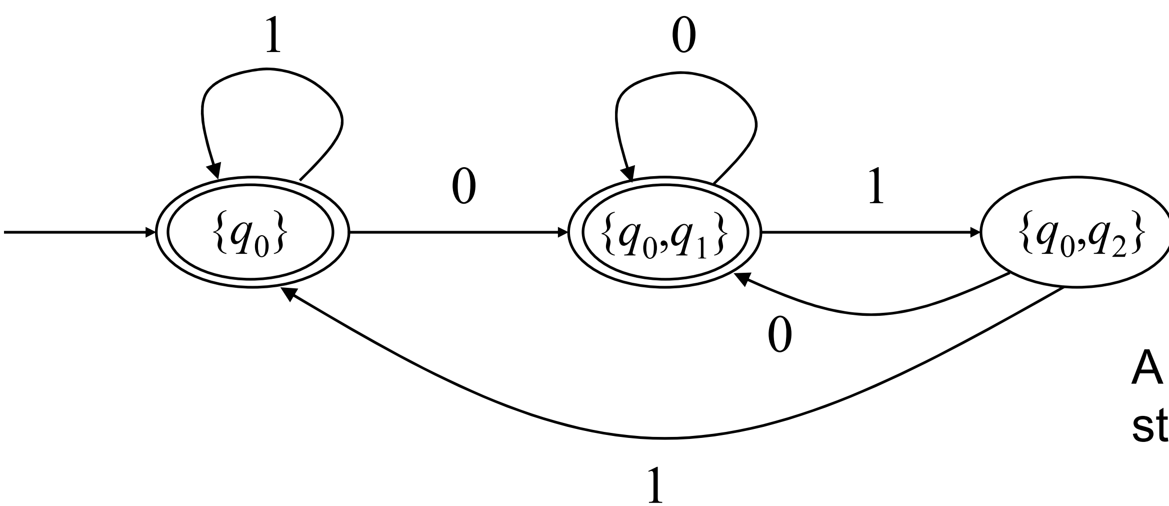
  If L and M are regular languages, then so is $L \cup M$.

  - Proof

Closure Under Complementation



A DFA accepting all strings that end in 01



A DFA accepting all strings that do ***not*** end in 01

Closure Under Complementation

- ***Theorem***

  If L is a regular language over alphabet Σ,

  then $\overline{L} = \Sigma^* - L$ is also a regular language.

  - Proof

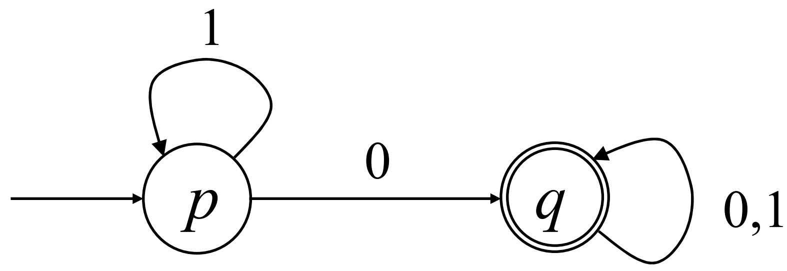
Example

- Consider the language M consisting of those strings of 0's and 1's that have an unequal number of 0's and 1's

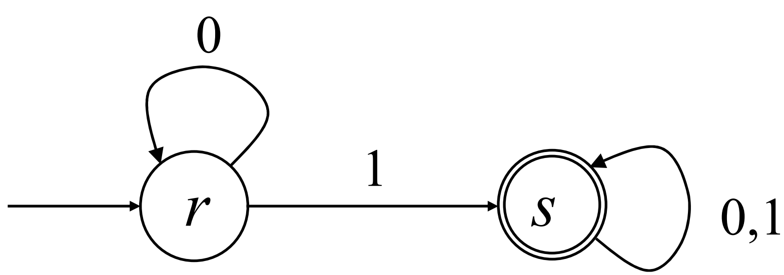
- Prove that M is not a regular language

- *Hint:* Combine the pumping lemma for $L_{eq} = \{w| \ w=0^n1^n\}$ and closure under complementation

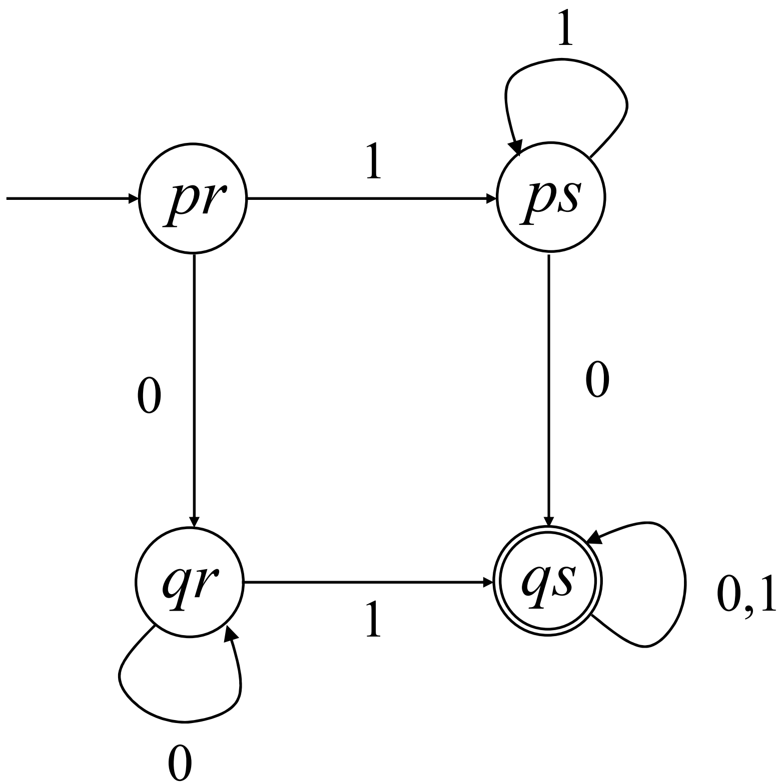
Closure Under Intersection



A DFA accepting all strings that have a 0



A DFA accepting all strings that have a 1



A DFA accepting all strings that have both a 0 and a 1

Closure Under Intersection

- ***Theorem***

  If L and M are regular languages, then so is $L \cap M$.

  - Proof

Closure Under Difference

- ***Theorem***

  If L and M are regular languages, then so is $L - M$.

  - Proof

Reversal

- The reversal of a string $w = a_1 a_2 \dots a_n$ is the string $w^R = a_n a_{n-1} \dots a_1$
- The reversal L^R of a language L consists of the reversal of all strings in L
- Suppose that L is regular and therefore, it is accepted by an automaton A, that is $L = L(A)$
- We can construct an automaton for L^R as follows:
  1. Reverse all the arcs in the transition diagram of A
  2. Make the start state of A the only new accepting state
  3. Create a new start state with epsilon-transitions to all the accepting states of A

Closure Under Reversal

- ***Theorem***

  If L is a regular language, so is L^R.

  - *Proof*

Example

- Let L be defined by the regular expression $(0+1)0^*$
- Compute the regular expression for L^R

Homomorphisms

- A *homomorphism* h on Σ is a function $h: \Sigma^* \rightarrow \Theta^*$ where Σ and Θ are alphabets
- Let $w = a_1 a_2 \ldots a_n \in \Sigma^*$
  - $h(\mathrm{w}) = h(\mathrm{a}_1)\, h(a_2) \ldots h(a_n)$
  - $h(L) = \{h(w): w \in L\}$
- Let $h: \{0,1\}^* \rightarrow \{a,b\}^*$ be defined by $h(0) = ab$ and $h(1) = \varepsilon$
  - $h(0011) = abab$
  - $h(L(10^*1)) = L((ab)^*)$

Closure Under Homomorphisms

- ***Theorem***

  If L is a regular language over alphabet Σ, and h is a homomorphism on Σ, then $h(L)$ is also regular.

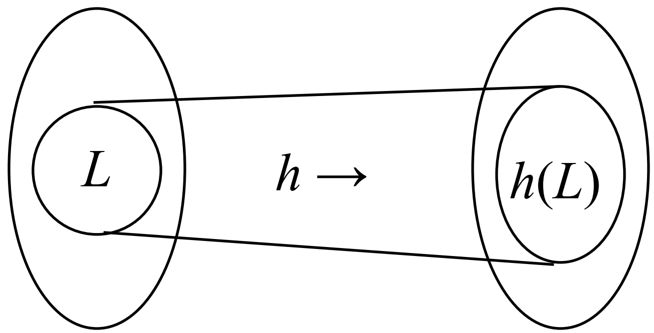
  - *Proof*

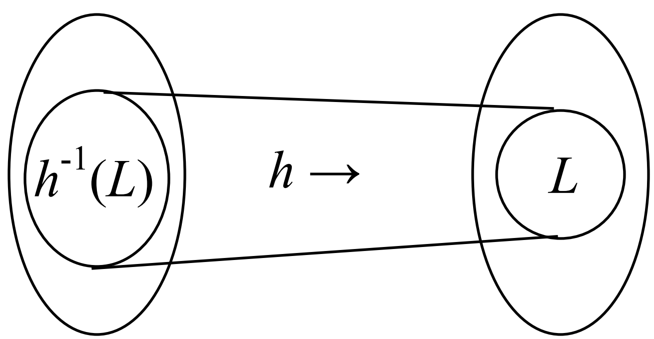
Example

- Suppose that h is defined by $h(0) = a$, $h(1) = ab$, $h(2) = ba$
  - Compute $h(0120)$
  - If L is the language $L(01^*2)$, compute $h(L)$

Inverse Homomorphisms

- Let $h: \Sigma^* \to \Theta^*$ be a homomorphism
- Consider a language L over Θ^*
- $h^{-1}(L) = \{w \in \Sigma^*: h(w) \in L\}$





Example

- Let h: $\{a,b\}^* \to \{0,1\}^*$ be defined by $h(a) = 01$ and $h(b) = 10$
- If $L = L((00+1)^*)$ then $h^{-1}(L) = L((ba)^*)$
- *Claim*: $h(w) \in L$ if and only if $w = (ba)^n$

Closure Under Inverse Homomorphisms

- ***Theorem***

  If h is a homomorphism from alphabet Σ to alphabet Θ, and L is a regular language over alphabet Θ, then $h^{-1}(L)$ is also a regular language.

  - *Proof*

Next lecture

- Decision properties of regular languages