COMPREHENSION AND DIAGONALIZATION

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Preliminaries. In Cantor's set theory a set was thought as being defined by a property. This conception was accordingly synthesized in a general statement, originally due to Frege, called the Axiom of Comprehension:

(*) *For any condition $\Pi(x)$ on x there exists a set which contains exactly those elements x which fulfill this condition.*

In symbols,

$$\forall z_1, \ldots, z_n \, \exists y \, \forall x \, \big(x \in y \iff \Pi(x)\big), \tag{1}$$

where $\Pi(x)$ is a first order well formed formula, $z_1, \ldots, z_n, x$ are the free variables of $\Pi(x)$ and y is not free in $\Pi(x)$.

As well-known (1) is inconsistent by Russell's antinomy, i.e. the logical truth

$$\neg \, \exists y \, \forall x \, \big(x \in y \iff x \notin x\big) \tag{2}$$

directly contradicts (1) by substituting $\Pi(x)$ with $x \notin x$ (see [6], 30-32). One of the outcomes, this paper is concerned with, is a solution of the problem which has never been proposed before, and which do not need to deal with classes nor types.

Before to go into the exposition of such results, we must emphasize that they hold on the set theory as originally conceived by Cantor and Frege,[1] i.e. the theory based on (*). We shall call it $\mathfrak{S}$.

Basic setup. Let us start recalling the definitions for the main concepts we shall be involved. In any case for more details the reader is referred to [5, 6]

$$\emptyset \text{ is the empty set,}$$
$$x \subseteq z \;=\; \forall w \, (w \in x \Longrightarrow w \in z),$$
$$\mathscr{P}(z) \;=\; \{x | x \subseteq z\},$$
$$x \cup y \;=\; \{z | z \in x \vee z \in y\},$$

[1] The so called *naive* set theory, raising from Cantor's definition of an "aggregate as a collection of elements into a whole of our thought" [1, 2]. In order to provide a foundation for arithmetic Frege attempted to formalize set theory [7, 8, 9], but the accomplishment of such work was hardly invalidated by (2). Furthermore, although we can attribute this conception as common to Cantor and Frege, in reality they did not share it without disagreed ([4], 220-228).

$$x \cap y \;=\; \{z | z \in x \land z \in y\},$$

$$\bigcup z \;=\; \{z | \exists y (x \in y \lor y \in z\},$$

$\bigcap z$ = the set of the members common to *all* members of z, if there is no x common to all members of z we have $\bigcap z = \emptyset$;

$$\overline{y} \;=\; \{x | x \notin y\}.$$

$\mathfrak{S}$ has two axioms only, (1) and Extensionality

$$\forall x \, \forall y \, \big[\forall z \, (z \in x \iff z \in y) \implies x = y \big].$$

1. For a new comprehension

Let us consider the formula

$$(\overset{\Leftrightarrow}{C}) \qquad \forall z_1, \ldots, z_n \, \forall y \, \forall x \, \big[(x \in y \iff \Pi(x)) \iff (x \notin y \iff \neg \Pi(x)) \big].$$

where Π, $z_1, \ldots, z_n$, x, y are as in (1).

One can easily see that $\overset{\Leftrightarrow}{C}$ is a logically valid formula. Table 1 displays its proof by means of the method of the semantic trees ([12], 110-115 and [13]). $\overset{\Leftrightarrow}{C}$ stands Russell's antinomy since

$$(2) \iff \overset{\Leftrightarrow}{C}$$

is logically valid again. $\overset{\Leftrightarrow}{C}$ holds for set theory exactly as (2) holds for set theory. We are then able to admit existential assertions which essentially refer to "the set of *all* those x such that ...".

Nothing prevents the use of $\overset{\Leftrightarrow}{C}$ as a general principle for set theory. For it one can define a set y, by means of a property Π, if and only if one defines its complement $\overline{y}$, by means of the complementary property $\neg\Pi$.

In ([6], 40-41), Fraenkel, Bar-Hillel and Levy provides a proof that Zermelo's *Aussonderung* (the Axiom of Subset, or of Separation) implies the statements that given any set x there is no set which is the complement of x and that there is no set to which every set belongs (see also [5], 21-22; [11], 61). The problem find a correct solution and a definition in $\overset{\Leftrightarrow}{C}$.

$$\neg\,\forall y\,\forall x\,[(x \in y \iff \Pi(x)) \iff (x \notin y \iff \neg\Pi(x))]$$

$$\exists y\,\exists x\,\neg\,[(x \in y \iff \Pi(x)) \iff (x \notin y \iff \neg\Pi(x))]$$

$$\neg\,[(a \in b \iff \Pi(a)) \iff (a \notin b \iff \neg\Pi(a))]$$

$$\begin{array}{cc} (a \in b \iff \Pi(a)) & \neg\,(a \in b \iff \Pi(a)) \\ \neg\,(a \notin b \iff \neg\Pi(a)) & (a \notin b \iff \neg\Pi(a)) \end{array}$$

$$\begin{array}{cccc} a \in b & a \notin b & a \in b & a \notin b \\ \Pi(a) & \neg\Pi(a) & \neg\Pi(a) & \Pi(a) \end{array}$$

$$\begin{array}{cccccccc} a \notin b & a \in b & a \notin b & a \in b & a \notin b & a \in b & a \notin b & a \in b \\ \neg\neg\Pi(a) & \neg\Pi(a) & \neg\neg\Pi(a) & \neg\Pi(a) & \neg\Pi(a) & \neg\neg\Pi(a) & \neg\Pi(a) & \neg\neg\Pi(a) \\ \times & \times & \times & \times & \times & \times & \times & \times \end{array}$$

Proof of $\overset{\Leftrightarrow}{C}$. All the branches of this tree, starting on the negation of $\overset{\Leftrightarrow}{C}$ (without $z_1, \ldots, z_n$, since they have not influence on the proof) close, hence $\overset{\Leftrightarrow}{C}$ is logically valid. The rules applied are in sequence: *i*) negation on quantifier, two times, *ii*) introduction of new individual constants, by the rule of the existential quantifier (Rule E), two times, *iii*) biconditional, five times ($\neg(a \notin b) = a \in b$, $\neg(a \in b) = a \notin b$).

TABLE 1

$$\neg\left[\,\exists y\,\forall x\,(x \in y \iff \Pi(x)) \quad\iff\quad \exists w\,\forall v\,(v \notin w \iff \neg\Pi(v))\,\right]$$

$$\begin{array}{c|c}
\begin{array}{c}\exists y\,\forall x\,(x \in y \iff \Pi(x)) \\ \neg\,\exists w\,\forall v\,(v \notin w \iff \neg\Pi(v))\end{array} & \begin{array}{c}\neg\,\exists y\,\forall x\,(x \in y \iff \Pi(x)) \\ \exists w\,\forall v\,(v \notin w \iff \neg\Pi(v))\end{array} \\
| & | \\
\forall x\,(x \in a \iff \Pi(x)) & \forall y\,\exists x\,\neg\,(x \in y \iff \Pi(x)) \\
| & | \\
\forall w\,\exists v\,\neg\,(v \notin w \iff \neg\Pi(v)) & \forall v\,(v \notin a \iff \neg\Pi(v)) \\
| & | \\
\exists v\,\neg\,(v \notin a \iff \neg\Pi(v)) & \exists x\,\neg\,(x \in a \iff \Pi(x)) \\
| & | \\
\neg\,(b \notin a \iff \neg\Pi(b)) & \neg\,(b \in a \iff \Pi(b)) \\
| & | \\
(b \in a \iff \Pi(b)) & (b \notin a \iff \neg\Pi(b))
\end{array}$$

$$\begin{array}{cc|cc}
\begin{array}{c} b \notin a \\ \Pi(b) \end{array} & \begin{array}{c} b \in a \\ \neg\Pi(b) \end{array} & \begin{array}{c} b \in a \\ \neg\Pi(b) \end{array} & \begin{array}{c} b \notin a \\ \Pi(b) \end{array}
\end{array}$$

$$\begin{array}{cccccccc}
b \in a & b \notin a & b \in a & b \notin a & b \notin a & b \in a & b \notin a & b \in a \\
\Pi(b) & \neg\Pi(b) & \Pi(b) & \neg\Pi(b) & \neg\Pi(b) & \Pi(b) & \neg\Pi(b) & \Pi(b) \\
\times & \times & \times & \times & \times & \times & \times & \times
\end{array}$$

Proof of (3). The negation of the biconditional produces two branches. The rules applied on the first one are: *i*) introduction of a new individual constant, a, by Rule E, *ii*) negation on quantifier two times, *iii*) application of the rule of the universal quantifier (Rule U) on a, *iv*) introduction of a new individual constant, b, by rule E, *v*) Rule U on b, *vi*) biconditional two times. The second branch is generated in a similar way. All the branches closes hence (3) is logically valid ($z_1, \ldots, z_n$ without effect, and $\neg(b \notin a) = b \in a$, $\neg(b \in a) = b \notin a$).

TABLE 2

There is one more first order formula (among many other[2]), we shall consider like a guiding principle in $\mathfrak{S}$ for its clear meaning:

$$(3)\qquad \begin{aligned} &\forall z_1,\dots,z_n\,\exists y\,\forall x\,\big[(x\in y \iff \Pi(x))\big] \iff \\ &\qquad\qquad \forall z_1,\dots,z_n\,\exists w\,\forall v\big[(v\notin w \iff \neg\Pi(v))\big]. \end{aligned}$$

For it one can assert in $\mathfrak{S}$ the existence of a set, y, as defined by means of a property, Π, only asserting the coexistence of its complement $\overline{y}$ as defined by means of the complementary property, $\neg\Pi$. A demonstration of its logical validity is given in table 2. By means of it we can reformulate $(*)$ in:

(**) *For any condition $\Pi(x)$ on x there exists a set which contains exactly those elements x which fulfill this condition if and only if there exists a set which do not contains exactly those elements x which do not fulfill this condition.*

In order to clarify inferences into this new settlement for $\mathfrak{S}$, we reproduce two deductions which starts from the definition of a set as founded on the relative property, following the rules of the predicate calculus ([12], 54-73).

i) $\exists\, y\,\forall\, x\,(x\in y\Leftrightarrow \Pi(x))$ (1)

ii) $\big(\exists\, y\,\forall\, x\,(x\in y\Leftrightarrow \Pi(x))\ \Leftrightarrow\ \exists\, w\,\forall\, v\,(v\notin w\Leftrightarrow\neg\Pi(v))\big)\Rightarrow$
$\big(\exists\, y\,\forall\, x\,(x\in y\Leftrightarrow \Pi(x))\ \Rightarrow\ \exists\, w\,\forall\, v\,(v\notin w\Leftrightarrow\neg\Pi(v))\big)$
tautology $(A\Leftrightarrow B)\Rightarrow(A\Rightarrow B)$.

iii) $\exists\, y\,\forall\, x\,(x\in y\Leftrightarrow \Pi(x))\ \Rightarrow\ \exists\, w\,\forall\, v\,(v\in w\Leftrightarrow \Pi(v))$ $ii,(3),$m.p.

iv) $\exists\, w\,\forall\, v\,(v\in w\Leftrightarrow \Pi(v))$ $i, iii,$m.p.

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[2] For example

$$\forall z_1,\dots,z_n\,\exists y\,\forall x\,\big[(x\in y \iff \Pi(x)) \iff (x\notin y \iff \neg\Pi(x))\big],$$

$$\forall z_1,\dots,z_n\,\forall y\,\forall x\,\big[(x\in y \Longrightarrow \Pi(x)) \vee (x\notin y \Longrightarrow \neg\Pi(x))\big],$$

and

$$\forall z_1,\dots,z_n\,\forall y\,\forall x\,\big[(x\in y \Longrightarrow \Pi(x)) \Longrightarrow (\neg\Pi(x) \Longrightarrow x\notin y)\big]$$

are all logical truths. Moreover we stress that neither

$$\forall z_1,\dots,z_n\,\exists y\,\forall x\,\big[(x\in y \iff \Pi(x)) \iff (x\in y \iff \neg\Pi(x))\big]$$

nor

$$\forall z_1,\dots,z_n\,\exists y\,\forall x\,\big[(x\in y \iff \Pi(x)) \iff (x\notin y \iff \neg\Phi(x))\big]$$

are logical truth.

$i)$	$\exists\, y\, \forall\, x\, (x \in y \Leftrightarrow \Pi(x))$	(1)
$ii)$	$\forall\, x\, (x \in a \Leftrightarrow \Pi(x))$	i, Rule C
$iii)$	$(a \in b \Leftrightarrow \Pi(a))$	ii, Rule A4
$iv)$	$(a \in b \Leftrightarrow \Pi(a)) \;\Leftrightarrow\; (a \notin b \Leftrightarrow \neg\Pi(a))$	$\overset{\Leftrightarrow}{C}$, Rule A4
$v)$	$(a \in b \Leftrightarrow \Pi(a)) \;\Leftrightarrow\; (a \notin b \Leftrightarrow \neg\Pi(a))$ $\Rightarrow (a \in b \Leftrightarrow \Pi(a)) \;\Rightarrow\; (a \notin b \Leftrightarrow \neg\Pi(a))$	tautology $(A \Leftrightarrow B) \Rightarrow (A \Rightarrow B)$
$vi)$	$(a \in b \Leftrightarrow \Pi(a)) \;\Rightarrow\; (a \notin b \Leftrightarrow \neg\Pi(a))$	iv, v, m.p.
$vii)$	$(a \notin b \Leftrightarrow \neg\Pi(a))$	iii, vi, m.p.
$viii)$	$\forall\, x\, (x \notin b \Leftrightarrow \neg\Pi(x))$	vii, Gen
$ix)$	$\exists\, y\, \forall\, x\, (x \notin y \Leftrightarrow \neg\Pi(x))$	$viii$, Rule E4

Consider $iii)$ and $vii)$, since $\Pi(a)$ is true $\neg\, \Pi(a)$ is false, nevertheless, we conclude $ix)$, the existence of a set which elements are qualified by a contradictory property.

Remark 1. Above, as in $\overset{\Leftrightarrow}{C}$ and in (3), it would be misleading to identify y in $x \notin y$, with a set without elements, i. e. $\emptyset$. In other terms it is not correct to assert that contradictory properties hold only for the empty set. Consider for instance

$$(4) \qquad \forall y\, \forall x\, \big[(x \in y \iff x < x\big) \iff (x \notin y \iff \neg(x < x))\big].$$

where $<$ is a partial ordering relation, for which $x < x$ is contradictory. Clearly (4) is a valid example of $\overset{\Leftrightarrow}{C}$.

2. The Power Set

Let us replace $\Pi(x)$ with $x \subseteq z$, in $\overset{\Leftrightarrow}{C}$, obtaining

$$(5) \qquad \forall z\, \forall y\, \forall x\, \big[(x \in y \iff x \subseteq z\big) \iff (x \notin y \iff \neg(x \subseteq z))\big].$$

From a comparison with the classical Axiom of Power Set[3]

[3] Axioms of Zermelo Fraenkel Skolem theory, the so called $\mathfrak{ZF}$, can be found in [3, 6]. See also original Zermelo's articles in [10]

(6) $$\forall z\, \exists y\, \forall x\, \big(x \in y \iff x \subseteq z\big).$$

where y is $\mathscr{P}(z)$, the power-set of z, it follows immediately that being (5) a logically valid formula, it holds for the power-set. (5) establishes the definition of $\mathscr{P}(z)$ as inseparable from the definition of its complement $\overline{\mathscr{P}}(z)$.

Let us consider also

$$\forall z\, \exists y\, \forall x\, \big[(x \in y \iff x \subseteq z) \iff (x \notin y \iff \neg(x \subseteq z))\big].$$

which is logically valid again, and states the existence of $\mathscr{P}(z)$ as inseparable from the existence of its complement $\overline{\mathscr{P}}(z)$.

On account of that Cantor's diagonalization[4] results to be invalid.

3. Diagonalization

Cantor's theorem, according to which, the power-set of a denumerable set contains absolutely more member than this set, is not valid.

The proof is straightforward. We shall recall here the demonstration given by Enderton in ([5], 132, see also [11], 179), which asserts *no set is equinumerous to its power-set.*

Let $\mathsf{g} : a \mapsto \mathscr{P}(a)$. *We will construct a subset b of a that is not in ran* g. *Specifically let*

i) $$b = \{x \in a \mid x \notin \mathsf{g}(x)\}.$$

Then $b \subseteq a$, but for each $x \in a$

ii) $$x \in b \iff x \notin \mathsf{g}(x).$$

Hence $b \neq \mathsf{g}(x)$.

An immediate consequence of $\overset{\Leftrightarrow}{C}$ and *ii)* is that

(7) $$x \notin b \iff x \in \mathsf{g}(x).$$

therefore $\overline{b} = \mathsf{g}(x)$.

For the sake of exactness the whole diagonalization is constructed on the use of the Aussonderung, so we shall state:

$$\forall x\, \big[(x \in b \iff x \in a \wedge x \notin \mathsf{g}(x)) \iff (x \notin b \iff \neg(x \in a \wedge x \notin \mathsf{g}(x)))\big].$$

from which it follows

$$\forall x\, \big[(x \in b \iff x \in a \wedge x \notin \mathsf{g}(x)) \iff (x \notin b \iff x \notin a \vee x \in \mathsf{g}(x))\big].$$

[4] Precisely, only one of the two Cantor's diagonalization. That one on reals will be object of a next investigation, see ending remarks.

The meaning of the latter lead to the case of something out of a accordingly to the consequences of (5).

We have established the following theorem.

Theorem 3.1. *By the logical validity of* $\overset{\Leftrightarrow}{C}$ *Cantor's diagonalization is incorrect.*

4. The Axiom of Choice

Let us recall the Axiom of Choice as defined in ([6], 39-40, 53-55)

(AC) *If t is a disjointed set which does not contain the null–set, its outer product $\mathcal{OP}t$ is different from the null–set.*

$\mathcal{OP}t$ exists only if $\mathscr{P}(\bigcup t)$, the set of the subsets of the union of t, exists. Immediately, by (5), AC is true, since, if t does not contain the null–set, $\mathscr{P}(\bigcup t)$ is never disjointed, therefore there are selection sets of t and $\mathcal{OP}t$ is different from the null–set.

The reasons are in the logical structure of (5), which defines $\mathscr{P}(z)$, namely y, as a set excluding those parts x for which hold $\neg\,(x \subseteq z)$ (see the component $(x \notin y \iff \neg\,(x \subseteq z))$ in (5)).

Let us consider (5), with $\emptyset \notin z$. $\neg\,(x \subseteq z)$ is true if $(z \subset x) \vee (x \cap z = \emptyset)$, hence $\mathscr{P}(z)$ is constructed in such a way that x and z are never disjointed (since $(\neg(z \subset x) \wedge \neg(x \cap z = \emptyset))$ holds for $\mathscr{P}(z)$ in (5)).

Now, consider t in AC. $\emptyset \notin t$ hence all the members of $\mathscr{P}(\bigcup t)$ are never disjointed, also if t is disjointed. Consequently $\mathcal{OP}t$ is different from the null–set.

The opposite for $\overline{\mathscr{P}}$, because the members of $\overline{\mathscr{P}}(\bigcup t)$ are always disjointed, and even if t does not contain the null–set its outer product is equal to the empty set. To visualize how $\overline{\mathscr{P}}$ causes this situation we can consider it to be defined by

$$\forall z\, \exists y\, \forall x\, \big[(x \in y \iff \neg(x \subseteq z)) \iff (x \notin y \iff \neg\neg(x \subseteq z))\big].$$

Since $\emptyset \notin z$, we can think y as excluding those parts x such that $x \subseteq z$ (and $\neg((z \subset x) \vee (x \cap z = \emptyset)))$. Thus for the parts x of y hold always $(x \cap z = \emptyset)$. We can then assert for $\mathfrak{S}$ the theorem:

Theorem 4.1.

$$\mathscr{P}(z) \Longrightarrow AC, \tag{8}$$

and

$$\overline{\mathscr{P}}(z) \Longrightarrow not\ AC. \tag{9}$$

Conclusive remarks. Our scope to demonstrate invalidity of Cantor diagonalization has been reached, hence we consider this paper in itself concluded. Nevertheless, many and theoretically profound are the open questions. We briefly describe the more technical of them:

- $\mathfrak{S}$ seems to be able to restore Frege system, with some *essential* variations on the role of the null – set and of the universal – set (in the literature, the null class and the universal class);
- Our results demonstrate that $\mathfrak{S}$ is closed under complementation. The adjoining of the null and universal sets approach $\mathfrak{S}$ to the algebra of set;
- The completion of $\mathfrak{S}$ require a correct, i.e. without diagonalization, definition of the continuum. Such task might be based on the fact that by $\overset{\Leftrightarrow}{C}$, densely and not densely ordered sets are complementary sets.

Such points, together with the accomplishment of $\mathfrak{S}$, will be the object of our next investigation.

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